Central Limit Theorem for discrete log–gases

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(based on joint work with Alexei Borodin and Alice Guionnet)

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Setup and overview

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N, \qquad \ell_i = \lambda_i + \theta i$$

Probability distributions on *discrete* N–tuples of the form.

$$\frac{1}{Z} \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod_{i=1}^{N} w(\ell_i; N),$$

Discrete log–gas.

We go **beyond** specific integrable weights.

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Discrete log–gas.
We go **beyond** specific integrable weights.

- Appearance in probabilistic models of statistical mechanics.
- Law of Large Numbers and Central Limit Theorem for global fluctuations as $N \to \infty$ under mild assumptions on $w(x; N)$.
- Our main tool: **discrete loop equations**.

Appearance of discrete log–gases

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

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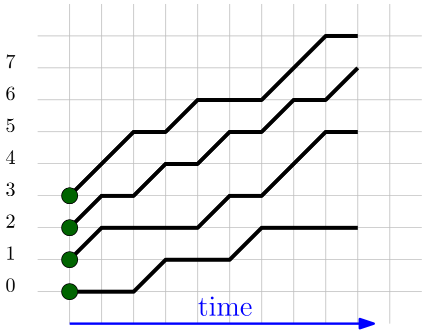
At $\theta=1$ becomes...

$$\frac{1}{Z}\prod_{1\le i<j\le N}(\ell_j-\ell_i)^2\prod_{i=1}^{N}w(\ell_i;N),$$

which frequently appears in natural stochastic systems.

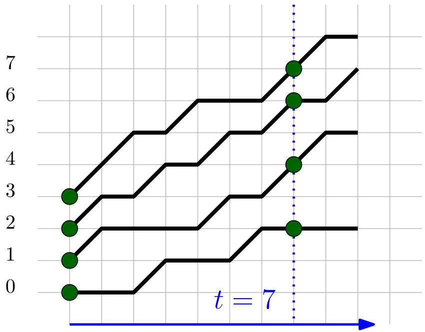
E.g.

Krawtchouk ensemble



- N independent simple random walks
- probability of jump p
- started at *adjacent* lattice points
- conditioned **never to collide**

Krawtchouk ensemble



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Claim. (Konig–O'Connel–Roch) Distribution of N walkers at time t

$$\frac{1}{Z} \prod_{1 \le i < j \le N} (\ell_j - \ell_i)^2 \prod_{i=1}^{N} \left[p^{\ell_i} (1-p)^{M-\ell_i} \binom{M}{\ell_i} \right], \qquad M = N + t - 1.$$

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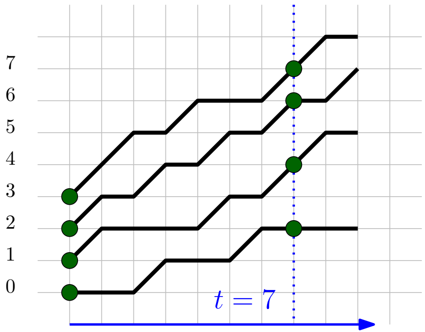

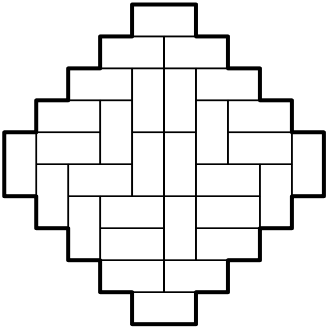

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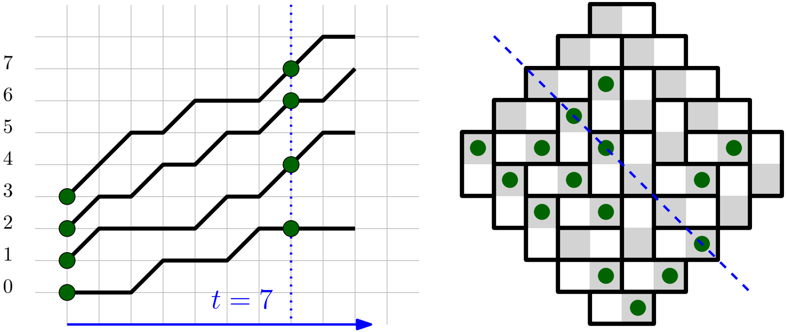

