The Poisson Voronoi tessellation in hyperbolic space

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Hyperbolic space

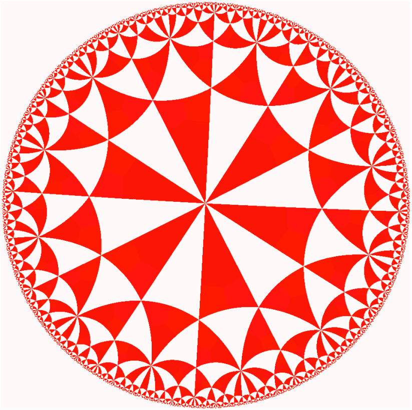

Figure : A tessellation of $\mathbb{H}$

Voronoi tessellation

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2. For a point $p_0 \in \mathcal{P}$, the cell with *nucleus* p_0 is given by

$$\left\{ z \in \mathcal{X} : d(z, p_0) = \min_{p \in \mathcal{P}} d(z, p) \right\}.$$

Hyperbolic space

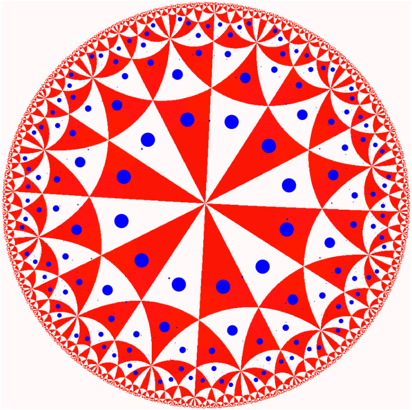

Figure : A tessellation of $\mathbb{H}$, with nuclei shown

Lattices

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Make the lattice into a graph by attaching two nuclei if and only if their Voronoi cells have a codimension-1 intersection.

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- $\mathbb{H}$ is nonamenable:

$$\inf_{\substack{V \subset \mathbb{H} \\ \partial V \text{ smooth} \\ \mathrm{Vol}_{\mathbb{H}}(V) < \infty}} \frac{|\partial V|}{\mathrm{Vol}_{\mathbb{H}}(V)} > 0.$$

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- Any lattice L in $\mathbb{H}$ is nonamenable:

$$\inf_{\substack{V \subset L \\ |V| < \infty}} \frac{|\partial V|}{|V|} > 0.$$

HPV

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Overarching question: does a "statistical lattice" still capture the space?

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Let $\mathscr{V}^\lambda$ be the dual graph of HPV.

HPV

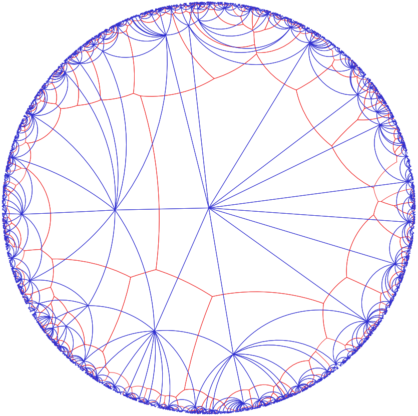

Figure : $\lambda = 0.2$ and $r = 0.9995$.

HPV

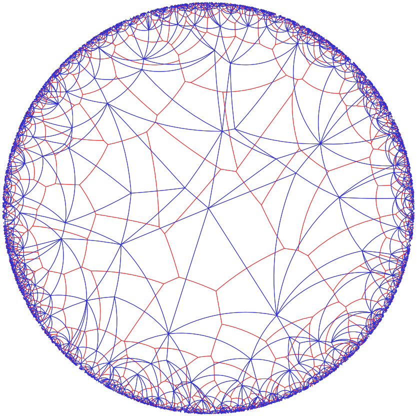

Figure : $\lambda = 1$ and $r = 0.9975$.

HPV properties

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Moreover:

Lemma

Let S_0 be the Voronoi cell with nucleus 0. There is a t_0 and a $\delta > 0$ so that for all $t > t_0$,

$$\mathbb{P}\left[S_0 \not\subset B_{\mathbb{H}}(0, r)\right] \leq e^{-\lambda e^{\delta r}},$$

where $B_{\mathbb{H}}(x, r)$ is the hyperbolic ball centered at x of radius r.

