On the chemical distance in critical percolation

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Setting

We consider bond percolation in $\mathbb{Z}^d$.

$$\mathbf{P}_p = \Pi_{e \in E(\mathbb{Z}^d)}(p\delta_0 + (1-p)\delta_1),$$

$(w(e))_{e \in E(\mathbb{Z}^d)}$ with distribution $\mathbf{P}_p$. An edge e is *open* if $w(e) = 1$, *closed* if $w(e) = 0$.

Order parameter:

$$\theta(p) = \mathbf{P}_p(\text{component of } 0 \text{ is infinite}).$$

In dimension $d = 2$,

$$p_c := \inf\{p : \theta(p) > 0\} = \frac{1}{2},$$

(Kesten, 1982).

Chemical distance

Chemical distance in percolation: the distance between two subsets A and B is the minimum number of edges in any open path between A and B.

When there is no path, define $\mathrm{dist}(A,B)=\infty$. In practice, we condition on the event that A and B are connected.

Example: crossing of a box of side length $2n$. A is the left side $\{-n\}\times[-n,n]$, B the right side. By Russo-Seymour-Welsh:

$$\mathbf{P}_p(\mathrm{dist}(A,B)<\infty)\geq C$$

for $p\geq\frac{1}{2}$.

Supercritical percolation: Antal-Pisztora

In the supercritical phase, distances are comparable to the Euclidean distance in all dimensions.

Theorem (Antal-Pisztora, 1996)

For any $p > p_c$ there is a constant $\rho(p,d)$ such that, $\mathbf{P}$-almost surely

$$\limsup_{|x|\to\infty} \frac{1}{|x|} \mathrm{dist}(0,x) \mathbf{1}_{\{0\leftrightarrow x\}} \leq \rho(p,d).$$

Large and moderate deviations by Garet-Marchand (2007, 2009).

The result is central in the study of random walk on percolation clusters; used in heat kernels estimates (Barlow, 2004), convergence of RW on infinite cluster to Brownian motion (Sidoravicius-Sznitman, 2004; Berger-Biskup, 2007; Mathieu-Piatnitski, 2007).

Critical percolation: high dimension

As part of their work on the Alexander-Orbach conjecture on the high-dimensional *incipient infinite cluster*, Kozma and Nachmias (2011) show that in high dimensions ($d \geq 19$ enough):

$$\mathbf{P}_{p_c}(0 \leftrightarrow \partial B_{\mathbb{Z}^d}(n)) \asymp n^{-2}.$$

van der Hofstad and Sapozhnikov (2013) show using another argument of Kozma-Nachmias (2009) that this implies, for $|x|$ large:

$$\mathbf{P}_{p_c}(0 \leftrightarrow \partial B_{\mathbb{Z}^d}(n) \text{ by a path with less than } \epsilon n^2 \text{ edges } \mid 0 \leftrightarrow x) \leq C\sqrt{\epsilon}.$$

Upper bound by Heydenreich-v.d. Hofstad-Hulsof (2014) gives *exponent* 2 for chemical distance.

Critical percolation: dimension 2

In low dimensions, few results are available. We concentrate on a special case: let $d = 2$ and S_n be the length of the shortest open crossing of $[-n, n]^2$.

Expect the existence of an exponent α such that

$$\mathbf{E}S_n \sim n^\alpha.$$

O. Schramm, *Conformally invariant scaling limits: an overview and a collection of problems* (2006), Problem 3.3.: "Does not seem accessible via an SLE analysis".

Physics predictions

Unlike other critical exponents, there is no exact prediction for α, only several competing conjectures (some "disproved" by numerics).

Early suggestion (Edwards-Kernstein 1985-1986): $\alpha = 1$ with logarithmic correction.

Numerical simulations (Hermann-Stanley 1988, Grassberger 1999, Zhou-Yang-Deng-Ziff 2012) suggest otherwise:

$$\alpha \approx 1.130\ldots$$

Aizenman-Burchard: "$\alpha > 1$"

In 1995, Aizenman and Burchard showed how to construct a scaling limit following Aizenman's idea of considering the percolation configuration as a system of random curves.

