Voter model percolation

Balázs Ráth (BME)
Joint work with
Daniel Valesin (Groningen)

May 16, 2015

Voter model

- Clifford & Sudbury, 1973
  Competition of biological species occupying space

Voter model

- Clifford & Sudbury, 1973
  Competition of biological species occupying space
- Holley & Liggett, 1975
  Opinion exchange among social agents

Voter model

- $(\xi_t)_{t\geq 0}$ Markov process on $\{0,1\}^{\mathbb{Z}^d}$

Voter model

- $(\xi_t)_{t\geq 0}$ Markov process on $\{0,1\}^{\mathbb{Z}^d}$
- $x \in \mathbb{Z}^d$: voter, 0,1: opinions

Voter model

- $(\xi_t)_{t\geq 0}$ Markov process on $\{0,1\}^{\mathbb{Z}^d}$
- $x \in \mathbb{Z}^d$: voter, 0,1: opinions
- Dynamics: each $x \in \mathbb{Z}^d$, with rate 1 chooses a vertex

$$y \in B(x,1) \setminus \{x\} = \{\, y \in \mathbb{Z}^d \;:\; \|x-y\|_1 \leq 1 \,\} \setminus \{x\}$$

and copies the opinion of y.

Voter model

- $(\xi_t)_{t\geq 0}$ Markov process on $\{0,1\}^{\mathbb{Z}^d}$
- $x \in \mathbb{Z}^d$: voter, 0,1: opinions
- Dynamics: each $x \in \mathbb{Z}^d$, with rate 1 chooses a vertex

$$y \in B(x,1) \setminus \{x\} = \{\, y \in \mathbb{Z}^d \ : \ \|x-y\|_1 \leq 1 \,\} \setminus \{x\}$$

  and copies the opinion of y.
- Spread-out model:

$$y \in B(x,R) \setminus \{x\}$$

Stationary distributions

Stationary distributions

- $d = 1, 2$: only trivial

Stationary distributions

- $d = 1, 2$: only trivial
- $d \geq 3$: Coexistence: $\forall\, \alpha \in [0, 1] \;\; \exists! \;\; \mu_\alpha$ with density α

Stationary distributions

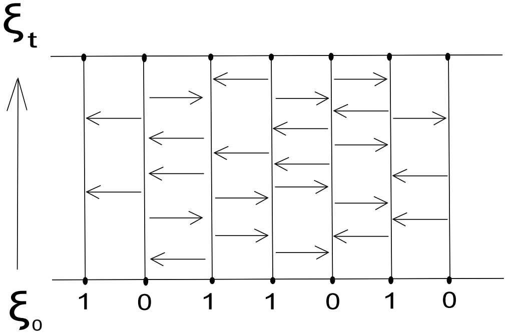
- $d = 1, 2$: only trivial
- $d \geq 3$: Coexistence: $\forall\, \alpha \in [0,1]$ $\exists!$ μ_α with density α

$$\mu_\alpha \overset{(d)}{=} \lim_{t\to\infty} \text{ process started from } \prod \mathrm{Ber}(\alpha)$$

Graphical construction

Graphical construction

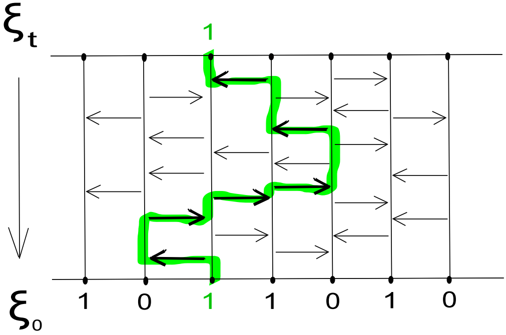
$(d = 1, R = 1)$, i.i.d. PPP of arrows



Forward in time: voter model

Graphical construction and duality

$(d = 1, R = 1)$, i.i.d. PPP of arrows



Backward in time in time: coalescing random walks

Construction of μ_α

Construction of μ_α

- run X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks

Construction of μ_α

- run X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks
- partition $\mathbb{Z}^d$: x and y in same block if $X_\cdot^x$ and $X_\cdot^y$ coalesce

Construction of μ_α

- run X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks
- partition $\mathbb{Z}^d$: x and y in same block if $X_\cdot^x$ and $X_\cdot^y$ coalesce
- given this partition: color blocks with i.i.d. $\mathrm{Ber}(\alpha)$ (0 or 1)
  (Note: each block size is infinite!)

