Continuity of the time and isoperimetric constants on supercritical percolation

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Joint work with: Biskup, Louidor and Rosenthal+Garet, Marchand and Théret

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Menu for this talk:

1. Time constant of first passage percolation.
2. Isoperimetric sets in planar percolation.
3. Shape theorem for random walk weight by its boundary.

First Passage Percolation

- Model for the spread of fluid in a porous medium (Hammersley and Welsh 65).
- For every edge e of $\mathbb{Z}^d$, associate a travel time $t(e) \geq 0$, $t(e) \sim \nu$ i.i.d.
- For a path $\gamma = (e_1, e_2, \ldots, e_n)$, let $T_\nu(\gamma) = \sum_{i=1}^n t(e_i)$.
- $\forall x, y \in \mathbb{Z}^d$ let $T_\nu(x, y) = \inf\{T_\nu(\gamma) : \gamma : x \mapsto y\}$.

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Time constant

- HW65 observed that T_ν is sub-additive $T_\nu(x,y) + T_\nu(y,z) \geq T_\nu(x,z)$.
- Used it to prove that if $\mathbf{E}[t_\nu(e)] < \infty$ then $\forall x \in \mathbb{Z}^d$ the limit of $\frac{T_\nu(0,nx)}{n} = \mu(x)$ exists in probability.
- Kingman proved a.s and L_1 (1968).
- If $\mathbf{E}[t_\nu(e)] = \infty$ then $\limsup_{n\to\infty} \frac{T_\nu(0,nx)}{n} = \infty$.

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Let ν be a probability measure on $[0,\infty]$, such that $\nu([0,\infty)) > p_c(\mathbb{Z}^d)$, then $\forall x \in \mathbb{Z}^d$,

$$\lim_{n\to\infty} \frac{T_\nu(\tilde{0}^{\mathcal{C}_M}, \widetilde{nx}^{\mathcal{C}_M})}{n} = \mu_\nu(x) \text{ a.s. and in } L_1.$$

Where $\nu[0,M] > p_c(\mathbb{Z}^d)$ and $\mu_\nu(x) > 0$ iff $\nu(\{0\}) < p_c(\mathbb{Z}^d)$.

Let $B_\nu(t) = \{x : T_\nu(\tilde{0}^{\mathcal{C}_M}, \widetilde{nx}^{\mathcal{C}_M}) < t\}$.

Theorem 2 (Cerf, Théret 2014)

$\frac{B_\nu(t)}{t} \longrightarrow \{x \in \mathbb{R}^d : \mu_\nu(x) \le 1\} =: B_\nu$ *in Hausdorff distance a.s.*

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Continuity result

Extension of Cox, Kesten 1981.

Theorem 3 (Garet, Marchand , P, Théret 2015)

Let ν_∞, ν_n be probability measures on $[0,\infty]$ s.t. for $n \in \mathbb{N} \cup \{\infty\}$, $\nu_n([0,\infty)) > p_c(\mathbb{Z}^d)$. If $\nu_n \xrightarrow{d} \nu_\infty$, then

$$\lim_{n\to\infty} \sup_{x \in \mathbb{S}^1} |\mu_{\nu_n}(x) - \mu_{\nu_\infty}(x)| = 0.$$

Corollary 4

$\lim_{n\to\infty} d_H(B_{\nu_n}, B_{\nu_\infty}) = 0.$

Special example: $\nu_p = p\delta_1 + (1-p)\delta_\infty$.

Open problem: Show $c(p)B_{\nu_p} \to B_{\|\cdot\|_2}$ as $p \downarrow p_c(\mathbb{Z}^d)$.

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Key step: Study effect of truncation

- Define the law of truncated passage times
  $\nu^K = \mathbf{1}_{[0,k)}\nu + \nu([K,\infty])\delta_K$. $t_{\nu^K}(e) = \min\{t_\nu(e), K\}$.

Theorem 5

Let $\nu([0,\infty)) > p_c(\mathbb{Z}^d)$. Then $\exists A(\nu), K_0(\nu)$ s.t. $\forall K > K_0$, $\forall x \in \mathbb{Z}^d$

$$\mu_{\nu^K}(x) \leq \mu_\nu(x) \leq \mu_{\nu^K}(x)\left(1 + \frac{A}{K}\right).$$

- One can approximate chemical distance time with finite passage times.