HPV properties

2. $\limsup_{r\to\infty} |B_{\mathbb{H}}(0,r) \cap \mathscr{V}^{\lambda}|^{1/r} < \infty$.

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4. $\mathscr{V}^{\lambda}$ is unimodular.

Anchored expansion

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Let

$$i^*(G) := \liminf_{\substack{|S|\to\infty \\ \rho\in S \\ G|_S \text{connected}}} \frac{|\partial S|}{\mathrm{Vol}_G(S)}. \tag{1}$$

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$$i^*(G) := \liminf_{\substack{|S|\to\infty \\ \rho\in S \\ G|_S \text{connected}}} \frac{|\partial S|}{\mathrm{Vol}_G(S)}. \tag{1}$$

Theorem

For $G = \mathscr{V}^\lambda$ there is a constant $c = c(\lambda) > 0$ so that $i^*(G) > c$ almost surely.

For $d = 2$, Benjamini-P-Pfeffer '14.
For $d \geq 2$, Benjamini-Krauz-P '15+.

Some consequences of anchored expansion

- In a graph with bounded degree and $i^*(G) > 0$, Virág ('00) shows that SRW X_k has

$$\liminf_{k\to\infty} \frac{d(\rho, X_k)}{k} > 0.$$

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- Infinite Bernoulli percolation clusters inherit positive anchored expansion for p sufficiently close to 1 (Chen, Peres, and Pete '03)
- The Ising model on G exhibits a phase transition with nonzero external field (Häggrström, Schonnman, Steif '00).

Stationary random graphs

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- (G, ρ) is called *reversible* if $(G, X_0, X_1) \stackrel{\mathcal{L}}{=} (G, X_1, X_0)$ as birooted random graphs.
- Let P be the law of (G, ρ), and define a measure Q by $\frac{dQ}{dP} = \frac{\deg \rho}{\mathbb{E}_P \deg \rho}$. For $\mathbb{E}_P \deg \rho < \infty$,

$$P \textit{ unimodular} \quad \Longleftrightarrow \quad Q \text{ reversible} .$$

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- An augmented Galton-Watson tree with positive offspring distribution is another example of a reversible random graph.

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- For example, the speed of random walk exists almost surely:
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exists.
- (Under the assumption of exponential growth) positive speed is equivalent to the existence of nonconstant bounded harmonic functions (Benjamini-Curien '12 and Piaggio-Lessa '15+).

Anchored expansion and positive speed

Theorem (Benjamini-P-Pfeffer '14)

Let (G, ρ) be a stationary random graph so that:

1. *(G, ρ) has positive anchored expansion almost surely and*
2. $\limsup_{r\to\infty} |B(\rho, r)|^{1/r} < \infty$ *almost surely.*

Then, simple random walk X_k started from ρ has positive speed, i.e.

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almost surely.

Hence simple random walk on $\mathscr{V}^\lambda$ has positive speed.

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Conjecture

The exponential growth assumption can be removed.

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Lemma (Benjamini-Eldan '12)

For any finite set $\mathbf{X} \subset \mathbb{H}$, $\mathrm{Vol}_{\mathbb{H}}(\mathrm{conv}_{\mathbb{H}}(\mathbf{X})) \leq 4\pi|\mathbf{X}|$, *where* $\mathrm{conv}_{\mathbb{H}}(\mathbf{X})$ *denotes the hyperbolic convex hull.*

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$$4\pi|\partial\,\mathrm{conv}_{\mathbb{H}}(\mathbf{X}') \cap \mathbf{X}'| \geq \mathrm{Vol}_{\mathbb{H}}(\mathrm{conv}_{\mathbb{H}}(\partial\,\mathrm{conv}_{\mathbb{H}}(\mathbf{X}') \cap \mathbf{X}')).$$

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Let $\mathbf{X} \subset \mathbb{H}$ be a finite set of nuclei in Lattice L.