Theorem (Aizenman, Burchard, 1995)

Critical percolation on $\frac{1}{n}\mathbb{Z}^2 \cap [-1,1]^2$*, viewed as a measure on the collection of random curves formed by all paths of open edges, has weak subsequential limits as* $n \to \infty$ *(in the uniform topology modulo reparametrization). Any limit is a measure on collections of continuum curves of Hausdorff dimension* $1 < d < 2$.

A lower bound on distances

In "pre-scaling limit" terms, the lower bound in Aizenman-Burchard gives: if $A, B \subset \mathbb{Z}^2 \cap ([-n, n]^2)$ such that

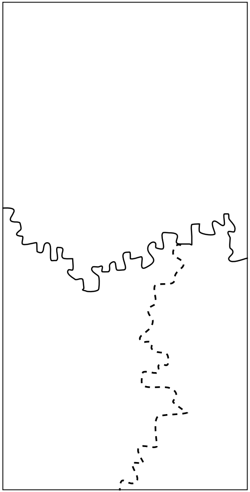
$$\mathrm{dist}_{\mathbb{Z}^2}(A, B) \geq n/10,$$

then there is a $\eta > 1$ such that for each $\epsilon > 0$

$$\limsup_{n \to \infty} \mathbf{P}\left(\mathrm{dist}_{\mathsf{chemical}}(A, B) \leq C(\epsilon) n^{\eta}\right) \leq \epsilon \qquad (1)$$

Pisztora observed that a block renormalization argument transforms (1) into an exponential (in n) estimate.

An upper bound: lowest crossing



The lowest open crossing of a rectangle is the one such that the area of the region "below" is minimal.

Characterization in terms of arms: two disjoint open arms to the sides and a closed dual arm to the bottom.

This implies:

$$\mathbf{E}\#(\text{lowest crossing}) \asymp n^2 \pi_3(n)$$

$n \approx$ size of the rectangle; $\pi_3(n)$: 3-arm probability.

Morrow-Zhang

Morrow and Zhang carried out the computation for all moments, and showed, for the triangular lattice:

Theorem (Morrow-Zhang, 2005)

For each k:

$$\mathbf{E}(L_n)^k = C_k(n^2\pi_3(n))^k = n^{\frac{4}{3}k+o(1)}.$$

L_n: length of lowest crossing of $[-n,n]^2$.

Kesten-Zhang

Kesten and Zhang (1992) outline an argument to show that, for some $\delta > 0$

$$\mathbf{P}(0 < L_n \leq n^{1+\delta}) \leq n^{-C}.$$

This is weaker than Aizenman-Burchard, but the proof is different.

A question of Kesten and Zhang

In their paper, Kesten and Zhang asked how the shortest crossing compares to the lowest crossing:

"It is not clear that $S_n/L_n \to 0$ in probability."

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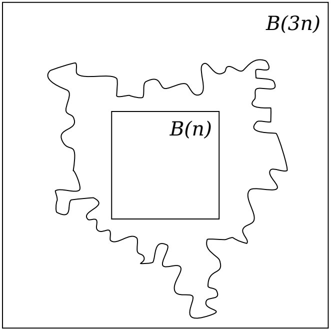
We answer Kesten and Zhang's question, and give a corresponding result in expectation. It is instructive to state it in terms of circuits in annuli.

Circuits in annuli

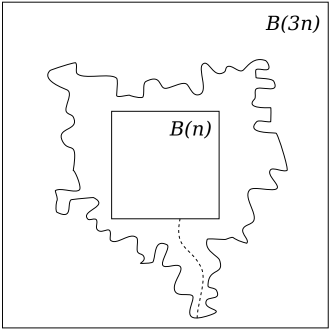


Let $B(n) = [-n, n]^2$.
Consider critical percolation in an annulus $A(n) = B(3n) \setminus B(n)$.