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Isoperimetric constant (you can join if you lost me already)

For a finite graph $G = (V(G), E(G))$, the Cheeger constant is defined as

$$\varphi_G = \min\left\{ \frac{|\partial A|}{|A|} : A \subset V(G), 0 < |A| \leq \frac{|V(G)|}{2} \right\},$$

where $\partial A = \{e = (x, y) \in E(G) : x \in A, y \notin A\}$.

Figure: bottle neck

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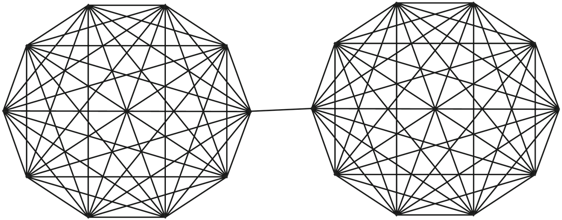

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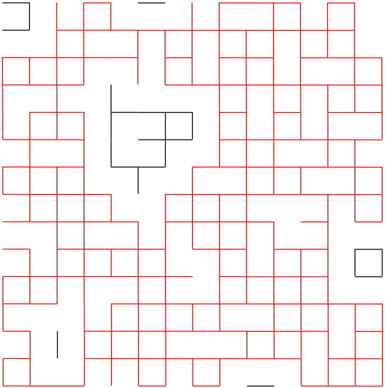
Percolation

Let $B(n) = \left[-\frac{n}{2}+1, \frac{n}{2}\right]^d \cap \mathbb{Z}^d$. For $p > p_c(\mathbb{Z}^d)$, w.h.p. there is a unique connected component in $B(n)$ of size order $|B(n)|$. Denote this component by $C_d(n)$.



- Let $\Phi_n = \min\left\{\frac{|\partial A|}{|A|} : A \subset V(C_d(n)), 0 < |A| \le \frac{|V(C_d(n))|}{2}\right\}$ denote the Cheeger constant of the giant component.
- In several works (Benjamini and Mossel 03, Mathieu and Remy 04, Berger, Biskup, Hoffman and Kozma 08, Pete 08) it was shown that

Theorem 6

$\exists c, C > 0$, depending on p and d only, such that $c < n\Phi_n < C$ a.a.s.

- This led Itai Benjamini to formulate:

Conjecture

For every $p > p_c(\mathbb{Z}^d)$, the limit $\lim_{n\to\infty} n\Phi_n$ exists a.s.

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Results

- Concentration around mean for $d \geq 3$:

Theorem 7 (P, Rosenthal 2011)

$\exists C(p,d) > 0$ *such that* $\mathbf{Var}(n\Phi_n) \leq Cn^{2-d}$.

- Proof of Benjamini's conjecture for $d = 2$. Set $\Phi_n = \varphi_{C_2(n)}$.

Theorem 8 (Biskup, Louidor, P, Rosenthal 2012)

There exists a constant $c(p) > 0$ *such that* $\mathbb{P}_p$ *almost surely,*

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Shape theorem

For the next statement $\widehat{W}_p$ is a fixed convex set that will be constructed shortly (Wulff shape). Let $\hat{\mathcal{U}}_{C_2(n)}$ be the set of Cheeger minimizing subsets of $C_2(n)$.

Theorem 9 (Biskup, Louidor, P, Rosenthal 2012)

$$\max_{U \in \hat{\mathcal{U}}_{C_2(n)}} \inf_{\substack{\xi \in \mathbb{R}^2: \\ \xi + \widehat{W}_p/\sqrt{2} \subseteq B(1)}} d_{\mathrm{H}}(n^{-1}U\,,\ \xi + \widehat{W}_p/\sqrt{2}) \underset{n\to\infty}{\longrightarrow} 0,$$

hold for $\mathbb{P}_p$ almost every realization of ω.

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$p \in (p_c(\mathbb{Z}^2), 1] \mapsto \widehat{W}_p$ is Hausdorff continuous.

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Construction of the Wulff shape via weighted paths

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Wulff construction

- Let β_p be a norm on $\mathbb{R}^2$ (p notation for later use).
- Let $\varphi_p =$ $\inf\left\{\mathrm{len}_{\beta_p}(\gamma) : \gamma \text{ is a simple curve in } \mathbb{R}^2, \mathrm{Leb}(\mathrm{int}(\gamma)) = 1\right\}$.
- The minimizing set is given by the Wulff construction

$$W_p := \bigcap_{\hat{n}:\ \|\hat{n}\|_2 = 1} \left\{x \in \mathbb{R}^2 : \hat{n} \cdot x \leq \beta_p(\hat{n})\right\},$$

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Wulff construction

- A quantitative uniqueness theorem.