Properties of μ_α

Properties of μ_α

- indeed $d = 1, 2$ trivial: they all coalesce

Properties of μ_α

- indeed $d = 1, 2$ trivial: they all coalesce
- if $d \geq 3$ then

$$\mathrm{Cov}_{\mu_\alpha}(\xi(x), \xi(y)) \asymp \|x - y\|^{2-d}$$

(decays like SRW Green function)

Properties of μ_α

- indeed $d = 1, 2$ trivial: they all coalesce
- if $d \geq 3$ then
$$\mathrm{Cov}_{\mu_\alpha}(\xi(x), \xi(y)) \asymp \|x - y\|^{2-d}$$
(decays like SRW Green function)
- strong correlations, clustering

Percolation under μ_α

Percolation under μ_α

- $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $0 =$ vacant, $1 =$ occupied

Percolation under μ_α

- $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $0 =$ vacant, $1 =$ occupied

$$\{0,1\}^{\mathbb{Z}^d} \supseteq A = \{\,\xi \;:\; \exists \text{ infinite cluster of 1's in } \xi\,\}$$

Percolation under μ_α

- $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $\quad 0 =$ vacant, $\quad 1 =$ occupied

$$\{0,1\}^{\mathbb{Z}^d} \supseteq A = \{\, \xi \ : \ \exists \ \text{infinite cluster of 1's in} \ \ \xi \,\}$$

- ergodicity + monotonicity $\implies \exists \, \alpha_c \in [0,1]$:

$$\begin{aligned} \alpha < \alpha_c \ &: \mu_\alpha(A) = 0, \\ \alpha > \alpha_c \ &: \mu_\alpha(A) = 1. \end{aligned}$$

Percolation under μ_α

- $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $0 =$ vacant, $1 =$ occupied

$$\{0,1\}^{\mathbb{Z}^d} \supseteq A = \{\, \xi \;:\; \exists \text{ infinite cluster of 1's in } \xi \,\}$$

- ergodicity + monotonicity $\implies \exists\, \alpha_c \in [0,1]$:

$$\begin{aligned} \alpha < \alpha_c &: \mu_\alpha(A) = 0, \\ \alpha > \alpha_c &: \mu_\alpha(A) = 1. \end{aligned}$$

Conjecture (Bricmont, Lebowitz, Maes, 1987)

If d=3, R=1 then $0 < \alpha_c < 1$.

- Non-triviality of percolation phase transition

Percolation under μ_α

- $\xi \in \{0,1\}^{\mathbb{Z}^d}$, $0 =$ vacant, $1 =$ occupied

$$\{0,1\}^{\mathbb{Z}^d} \supseteq A = \{\, \xi \,:\, \exists \text{ infinite cluster of 1's in } \xi \,\}$$

- ergodicity + monotonicity $\implies \exists\, \alpha_c \in [0,1]$:

$$\begin{aligned} \alpha < \alpha_c &: \mu_\alpha(A) = 0, \\ \alpha > \alpha_c &: \mu_\alpha(A) = 1. \end{aligned}$$

Conjecture (Bricmont, Lebowitz, Maes, 1987)

If d=3, R=1 then $0 < \alpha_c < 1$.

- Non-triviality of percolation phase transition
- Physics prediction [Halperin, Weinrib, 1983]:
  Different critical exponents than Bernoulli percolation

Context

Percolation models with "Green" decay of correlations on $\mathbb{Z}^d, d \geq 3$:

Context

Percolation models with ”Green” decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$

Context

Percolation models with ”Green” decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$
  Non-triv phase transition:
  [Sznitman, 2010], [Sidoravicius, Sznitman, 2009]

Context

Percolation models with ”Green” decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$
  Non-triv phase transition:
  [Sznitman, 2010], [Sidoravicius, Sznitman, 2009]
- Level set of Gaussian free field

Context

Percolation models with "Green" decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$
  Non-triv phase transition:
  [Sznitman, 2010], [Sidoravicius, Sznitman, 2009]
- Level set of Gaussian free field
  Non-triv phase transition:
  [Rodriguez, Sznitman, 2013]

Context

Percolation models with "Green" decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$
  Non-triv phase transition:
  [Sznitman, 2010], [Sidoravicius, Sznitman, 2009]
- Level set of Gaussian free field
  Non-triv phase transition:
  [Rodriguez, Sznitman, 2013]
- Random interlacements $\mathcal{I}^u$