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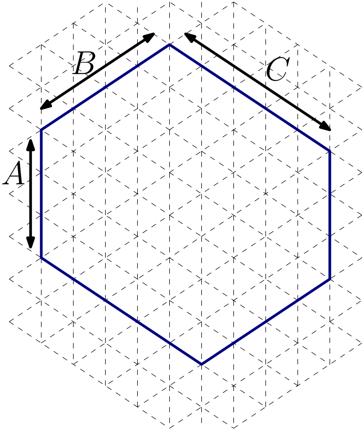
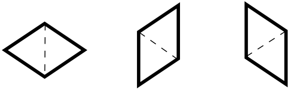


- Regular $A \times B \times C$ hexagon
- 3 types of **lozenges**



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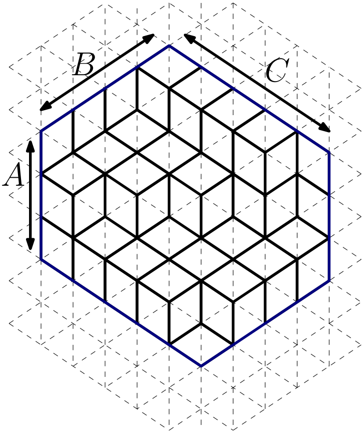

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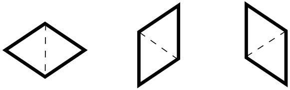

- **uniformly random** tiling

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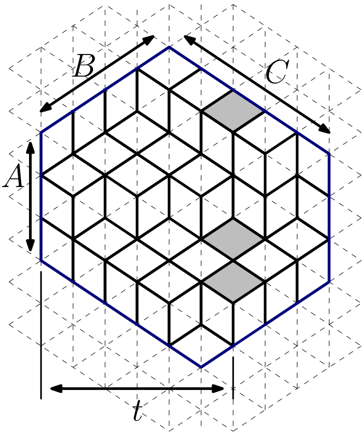

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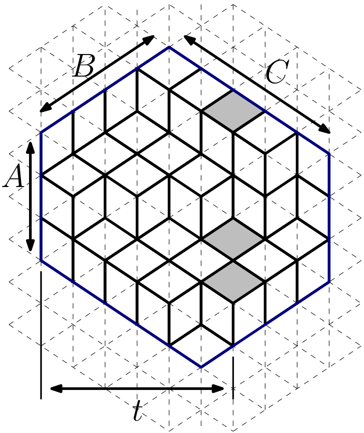

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$$N = B + C - t$$

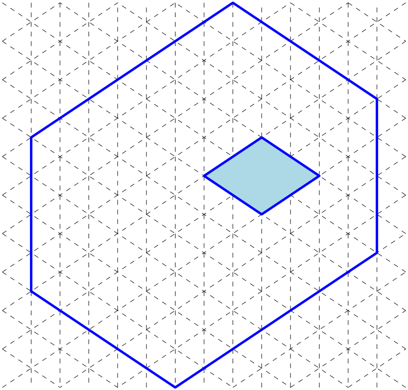
$$t > \max(B, C)$$

$$(a)_n = a(a+1)\dots(a+n-1)$$

Claim. (Cohn–Larsen–Propp)

$$\frac{1}{Z}\prod_{i<j}(\ell_i - \ell_j)^2 \prod_{i=1}^{N}\Big[(A+B+C+1-t-\ell_i)_{t-B}\,(\ell_i)_{t-C}\Big]$$

Two–interval support



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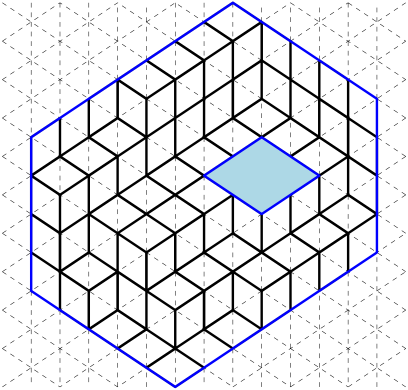