Russo-Seymour-Welsh: there is an open circuit around $B(n)$ inside $A(n)$ with positive probability.

If an open circuit exists, define the innermost circuit γ_n: the circuit such that the interior is minimal.

The innermost circuit



3-arm characterization: $e \in \gamma_n$ if e is on an open circuit and e^* is connected to $\partial B(n)$ by a closed dual path.

In particular:

$$\mathbf{E}\#\gamma_n \asymp n^2 \pi_3(n)$$

The shortest circuit

Let $\tilde{S}_n$ be the number of edges on the shortest circuit around $B(n)$ inside $A(n)$, ($\tilde{S}(n) = 0$ if there is none).

Theorem (Damron, Hanson, S., 2015)

$$\mathbf{E}\tilde{S}_n = o(n^2 \pi_3(n)).$$

This shows the "fractal nature" of percolation: shortest circuit is much shorter than the innermost, even though it encloses a bigger area.

With positive probability, the two circuits come close to each other many times (abundance of 4-arm points).

An answer to Kesten-Zhang

Using the previous result (adapted to crossing of a square) we obtain:

Theorem (Damron, Hanson, S., 2015)

Let $H_n = \{$*there is an open crossing of* $[-n,n]^2\}$. *Then, conditionally on* H_n,

$$S_n/L_n \to 0.$$

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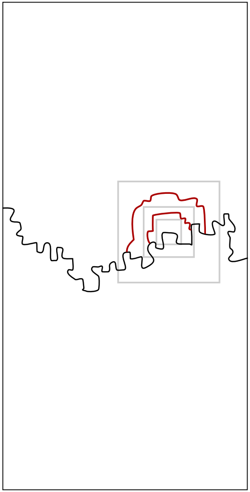
$$S_n/L_n \to 0.$$

To deduce this from $\mathbf{E}S_n = o(\mathbf{E}L_n)$, need that L_n cannot be much smaller than $\mathbf{E}L_n$ (unless $L_n = 0$):

$$\mathbf{P}(0 < L_n < n^{-\epsilon}\mathbf{E}L_n) \to 0.$$

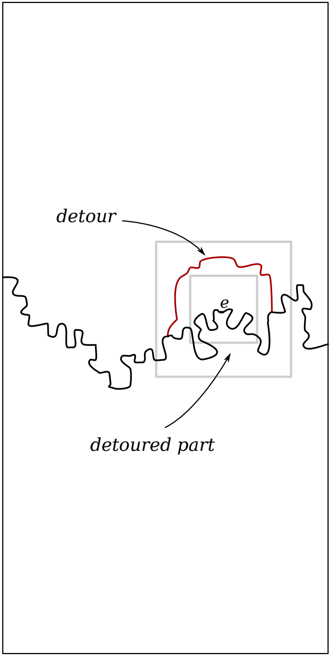
A lower tail bound of this type for another set, the Fourier spectrum of percolation (related to pivotals), was obtained by Garban-Pete-Schramm. Our strategy of proof is different.

Some proof ideas

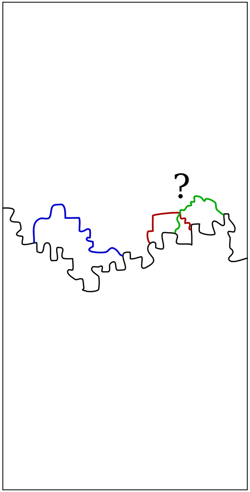


Obvious approach: try to modify the lowest path by short detours.

The lowest path is constrained by the existence of closed arm;
Russo-Seymour-Welsh implies the existence of a lot of "forks" in the road. Expect some of them to be shortcuts.


detour
e
detoured part

Difficulties



Different shortcuts might conflict. Need to keep track of the location of endpoints.

It is not clear that you can find really significant shortcuts by taking "detours". Is taking the minimum over a number of detours given by RSW sufficient?

Some proof ideas: how to make short paths

Quasimultiplicativity and fractal dimension of lowest path imply that *thin* paths are *short*.