Context

Percolation models with "Green" decay of correlations on $\mathbb{Z}^d$, $d \geq 3$:

- Vacant set of random interlacements $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$
  Non-triv phase transition:
  [Sznitman, 2010], [Sidoravicius, Sznitman, 2009]
- Level set of Gaussian free field
  Non-triv phase transition:
  [Rodriguez, Sznitman, 2013]
- Random interlacements $\mathcal{I}^u$
  No phase transition: $\mathcal{I}^u$ is connected for any $u > 0$

Our result on voter model percolation

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if* $d \geq 5$, $R \geq 1$

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if* $d \geq 5$, $R \geq 1$
- *if* $d \geq 3$, $R \gg 1$

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if* $d \geq 5$, $R \geq 1$
- *if* $d \geq 3$, $R \gg 1$

Low dimensions, $R = 1$?

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if* $d \geq 5$, $R \geq 1$
- *if* $d \geq 3$, $R \gg 1$

Low dimensions, $R = 1$?

- $d = 4$ is the "critical dimension" for our method

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if $d \geq 5$, $R \geq 1$*
- *if $d \geq 3$, $R \gg 1$*

Low dimensions, $R = 1$?

- $d = 4$ is the ”critical dimension” for our method
- $d = 3$???

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if* $d \geq 5$, $R \geq 1$
- *if* $d \geq 3$, $R \gg 1$

Low dimensions, $R = 1$?

- $d = 4$ is the "critical dimension" for our method
- $d = 3$??? Simulations [Marinov, Lebowitz, 2006]: Non-triv phase transition

Our result on voter model percolation

Theorem (Ráth, Valesin, 2015)

$0 < \alpha_c < 1$

- *if $d \geq 5$, $R \geq 1$*
- *if $d \geq 3$, $R \gg 1$*

Low dimensions, $R = 1$?

- $d = 4$ is the "critical dimension" for our method
- $d = 3$??? Simulations [Marinov, Lebowitz, 2006]: Non-triv phase transition
- Our proof:

Multi-scale renormalization $=$
bottom level estimates $+$ decorrelation inequalities

Bottom-level estimates

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$
- Annulus: $A(L) = B(0, 2L) \setminus B(0, L)$

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$
- Annulus: $A(L) = B(0, 2L) \setminus B(0, L)$
- Block of $x \in \mathbb{Z}^d$ at time T:

$$\mathcal{B}_T^x = \{\, y \in \mathbb{Z}^d \;:\; X_T^y = X_T^x \,\}$$

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$
- Annulus: $A(L) = B(0, 2L) \setminus B(0, L)$
- Block of $x \in \mathbb{Z}^d$ at time T:

$$\mathcal{B}_T^x = \{ y \in \mathbb{Z}^d \ : \ X_T^y = X_T^x \}$$

We only have a chance to show $\alpha_c > 0$ if

$$\lim_{L \to \infty} \mathbb{P} \left(\mathcal{B}_\infty^0 \text{ crosses } A(L) \right) = 0$$

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$
- Annulus: $A(L) = B(0, 2L) \setminus B(0, L)$
- Block of $x \in \mathbb{Z}^d$ at time T:

$$\mathcal{B}_T^x = \{\, y \in \mathbb{Z}^d \;:\; X_T^y = X_T^x \,\}$$

We only have a chance to show $\alpha_c > 0$ if

$$\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}_\infty^0 \text{ crosses } A(L)\right) = 0$$

- $d = 3$???
- $d \geq 5$ OK !!!

Bottom-level estimates

- X_t^x, $x \in \mathbb{Z}^d$ coalescing random walks on $\mathbb{Z}^d$
- Annulus: $A(L) = B(0, 2L) \setminus B(0, L)$
- Block of $x \in \mathbb{Z}^d$ at time T:

$$\mathcal{B}_T^x = \{\, y \in \mathbb{Z}^d \ : \ X_T^y = X_T^x \,\}$$

We only have a chance to show $\alpha_c > 0$ if

$$\lim_{L \to \infty} \mathbb{P}\left(\mathcal{B}_\infty^0 \ \text{crosses} \ A(L) \right) = 0$$

- $d = 3$???
- $d \geq 5$ OK !!! Why?

Why $d \geq 5$?