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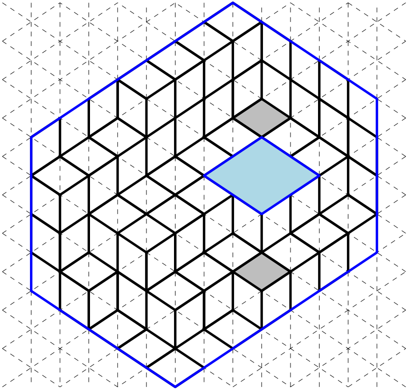

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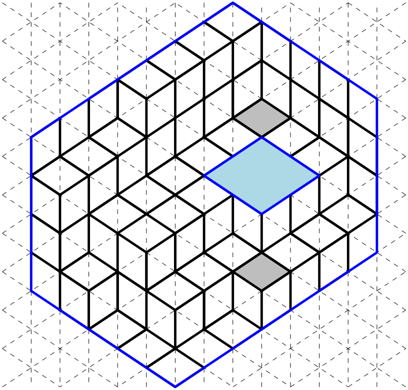

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Claim. It is:
(and similarly for k holes)

$$\prod_{i<j}(\ell_i - \ell_j)^2 \prod_{i=1}^{N}\Big[(A+B+C+1-t-\ell_i)_{t-B}\,(\ell_i)_{t-C}\,(H-\ell_i)_D\,(H-\ell_i)_D\Big]$$

General θ case

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

- $\ell_i = L\cdot x_i, \quad L\to\infty, \quad \beta=2\theta.$

$$\frac{1}{Z}\prod_{1\le i<j\le N}(x_j-x_i)^{\beta}\prod_{i=1}^{N} w(\ell_i;N).$$

Eigenvalue ensembles of **random matrix theory**.

$\beta=1,2,4$ corresponds to real/complex/quaternion matrices.

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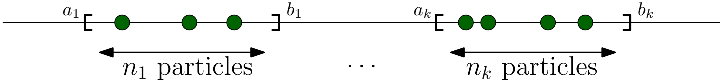
- Another appearance — **asymptotic representation theory**

(Olshanski: (z,w)-measures).
Factor $\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}$ links to evaluation formulas for **Jack** symmetric polynomials.

Large N setup

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

k regions with prescribed **filling fractions**



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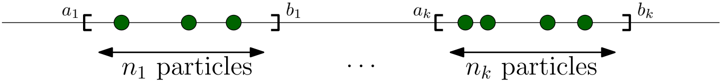

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2. All data **regularly** depends on $N\to\infty$

$$a_i=\alpha_i N+\ldots,\quad b_i=\beta_i N+\ldots,\quad n_i=\hat{n}_i N+\ldots$$

$$w(x;N)=\exp\left(NV_N\left(\frac{x}{N}\right)\right),\quad NV_N(z)=NV(z)+\ldots$$

Potential $V(z)$ should have bounded derivative

(except at end–points, where we allow $V(z)\approx c\cdot z\ln(z)$).

Law of Large Numbers

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

Theorem. Suppose that all data **regularly** depends on $N\to\infty$, then the LLN holds: There exists $\mu(x)dx$ with $0\le\mu(x)\le\theta^{-1}$, such that for any Lipshitz f and any $\varepsilon>0$

$$\lim_{N\to\infty} N^{1/2-\varepsilon}\left|\frac{1}{N}\sum_{i=1}^{N} f\left(\frac{\ell_i}{N}\right)-\int f(x)\mu(x)dx\right|=0$$

In fact the difference is $O(1/N)$.

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$\mu(x)dx$ is the **unique maximizer** of the functional I_V

$$I_V[\rho]=\theta\iint_{x\neq y}\ln|x-y|\rho(dx)\rho(dy)-\int_{-\infty}^{\infty}V(x)\rho(dx).$$

in appropriate class of measures taking into account filling fractions

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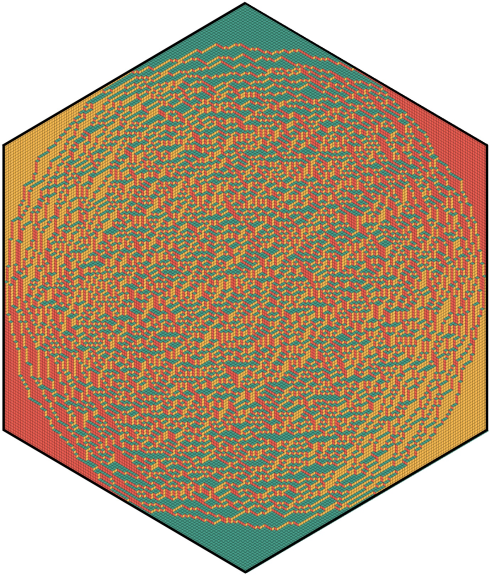
This is a **very general** statement. Lots of analogues.