Theorem 11 (Dobrushin, Kotecký and Shlosman 92)

For any simple rectifiable curve γ enclosing a region of unit Lebesgue area,

$$\inf_{x \in \mathbb{R}^2} d_{\mathrm{H}}(\gamma + x, \partial \widehat{W}_p) \leq C_p \frac{\sqrt{\mathrm{len}_{\beta_p}(\gamma)^2 - \mathrm{len}_{\beta_p}(\partial \widehat{W}_p)^2}}{\mathrm{len}_{\beta_p}(\partial \widehat{W}_p)^2}.$$

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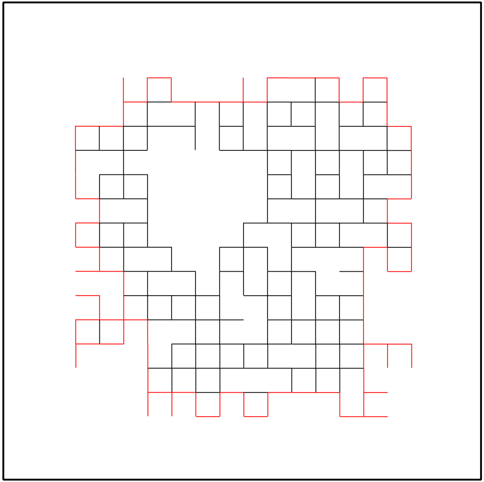
Geometry of $\mathbb{Z}^2$ envelopes

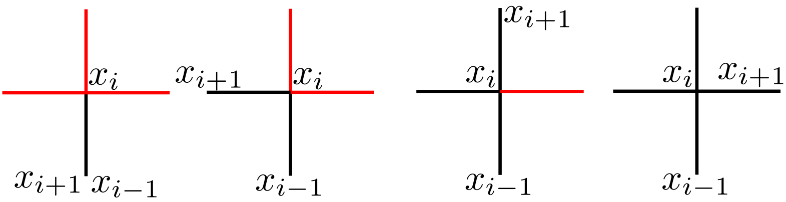
We wish to define a surface tension (the norm β_p) for the Wulff shape, that holds information on boundary of sets. First we need to characterize the envelopes of sets in $\mathbb{Z}^2$.



Boundary norm

Definition (Right boundary edge)

1. *Let $\gamma = (x_0, x_1, \ldots, x_n)$ be a path. An oriented edge (z',z) is said to be a right-boundary edge if $z' = x_i$, and z is a neighbor of z' between x_{i+1} and x_{i-1} in the clockwise direction.*
2. *The right boundary $\partial^+\gamma$ of γ is the set of right-boundary edges.*



Boundary norm

Definition (Right boundary edge)

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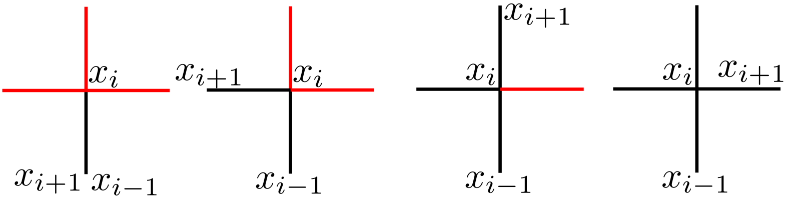
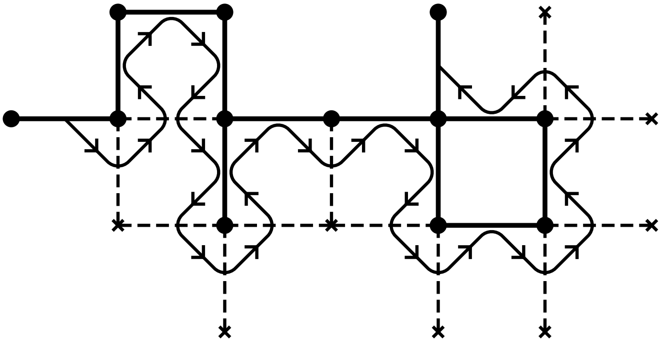