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}(E_L) = 1, \quad E_L = \{\forall\, x \in A(L) : \operatorname{diam}(\mathcal{B}^x_T) < L\}$$

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}(E_L) = 1, \quad E_L = \{\forall\, x \in A(L) : \operatorname{diam}(\mathcal{B}^x_T) < L\}$$

- If $\mathcal{B}^0_\infty$ crosses $A(L)$ and E_L occurs, then

$$\exists\, x \sim y \in A(L) \ : \ X^x_T \neq X^y_T, \quad \text{but} \quad X^x_\infty = X^y_\infty = X^0_\infty$$

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}\,(E_L) = 1, \quad E_L = \{\,\forall\, x \in A(L) :\ \mathrm{diam}(\mathcal{B}^x_T) < L\}$$

- If $\mathcal{B}^0_\infty$ crosses $A(L)$ and E_L occurs, then

$$\exists\, x \sim y \in A(L)\ :\ X^x_T \neq X^y_T, \quad \text{but} \quad X^x_\infty = X^y_\infty = X^0_\infty$$

- $\mathbb{P}(\,X^x_\infty = X^y_\infty \,|\, X^x_T \neq X^y_T\,) \asymp \sqrt{T}^{2-d}$

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}\,(E_L) = 1, \quad E_L = \{\,\forall\, x \in A(L) : \ \mathrm{diam}(\mathcal{B}^x_T) < L\}$$

- If $\mathcal{B}^0_\infty$ crosses $A(L)$ and E_L occurs, then

$$\exists\, x \sim y \in A(L) \ : \ X^x_T \neq X^y_T, \quad \text{but} \quad X^x_\infty = X^y_\infty = X^0_\infty$$

- $\mathbb{P}(\,X^x_\infty = X^y_\infty \,|\, X^x_T \neq X^y_T\,) \asymp \sqrt{T}^{2-d}$
- number of choices of $x \sim y \in A(L)$ is roughly L^d

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}(E_L) = 1, \quad E_L = \{\forall\, x \in A(L) :\ \mathrm{diam}(\mathcal{B}^x_T) < L\}$$

- If $\mathcal{B}^0_\infty$ crosses $A(L)$ and E_L occurs, then

$$\exists\, x \sim y \in A(L) \ :\ X^x_T \neq X^y_T, \quad \text{but} \quad X^x_\infty = X^y_\infty = X^0_\infty$$

- $\mathbb{P}(\,X^x_\infty = X^y_\infty \,|\, X^x_T \neq X^y_T\,) \asymp \sqrt{T}^{2-d}$
- number of choices of $x \sim y \in A(L)$ is roughly L^d
- $\mathbb{P}(X^x_\infty = X^0_\infty) \asymp L^{2-d}$

Why $d \geq 5$?

- Want: $\lim_{L\to\infty} \mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L)\right) = 0$
- Let $T = L^{2-\varepsilon}$, then

$$\lim_{L\to\infty} \mathbb{P}(E_L) = 1, \quad E_L = \{\forall\, x \in A(L) : \operatorname{diam}(\mathcal{B}^x_T) < L\}$$

- If $\mathcal{B}^0_\infty$ crosses $A(L)$ and E_L occurs, then

$$\exists\, x \sim y \in A(L) \ : \ X^x_T \neq X^y_T, \quad \text{but} \quad X^x_\infty = X^y_\infty = X^0_\infty$$

- $\mathbb{P}(\, X^x_\infty = X^y_\infty \,|\, X^x_T \neq X^y_T\,) \asymp \sqrt{T}^{2-d}$
- number of choices of $x \sim y \in A(L)$ is roughly L^d
- $\mathbb{P}(X^x_\infty = X^0_\infty) \asymp L^{2-d}$

$$\mathbb{P}\left(\mathcal{B}^0_\infty \text{ crosses } A(L) \,|\, E_L\right) \leq L^d \cdot \sqrt{T}^{2-d} \cdot L^{2-d} = L^{4-d} \cdot L^{(d/2-1)\varepsilon}$$

Decorrelation

Decorrelation

If

$$K \subset\subset \mathbb{Z}^d, \qquad 0 < \alpha < 1$$

then

$$\mu_\alpha\left(\xi|_K \equiv 1\right) = \mathbb{E}\left(\alpha^{N_\infty}\right)$$

where

$$N_t = \#\{\, X_t^x \,:\, x \in K \,\}, \qquad N_t \searrow N_\infty$$

Decorrelation

If

$$K \subset\subset \mathbb{Z}^d, \qquad 0 < \alpha < 1$$

then

$$\mu_\alpha\left(\xi|_K \equiv 1\right) = \mathbb{E}\left(\alpha^{N_\infty}\right)$$

where

$$N_t = \#\{\, X_t^x \,:\, x \in K \,\}, \qquad N_t \searrow N_\infty$$

Question:

$$\mathbb{E}\left(\alpha^{N_\infty}\right) \leq ??? \qquad N_\infty \geq ???$$

Decorrelation ($d \geq 3$, $R \gg 1$)