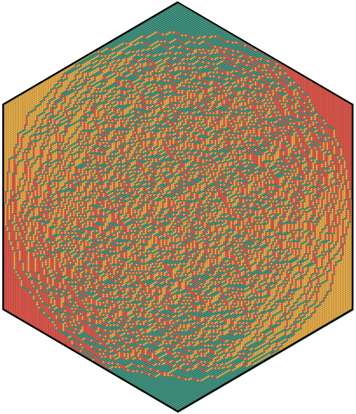
Law of Large Numbers: example

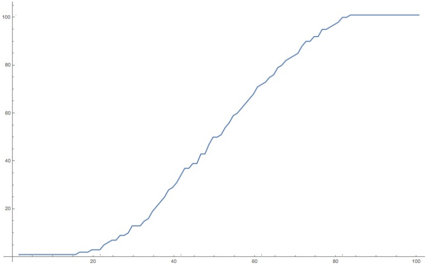


(Pictures by L. Petrov)

Law of Large Numbers: example

Graph of $\lambda_i = \ell_i - i$ (green lozenges) along the middle vertical





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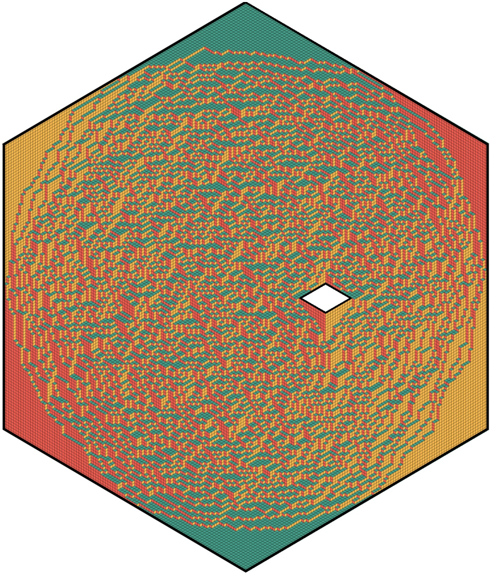
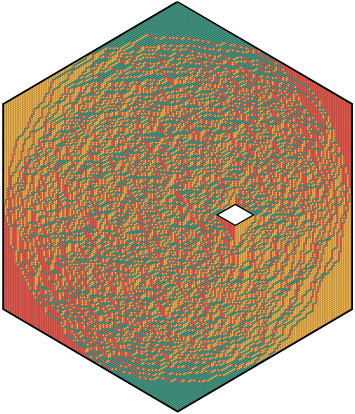
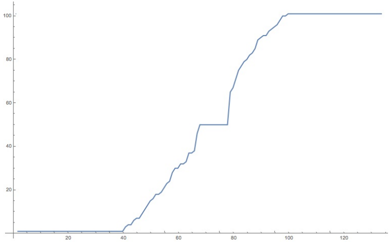


(Pictures by L. Petrov)

Law of Large Numbers: example

Graph of $\lambda_i = \ell_i - i$ (green lozenges) along the vertical axis of hole

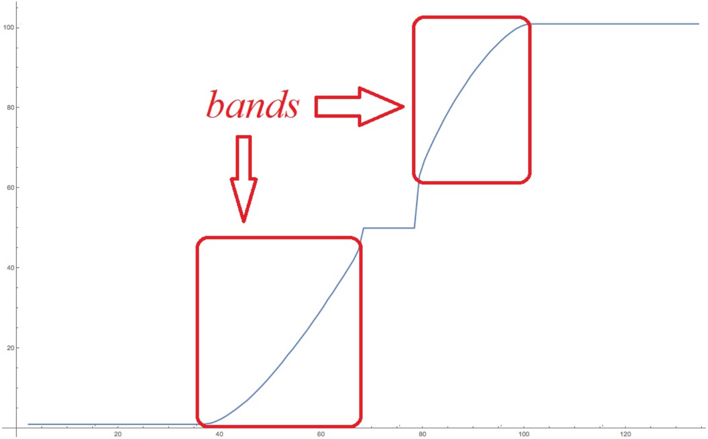




The filling fractions above and below the hole are **fixed**.