Boundary norm

Definition (Right most path)

1. *A nearest neighbor path* $\gamma = (x_0, x_1, \ldots, x_n)$ *is simple if it uses every edge at most once in each orientation.*
2. *A path is right-most if it is simple and it doesn't contain right-boundary edges.*



Boundary norm (back to percolation)

Definition (Right boundary distance)

1. *Let $\mathcal{R}(x, y)$ be the set of right-most paths from x to y.*
2. *For $\omega \in \Omega$ and a rightmost path γ, let* $b(\gamma) = |\{e \in \partial^{+}\gamma : \omega(e) = 1\}|$.
3. *If $x \sim_{\omega} y$ the the rightmost distance between them is*

$$b(x, y) = \inf\{b(\gamma) : \gamma \in \mathcal{R}(x, y),\ open\}.$$

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Boundary norm

Theorem 12 (The boundary norm)

For any $p > p_c(\mathbb{Z}^2)$ and any $x \in \mathbb{R}^2$, the limit

$$\beta_p(x) := \lim_{n\to\infty} \frac{b(\tilde{0}^{\mathcal{C}_\infty}, \widetilde{nx}^{\mathcal{C}_\infty})}{n} \qquad \textit{exists } \mathbb{P}_p\textit{-a.s.}$$

with $0 \underset{x\neq 0}{<} \beta_p(x) < \infty$. The limit also exists in L^1 and the convergence is uniform on $\{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$.

Moreover, β_p is a norm on $\mathbb{R}^2$.

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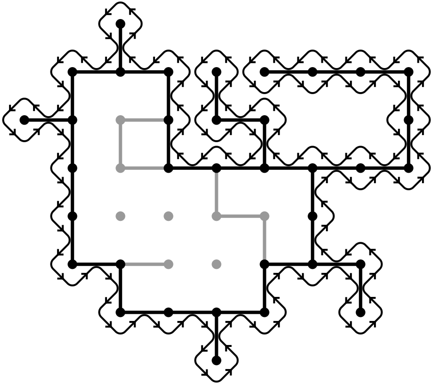
Approximating circuits by simple closed curves in $\mathbb{R}^2$

Definition (Right-most circuit)

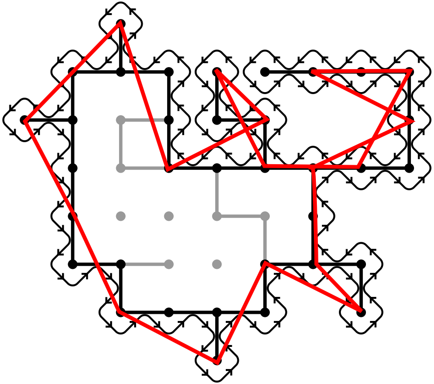
A right-most circuit is a closed right-most path.

It is always possible to find a simple closed curve close to a right-most circuit.



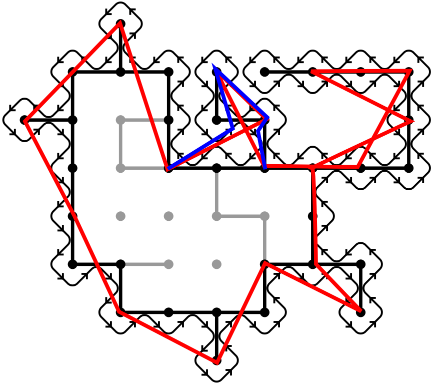
Approximating circuits by simple closed curves in $\mathbb{R}^2$

By first taking a polygonal approximation.

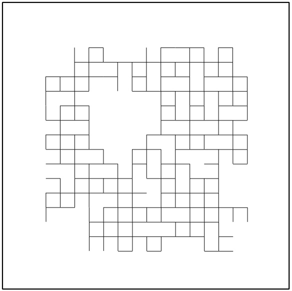


Approximating circuits by simple closed curves in $\mathbb{R}^2$

Then "simplifying" it.

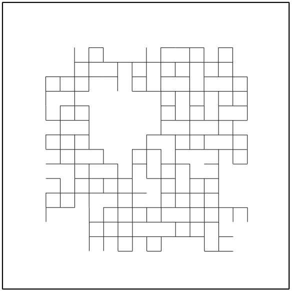


Proof sketch: Lower bound



- Begin with the a set A_n giving the Cheeger constant, which is connected and of size larger than cn^2.
- $\Phi_n = \frac{|\partial A_n|}{|A_n|}$.