Trick: Let $Y_t^x, x \in K$ be annihilating random walks. Coupling:

$$\{\, Y_t^x \;:\; x \in K \,\} \subseteq \{\, X_t^x \;:\; x \in K \,\}$$

Decorrelation ($d \geq 3$, $R \gg 1$)

Trick: Let $Y_t^x, x \in K$ be annihilating random walks. Coupling:

$$\{\, Y_t^x \;:\; x \in K \,\} \subseteq \{\, X_t^x \;:\; x \in K \,\}$$

$$A_\infty = \text{total number of annihilations}$$

$$N_\infty \geq |K| - 2A_\infty$$

Decorrelation ($d \geq 3$, $R \gg 1$)

Trick: Let $Y_t^x, x \in K$ be annihilating random walks. Coupling:

$$\{ Y_t^x \ : \ x \in K \} \subseteq \{ X_t^x \ : \ x \in K \}$$

$$A_\infty = \text{total number of annihilations}$$

$$N_\infty \geq |K| - 2A_\infty$$

$$\mathbb{E}\left(\left(\frac{1}{\alpha^2}\right)^{A_\infty}\right) \leq ??? \quad A_\infty \leq ???$$

Decorrelation ($d \geq 3$, $R \gg 1$)

Trick: Let $Y_t^x, x \in K$ be annihilating random walks. Coupling:

$$\{ Y_t^x \ : \ x \in K \} \subseteq \{ X_t^x \ : \ x \in K \}$$

$$A_\infty = \text{total number of annihilations}$$

$$N_\infty \geq |K| - 2A_\infty$$

$$\mathbb{E}\left(\left(\frac{1}{\alpha^2}\right)^{A_\infty}\right) \leq ??? \quad A_\infty \leq ???$$

$$\mathbb{E}\left(\beta^{A_\infty}\right) \leq ??? \qquad \beta > 1$$

Decorrelation ($d \geq 3$, $R \gg 1$)

$$x, y \in K, \quad \{x, y\} = e, \quad \eta_e = \mathbb{1}[\, Y_\cdot^x \text{ annihilates } Y_\cdot^y \,]$$

$$A_\infty = \sum_{e \in \binom{K}{2}} \eta_e$$

Decorrelation ($d \geq 3$, $R \gg 1$)

$$x, y \in K, \quad \{x, y\} = e, \quad \eta_e = \mathbb{1}[\, Y^x_\cdot \text{ annihilates } Y^y_\cdot \,]$$

$$A_\infty = \sum_{e \in \binom{K}{2}} \eta_e$$

$$\eta_e, e \in \binom{K}{2} \quad \text{are} \quad \text{negatively correlated:}$$

$$\mathbb{E}\left(\beta^{A_\infty}\right) \leq \mathbb{E}\left(\beta^{A^*_\infty}\right),$$

where $\eta^*_e, e \in \binom{K}{2}$ independent with $\mathbb{E}(\eta^*_e) = \mathbb{E}(\eta_e)$ and

$$A^*_\infty = \sum_{e \in \binom{K}{2}} \eta_e$$

Decorrelation ($d \geq 3$, $R \gg 1$)

$$x, y \in K, \quad \{x, y\} = e, \quad \eta_e = \mathbb{1}[\, Y^x_\cdot \text{ annihilates } Y^y_\cdot \,]$$

$$A_\infty = \sum_{e \in \binom{K}{2}} \eta_e$$

$$\eta_e, e \in \binom{K}{2} \quad \text{are} \quad \text{negatively correlated:}$$

$$\mathbb{E}\left(\beta^{A_\infty}\right) \leq \mathbb{E}\left(\beta^{A^*_\infty}\right),$$

where $\eta^*_e, e \in \binom{K}{2}$ independent with $\mathbb{E}(\eta^*_e) = \mathbb{E}(\eta_e)$ and

$$A^*_\infty = \sum_{e \in \binom{K}{2}} \eta_e$$

This trick only helpful in spread-out case

Thank you for your attention!