Law of Large Numbers: example

Averaged $\lambda_i = \ell_i - i$ (green lozenges) along the vertical axis of hole



- Frozen region: void. No particles, $\mu(x) = 0$.
- Frozen region: saturation. Dense packing, $\mu(x) = \theta^{-1}$.
- **Band.**

Central Limit Theorem?

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Is there a next order, as in CLT?

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left[f\left(\frac{\ell_i}{N}\right) - \mathbb{E} f\left(\frac{\ell_i}{N}\right)\right] \quad ?$$

Central Limit Theorem?

$$\frac{1}{Z} \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp(-NV(x_i))$$

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- In continuous setting of RMT theory — yes, **CLT**.
  (Johansson–1998) one cut/one band, quite general $V(x)$.
  ...
  (Borot–Guionnet–2013) generic $V(x)$, fixed filling fractions in each band. If not fixed $\Rightarrow$ discrete component.

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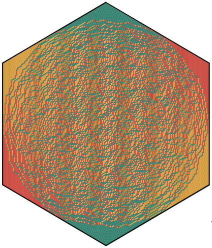

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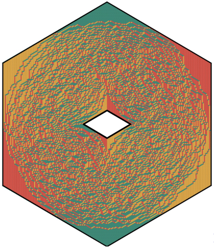

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- (Kenyon–2006), (Petrov–2012) CLT (GFF) for tilings of some simply–connected domains. What if there are holes?
- Several other discrete CLT's exploit specific **integrability**. Methods not suitable for generic models. Approach of Johansson seems to miss a critical ingredient in discrete world.

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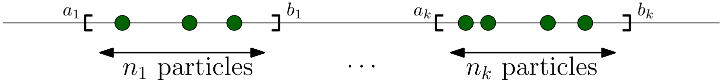

Theorem. Assume that $w(\cdot;N)$ and $V(\cdot)$ are analytic ($x\ln(x)$ behavior of V at end–points is ok), all data depends on N regularly, and $\mu(x)dx$ is such that there is **one** band in each region. Then under *technical assumptions*, for analytic $f_1(x),\dots,f_m(x)$

$$\lim_{N\to\infty}\sum_{i=1}^{N}\left[f_j\left(\frac{\ell_i}{N}\right)-\mathbb{E}f_j\left(\frac{\ell_i}{N}\right)\right],\quad j=1,\dots,m.$$

are jointly **Gaussian** with explicit covariance.

Central Limit Theorem

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- In all the examples shown so far the technical assumption is easy to check. Always holds for convex $V(x)$ with one band.

Central Limit Theorem

Theorem. Assume that all data depends on N regularly, $V(x)$ is analytic (expect for possible $x\ln(x)$ behavior at end–points), and $\mu(x)dx$ is such that there is **one** band in each region. Then under *technical assumptions*, for analytic $f_1(x), \dots, f_m(x)$

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- The covariance depends only on **end–points** of the bands. A log–correlated (generalized) Gaussian field. Section of 2d GFF.
- The result coincides with **universal** behavior in random matrices / continuous β log–gases. (Johansson), (Bonnet–David–Eynard; Scherbina; Borot–Guionnet).

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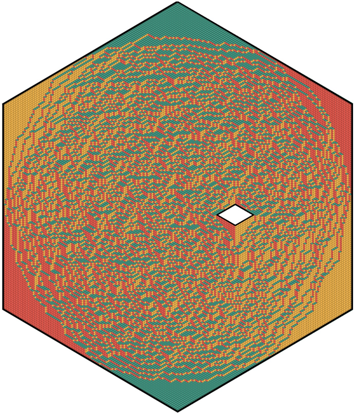
are jointly **Gaussian** with explicit covariance.

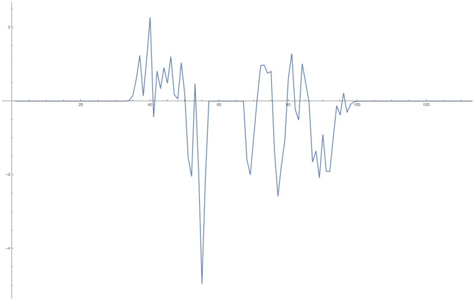
- For a number of **particular** models the result was established before.
- However this is the first **generic** results even at $\theta=1$.