Proof sketch: Lower bound



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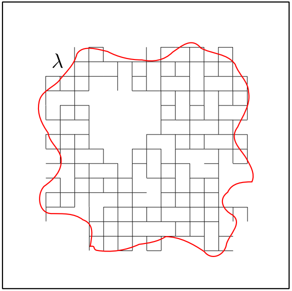

- There exists a closed curve λ, such that: $b(\gamma) \geq (1-\epsilon)\mathrm{len}_{\beta_p}(\lambda)$, where $\gamma = \mathrm{env}(A_n)$.
- Note that rate of convergence to the norm is needed for this approximation on long polygonal lines.

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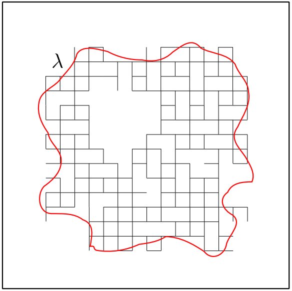

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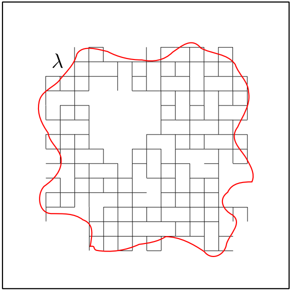

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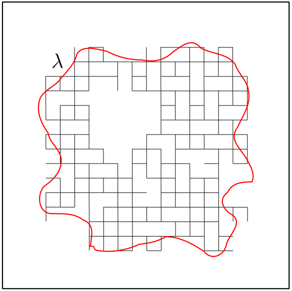

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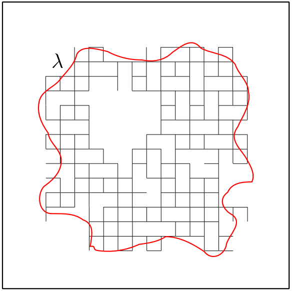

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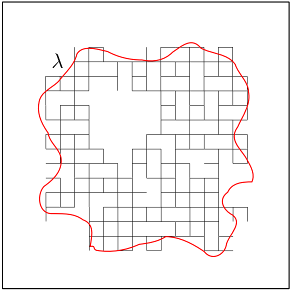

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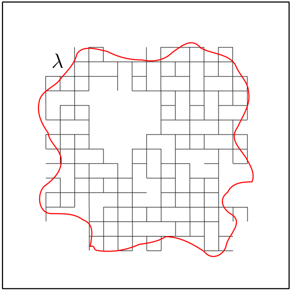

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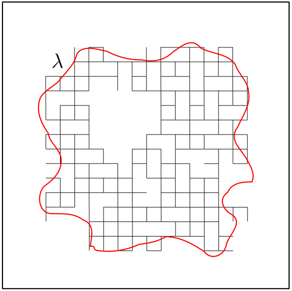

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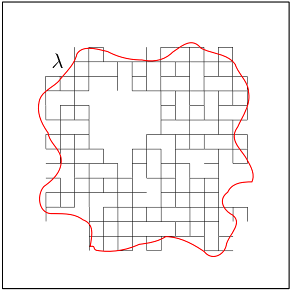

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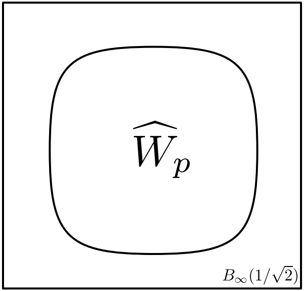
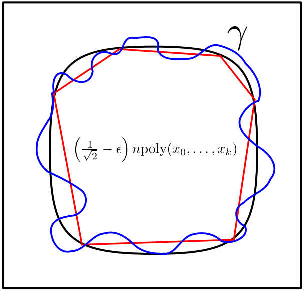
Proof sketch: Upper bound



Begin with the Wulff shape. $\varphi_p = \mathrm{len}(\partial \widehat{W}_p)$.