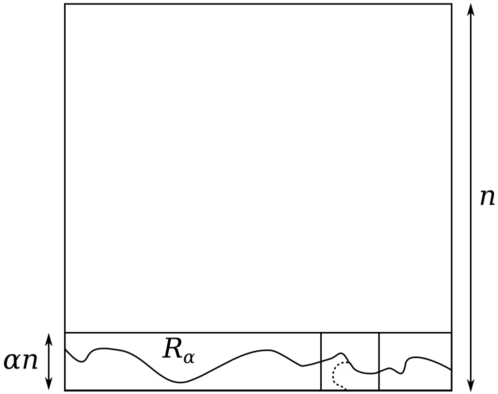
This observation is the basis of our proof of the following statement:

Lemma (Fluctuations of the lowest path are $O(n^2\pi_3(n))$)

Let L_n be the length of the lowest crossing of $B(n)$. For every $\epsilon > 0$:

$$\mathbf{P}(0 < L_n \leq \epsilon \mathbf{E} L_n) \geq c(\epsilon) > 0.$$

(M. Aizenman suggested $\mathbf{Var}(L_n) \sim (\mathbf{E}L_n)^2$.)



n
αn
R_α

Volume of the lowest crossing in the rectangle R_α:

$$\mathbf{E}[L_{\alpha n} \mid C_\alpha] \le C \underbrace{\alpha n}_{\text{height}} \times \underbrace{n}_{\text{width}} \times \underbrace{\pi_3(n\alpha)}_{\text{3 arms in a small square}} .$$

Aizenman-Burchard (or Kesten-Zhang):

$$k^{1+\delta} \le C\mathbf{E}L_k \sim k^2 \pi_3(k),$$

for some $\delta > 0$, so

$$\pi_3(k) \ge k^{-1+\delta}/C.$$

From this (using quasimultiplicativity):

$$\frac{\pi_3(\alpha n)}{\pi_3(n)} \le C\alpha^{-1+\delta}.$$

Volume of the lowest crossing in the rectangle R_α:

$$\begin{aligned}\mathbf{E}[L_{\alpha n} \quad | \quad \underbrace{C_\alpha}_{\text{existence of crossing}} \quad] &\le C\alpha n^2 \,\frac{\pi_3(\alpha n)}{\pi_3(n)}\, \pi_3(n) \\ &\le C\alpha^\delta n^2 \pi_3(n).\end{aligned}$$

Markov's inequality:

$$\mathbf{P}(0 < L_{n\alpha} \le f(\alpha)\mathbf{E}L_n) \ge c(\alpha),$$

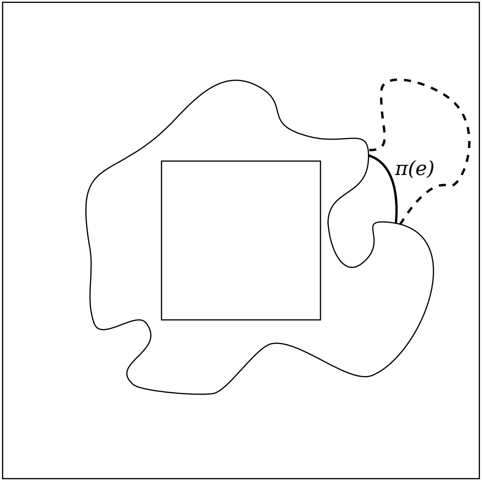
with

$$f(\alpha) \downarrow 0$$

as

$$\alpha \to 0.$$

How to avoid overcounting: shielded detours



Key estimate

The heart of the proof is to show that edges away from the boundary have short shielded detours with high probability.

$\pi(e)$: *short detour* around e, such that $\#\pi(e) \leq \epsilon\#(\text{detoured portion})$.
$\pi(e) = \emptyset$ if there is no detour.