Central Limit Theorem: example

Graph of $\ell_i - \mathbb{E}\ell_i$ (green lozenges) along the vertical axis of hole



The rough fluctuations are *smoothed* in CLT



- The filling fractions above and below the hole are **fixed**.
- Comparison with RMT predicts that if we do not fix them, then a **discrete** component would appear.

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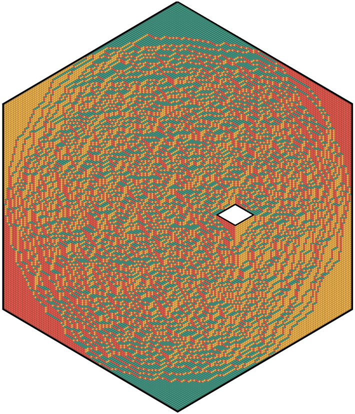

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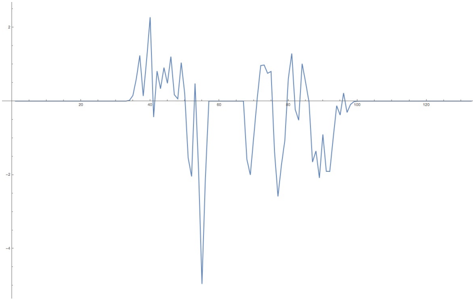

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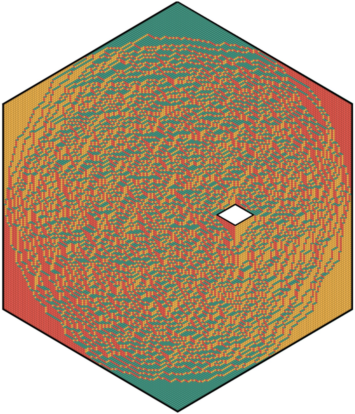

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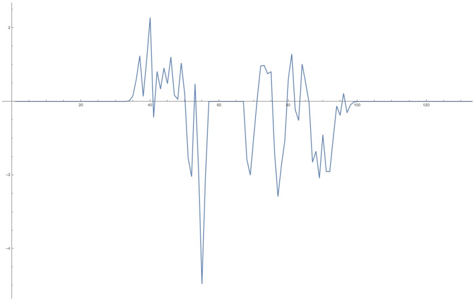

- Comparison with RMT predicts that if we do not fix them, then a **discrete** component would appear. **Why?**
- Jump of one particle through the hole leads to a macroscopic fluctuation of $\sum_{i=1}^{N}[f(\ell_i/N) - \mathbb{E}f(\ell_i/N)]$

Form of measure

$$\frac{1}{Z} \prod_{1 \le i < j \le N} \frac{\Gamma(\ell_j - \ell_i + 1)\Gamma(\ell_j - \ell_i + \theta)}{\Gamma(\ell_j - \ell_i)\Gamma(\ell_j - \ell_i + 1 - \theta)} \prod_{i=1}^{N} w(\ell_i; N),$$

What's so special about this measure? Why not $\prod_{i<j}(\ell_j - \ell_i)^\beta$?

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Recall: Johansson's CLT in RMT is based on **loop equation**

$$\frac{1}{Z}\prod_{1\le i<j\le N}|x_j-x_i|^\beta\prod_{i=1}^{N}\exp(-NV(x_i)).$$

$$G_N(z)=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-x_i}.$$

$$\left[\mathbb{E}G_N(z)\right]^2+\frac{2}{\beta}V'(z)\left[\mathbb{E}G_N(z)\right]+(analytic)=\frac{1}{N}(\dots)$$

Obtained by clever **integration by parts**.

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It also has applications far beyond. E.g. recently in edge universality in RMT (Bourgade–Erdos–Yau), (Bekerman–Figalli–Guionnet)

Discrete CLT was long **blocked** by absence of a discrete analogue.

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Form of discrete measure, for which an analogue could exist?