Proof sketch: Upper bound



- Approximate by a polygon, blow it up by $\frac{n}{\sqrt{2}}$ and find an open circuit with smaller boundary (close to the polygon) with larger length. (With paths that are almost optimal)
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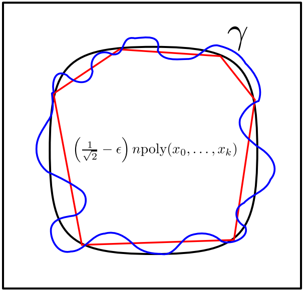

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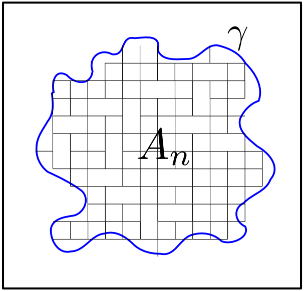

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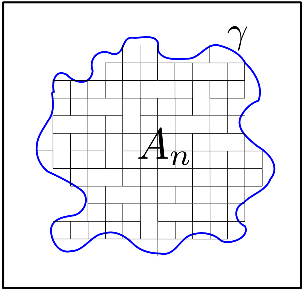

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$\square$

Work in progress (joint with Biskup):

The boys who cried Wulff

(the title is still under debate with Biskup)

The model

- $\{X_t : t \geq 0\}$ is a continuous time RW on $\mathbb{Z}^2$ with law $\mathbf{P}^0$.
- Define the local time $l_t(x) = \int_0^t \mathbf{1}_{\{X_s = x\}} ds$.
- Define $R(t) = \text{hull}\{x \in \mathbb{Z}^d : l_t(x) > 0\}$.
- For $\beta > 0$ consider the Gibbs measure on the path space

$$Q^0_{\beta,t}(A) = \frac{1}{Z(\beta,t)} \mathbf{E}^0 \left(\mathbf{1}_A e^{-\beta|\partial R(t)|} \right).$$

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History

- Berestycki and Yadin (2013) studied a related model (interaction through entire inner boundary).
- Showed that the path is confined (no shape theorem) on the spatial scale

$$r(t,\beta) = \left(\frac{t}{\beta}\right)^{\frac{1}{d+1}}.$$

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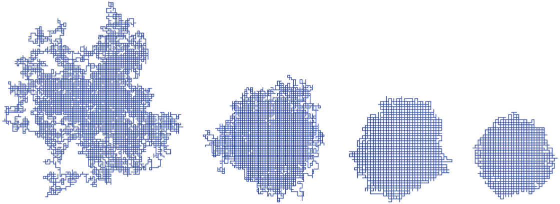

Figure 1: Simulations using a Gibbs sampler algorithm of a random walk cluster with $t =$ 25,000 steps, corresponding to $\beta = 0.01$, $\beta = 0.1$, $\beta = 1$, and $\beta = 2$.

History

- Bolthausen (1994) studied the interaction proportional to the cardinality of the range.
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- The limiting shape is the result of the variational problem

$$\inf\{\lambda(U) : U \subset \mathbb{R}^d \text{ open}, \ |U| = 1\}.$$

Euclidean ball by Faber-Krahn

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Euclidean ball by Faber-Krahn

The modified Berestycki, Yadin Model in $\mathbb{Z}^2$

- The limiting shape is determined by the variational formula

$$\inf\{\lambda(U) + \mathrm{len}_{\|\cdot\|_1}(\partial U) : U \subset \mathbb{R}^d \text{ open with rectifiable } \partial U\}.$$

No volume constraint. An interpolation of the two previous variational formulas.

- We show the variational problem has a unique minimizer, up to shifts, U_0 (convex).

Theorem 13 (Biskup, P 2015)

For every $\epsilon > 0$, there is $\beta_0(\epsilon) < \infty$ such that for all $\beta > \beta_0(\epsilon)$,

$$\lim_{t\to\infty} Q^0_{\beta,t}\left(\inf_{x\in\mathbb{R}^d} dist_H\left(r(t,\beta)^{-1}R(t),\, x+U_0\right) > \epsilon\right) = 0. \quad (1)$$

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The modified Berestycki, Yadin Model in $\mathbb{Z}^2$

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- We show the variational problem has a unique minimizer, up to shifts, U_0 (convex).

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Why this variational problem?

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- First term $\geq e^{-|\partial R(t)|}$. Second term equals $P^0(\tau_S>t)\approx e^{-t\lambda_S}$.
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Questions?

Anyone still awake?