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$$C|\partial_L \mathbf{X}| \geq \mathrm{Vol}_{\mathbb{H}}(\mathrm{conv}_{\mathbb{H}}(\mathbf{X}')) \geq c\left|\left\{\text{Delaunay triangles in } \mathrm{conv}_{\mathbb{H}}(\mathbf{X}')\right\}\right|.$$

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Proof 1 (BPP, $d = 2$)

Proposition

There is a constant $c > 0$ and a $k_0 > 0$ random so that for all collections of Delaunay triangles $t_1, t_2, \ldots, t_k$ whose union $\cup_{i=1}^{k} t_i$ is simply connected and contains 0,

$$\sum_{i=1}^{k} \mathrm{Vol}_{\mathbb{H}}(t_i) > ck.$$

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Problem

Show

$$\mathbb{P}\left[\sum_{i=0}^{k-2} \mathrm{Vol}_{\mathbb{H}}(\Delta(x_i, x_{i+1}, x_{i+2})) \leq \epsilon k\right] \approx \exp(k\Theta(\log \epsilon)).$$

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Caveat: we need a bound that is good enough that this estimate beats the number of k-element subsets of points from $\Pi^{\lambda} \cap B_{\mathbb{H}}(0, r)$.

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Caveat: we need a bound that is good enough that this estimate beats the number of k-element subsets of points from $\Pi^\lambda \cap B_{\mathbb{H}}(0, r)$. Naïvely, we need $r \approx k$, and so this is $\approx e^{\Theta(k^2)}$.

Toy problem 2

Let $\mathcal{E}$ be the event that for all i, $1 \le i \le k-2$, $\{x_i, x_{i+1}, x_{i+2}\}$ have a finite circumdisc (as all Delaunay triangles do).

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This approach leads to a proof of the area lower bound for Delaunay triangles.

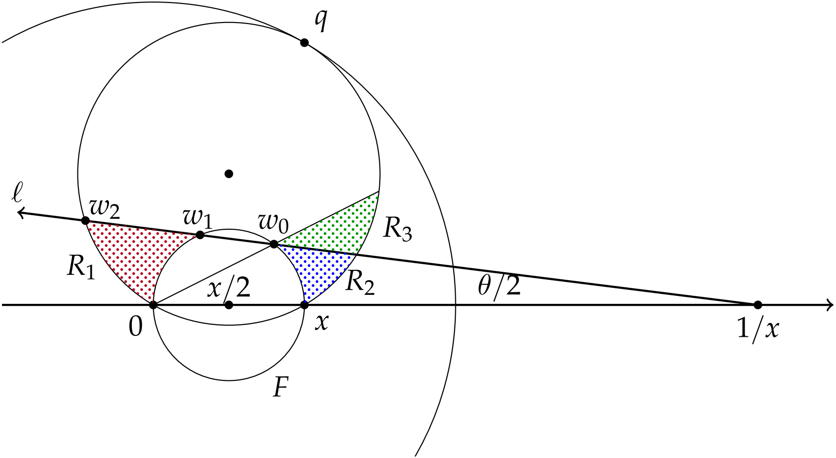
Geometric ingredient

Proposition

Suppose that $r > 0$ is fixed. Let y be a point that is picked uniformly from the $B_{\mathbb{H}}(0, r)$ according to hyperbolic area measure. There is an absolute constant $C > 0$ so that

$$\mathbb{P}\left[|\Delta(0, x, y)| \leq \theta and \ \mathrm{CD}_{\mathbb{H}}(0, x, y) \ exists\right] \leq \frac{C\theta}{d_{\mathbb{H}}(0, x)|B_{\mathbb{H}}(0, r)|}.$$

Poof of geometric ingredient



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Anchored expansion for discrete random graphs is stable with respect to random perturbation. This phenomenon should hold as well for other randomly discretized symmetric spaces.

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It's straightforward to show that SRW on $\mathscr{V}^\lambda$ converges, as a sequence of points in $\mathbb{C}$, to a point on the unit circle. Let ν_0 be the harmonic measure on S^1 of SRW started from 0.

Conjecture

For almost every realization of $\mathscr{V}^\lambda$, ν_0 is singular with respect to Lebesgue measure on S^1.