Can be hinted by **discrete Selberg integrals.**

$$\int_{\mathbb{R}^N}\prod_{1\le i<j\le N}|x_j-x_i|^\beta\prod_{i=1}^{N}w(x),\quad w(x)=\begin{cases}x^a(1-x)^b\,\mathbf{1}_{0<x<1},\\ x^a e^{-x}\,\mathbf{1}_{x>0},\\ e^{-x^2}.\end{cases}$$

Known **explicit** formula manifests integrability of β log–gases.

Form of measure

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Is known **only** at $\beta=2$, but...

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$$\ell_i=\lambda_i+(i-1)\theta, \quad 0\le\lambda_1\le\lambda_2\le\dots\le\lambda_N \text{ — integers}$$

$$\sum\prod_{i<j}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N}\frac{c^x}{\Gamma(\ell_i+1)}.$$

is explicit **for all $\theta>0$** via Jack polynomials (+2 "binomial" $w(x)$).

Main tool: Nekrasov equation

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

Theorem. Assume

$$\frac{w(x;N)}{w(x-1;N)}=\frac{\phi_N^+(x)}{\phi_N^-(x)},\qquad \text{for analytic } \phi_N^\pm.$$

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$$\phi_N^-(\xi)\cdot\mathbb{E}\left[\prod_{i=1}^{N}\left(1-\frac{\theta}{\xi-\ell_i}\right)\right]+\phi_N^+(\xi)\cdot\mathbb{E}\left[\prod_{i=1}^{N}\left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right].$$

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- This is a modification of (Nekrasov–Pestun), (Nekrasov–Shatashvili), (Nekrasov)
- Knowing the statement, the proof is elementary.
- Discrete analogue of loop / Schwinger–Dyson equations.

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How to use this theorem for asymptotic study?

- $\phi^\pm$ — small degree polynomials (linear?), then the result is also a polynomial. Find it to get equations.
- As degree grows, not very helpful. Need another approach.

Functions R_μ and Q_μ

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

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Regularity of data as $N\to\infty$ includes and implies

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Then $\xi=Nz$, $N\to\infty$ leads to analyticity of

$$R_\mu(z)=\phi^-(z)\exp\big(-\theta G_\mu(z)\big)+\phi^+(z)\exp\big(\theta G_\mu(z)\big)$$

G_μ is the **Stieltjes transform** of limiting density.

$$G_\mu(z)=\int\frac{1}{z-x}\mu(x)dx.$$

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$$R_\mu(z) = \phi^-(z)\exp\big(-\theta G_\mu(z)\big) + \phi^+(z)\exp\big(\theta G_\mu(z)\big)$$

We also need

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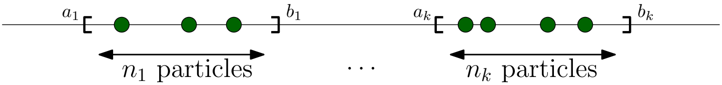

Key technical assumption: for analytic $H(z)$

$$Q_\mu(z) = H(z)\prod_{i=1}^{k}\sqrt{(z-u_i)(z-v_i)}, \qquad H(z)\neq 0.$$

- Quadratic singularities: $Q_\mu(z) = \sqrt{R_\mu(z)^2 - 4\phi^+(z)\phi^-(z)}$.

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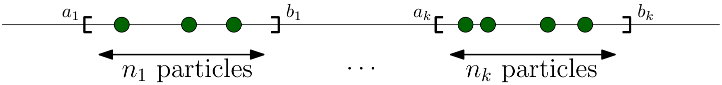

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- u_i and v_i must be end–points of bands.

Second order expansion

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Second order expansion as $N \to \infty$ gives

$$Q_\mu(z) \cdot N\mathbb{E}(G_N(z) - G_\mu(z)) = (\text{explicit}) + (\text{analytic}) + (\text{small}).$$

Here $\quad G_\mu(z) = \int \frac{1}{z-x}\mu(x)dx, \quad G_N(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z - \ell_i/N}.$

(small) requires non-trivial technical work

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Integrate around $\bigcup_{i=1}^{k}[u_i, v_i]$ to get $\lim_{N\to\infty} N\mathbb{E}(G_N(y) - G_\mu(y))$.

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Integrate around $\bigcup_{i=1}^{k}[u_i, v_i]$ to get $\lim_{N\to\infty} N\mathbb{E}(G_N(y)-G_\mu(y))$.