Lemma

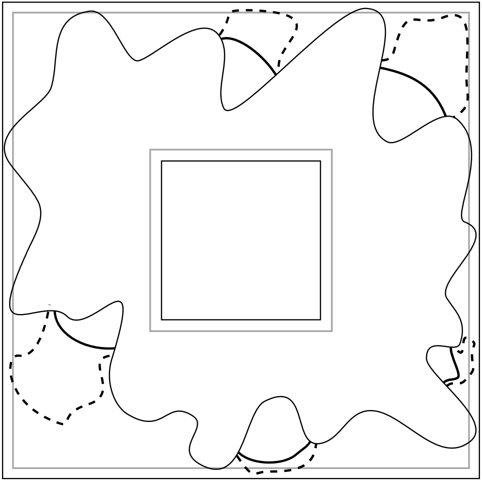
Let $0 < C_1 < 1$. *If* $\operatorname{dist}(e, \partial A(n)) \geq n^{C_1}$, *there is a constant* C_2 *such that*

$$\mathbf{P}(\pi(e) = \emptyset \mid e \in \gamma_n) \leq n^{-C_2}.$$

Detour construction

To construct a path σ_n that is shorter than γ_n, we choose a maximal collection of detours, subject to the constraint that the detoured portions are disjoint.

$$\begin{aligned}\sigma_n = &\{\text{maximal collection of detours}\} \\ &\cup (\hat{\gamma}_n \setminus \{\text{detoured portions of } \gamma_n\})\end{aligned}$$



Estimating $\mathbf{E}S_n$

By construction:

$$\#\sigma_n \le \#\{e \text{ in a maximal collection of short detours}\}$$
$$+ \#\{e \in \hat\gamma_n : e \text{ has no short detour}\} + (\text{boundary contribution}).$$

First term $\le \epsilon \# \gamma_n$ by definition.

Second term is bounded using the *key estimate*:

$$\begin{aligned} &\mathbf{E}\#\{e \in \hat\gamma_n : e \text{ has no short detour}\} \\ \le & \sum_{e \in E(A(n)) \setminus \text{boundary}} \mathbf{P}(\pi(e) = \emptyset \mid e \in \gamma_n)\mathbf{P}(e \in \gamma_n) \\ \le & n^{-C_2} \sum_e \mathbf{P}(e \in \gamma_n) = n^{-C_2}\mathbf{E}\#\gamma_n. \end{aligned}$$

Showing shielded detours exist

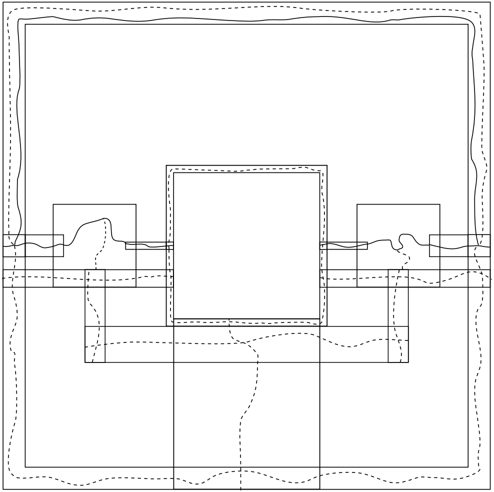
We define events $E_k(e)$ depending only on the annulus $B(e, 3^{k+2}) \setminus B(e, 3^k)$, $3^{k+2} \leq n^{C_1}$ such that

$$E_k(e) \Rightarrow \pi(e) \neq \emptyset.$$

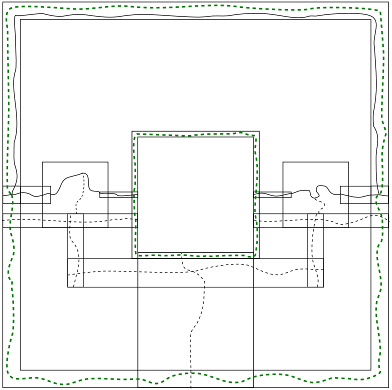
If we could replace $\mathbf{P}(\cdot \mid e \in \hat{\gamma}_n)$ with $\mathbf{P}(\cdot)$, the E_k are $\mathbf{P}$-independent, and we would be done. In fact, we need an additional arms separation argument.