- We use **one band per interval**, as otherwise we can not integrate due to singularities of G_N.
- We use **fixed filling fractions**, to resolve the contribution of the residue at ∞.
- We use $H(z) \neq 0$, as otherwise the unknown (analytic) would contribute.

Proof of Central Limit Theorem

$$G_\mu(z) = \int \frac{1}{z-x}\mu(x)dx, \quad G_N(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\ell_i/N}.$$

We explicitly found $\lim_{N\to\infty} N\mathbb{E}(G_N(y) - G_\mu(y))$.

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$$w(x;N) \to w(x;N)\prod_{a=1}^{m}\left(1 + \frac{t_a}{y_a - x/N}\right).$$

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The deformed measure is in the same class. *If we justify* interchange of derivation and $N \to \infty$ limit, then the cumulants yield asymptotic Gaussianity and the expression for covariance.

Proof of Central Limit Theorem

$$G_\mu(z) = \int \frac{1}{z-x}\mu(x)dx, \quad G_N(z) = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z-\ell_i/N}.$$

Proposition. Deform the weight by m factors

$$w(x;N) \to w(x;N) \prod_{a=1}^{m} \left(1 + \frac{t_a}{y_a - x/N}\right).$$

Then $\lim_{N\to\infty}$ of mixed t_a derivative at 0 of $N\mathbb{E}(G_N(y) - G_\mu(y))$ gives **joint cumulants** of $N\mathbb{E}(G_N(y_a) - G_\mu(y_a))$

Result: $\lim N\mathbb{E}(G_N(y) - \mathbb{E}G_N(y))$ — Gaussian. **One band** $[u,v]$:

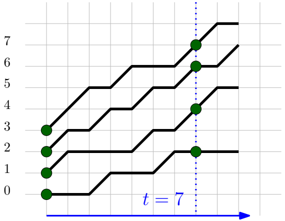
$$\lim_{N\to\infty} N^2\mathbb{E}\left[G_N(y)G_N(z) - \mathbb{E}G_N(y)\mathbb{E}G_N(z)\right]$$
$$= -\frac{1}{2(y-z)^2}\left(1 - \frac{yz - \frac{1}{2}(u+v)(y+z) + u + v}{\sqrt{(y-u)(y-v)}\sqrt{(z-u)(z-v)}}\right),$$

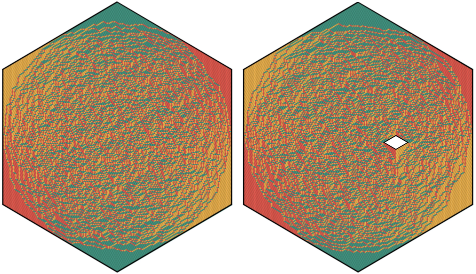
An explicit integral expression for k bands.

Summary

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

1. Central limit theorem with **universal** covariance under
   - One band per interval of support.
   - Technical assumption, which holds in many cases, e.g.





(z,w)–measures of asymptotic representation theory
$w(x;N)=\exp(NV(x/N))$ with convex V
Conjecture (work in progress). In *generic* situation.

Summary

$$\frac{1}{Z}\prod_{1\le i<j\le N}\frac{\Gamma(\ell_j-\ell_i+1)\Gamma(\ell_j-\ell_i+\theta)}{\Gamma(\ell_j-\ell_i)\Gamma(\ell_j-\ell_i+1-\theta)}\prod_{i=1}^{N} w(\ell_i;N),$$

1. Central limit theorem with **universal** covariance under
   - One band per interval of support.
   - Technical assumption, which holds in many cases.
     Conjecture (work in progress). In *generic* situation.
2. An important ingredient of the proof is Nekrasov equation (discrete loop / Schwinger–Dyson equation)

$$\frac{w(x;N)}{w(x-1;N)}=\frac{\phi_N^+(x)}{\phi_N^-(x)}, \qquad \text{for analytic } \phi_N^\pm.$$

$$\phi_N^-(\xi)\cdot\mathbb{E}\left[\prod_{i=1}^{N}\left(1-\frac{\theta}{\xi-\ell_i}\right)\right]+\phi_N^+(\xi)\cdot\mathbb{E}\left[\prod_{i=1}^{N}\left(1+\frac{\theta}{\xi-\ell_i-1}\right)\right]$$

is **analytic** in $\mathcal{D}\subset\mathbb{C}$, where $\phi_N^\pm$ are.