I will explain some features of E_k, and outline why $\mathbf{P}(E_k) \geq C$, for k-independent C.

E_k: A sketch



E_k: Shielding circuits

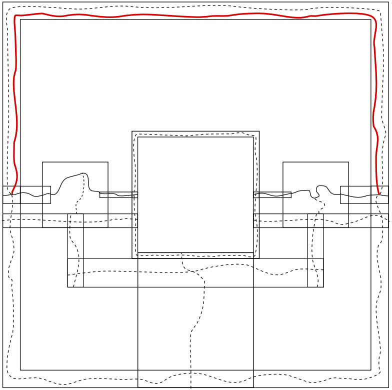


Closed dual circuits with 2 defects around $B(3^k)$ and $B(3^{k-1})$ isolate the events inside the annulus from the outside. The outer circuit acts as the shielding path.

If there is an edge of γ_n in $B(3^{k-1})$, the open arms are forced through the defects.

$O(1)$ probability cost by RSW.

E_k: Short detour path

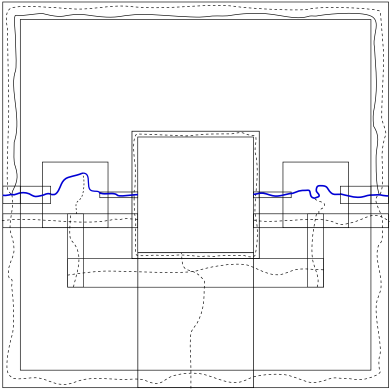


Open arc in a corridor of size $\delta 3^k$ inside the outer path. Taking outermost open path, volume is estimated by $\epsilon 3^{2k} \pi_3(3^k)$ if δ is small.

Endpoints are at the defects, so this is a detour.

$O(1)$ probability cost by RSW. (Constant depends δ.)

E_k: Lower bound for γ_n

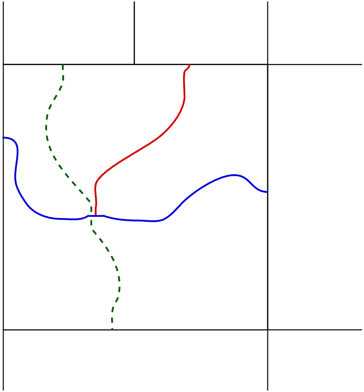


We connect on the order of $C3^{2k}\pi_3(3^k)$ edges to the defects by open paths, and to the outer closed circuit.

If γ_n intersects the inside of $B(3^{k-1})$, these lie on γ_n.

$O(1)$ probability cost (Second moment method.)

Connecting all the above



To connect the constructions above at the defects of the outer dual circuits, we use 5 arm points (avoids conditioning on a rough boundary).

Simpler 3 arm construction at the defects of the inner dual circuit.

O(1) probability (Universal 5-arm exponent $= 2$, and $\mathbf{E}Z = \mathbf{P}(Z > 0)$.)

Wrapping up

All the above gives $\mathbf{P}(E_k) \geq C(\delta) > 0$.

An easy gluing construction shows

$$\mathbf{P}(E_k \mid A_3(n)) \geq C > 0,$$

where $n \geq 3^{k+2}$. $A_3(n)$: three arms to distance n in a box.

Slightly more elaborate arms separation and gluing shows that

$$\mathbf{P}(E_k(e) \mid A_3(e, n)) \asymp \mathbf{P}(E_k(e) \mid e \in \gamma_n).$$

Under $\mathbf{P}(\cdot \mid A_3(n))$ the events are no longer independent. Using an arms separation argument from Damron-Sapozhnikov (2009), we are still able to upgrade the estimate

$$\mathbf{P}(E_k \mid A_3(n)) \geq C > 0$$

into

$$\mathbf{P}(\text{less than } cn \text{ of the } E_k \mid A_3(n)) \leq Cn^{-cC_3}$$

for some constant C_3.

Thank you for your attention!