From the mesoscopic to microscopic scale in random matrix theory

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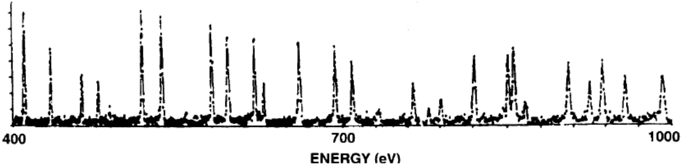
Seymour Sherman conference

A spacially confined quantum mechanical system can only take certain discrete values of energy. Uranium-238 :



Quantum mechanics postulates that these values are eigenvalues of a certain Hermitian matrix (or operator) H, the Hamiltonian of the system.

The matrix elements H_{ij} represent quantum transition rates between states labelled by i and j.

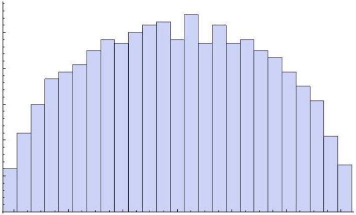
Wigner's universality idea (1956). *Perhaps I am too courageous when I try to guess the distribution of the distances between successive levels. The situation is quite simple if one attacks the problem in a simpleminded fashion. The question is simply what are the distances of the characteristic values of a symmetric matrix with random coefficients.*

Wigner's model : the Gaussian Orthogonal Ensemble,
(a) Invariance by $H \mapsto U^* H U$, $U \in \mathrm{O}(N)$.
(b) Independence of the $H_{i,j}$'s, $i \leq j$.
The entries are Gaussian and the spectral density is

$$\frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum_i \lambda_i^2}$$

with $\beta = 1$ (2, 4 for invariance under unitary or symplectic conjugacy).

- Semicircle law as $N \to \infty$.
- Smooth statistics are Gaussian without normalization.



Fundamental belief in universality : macroscopic statistics depend on the model, but microscopic statistics only depend on the symmetries.

- GOE : Hamiltonians of systems with time reversal invariance
- GUE : no time reversal symmetry (e.g. application of a magnetic field)
- GSE : time reversal but no rotational symmetry

This is not proved for any realistic Hamiltonian.

The local universality is now **known for random matrices**. In the definition of the Gaussian ensembles, either keep :

- the independence of the entries (Wigner ensembles) ;
- or the conjugacy invariance (Invariant ensembles).

This talk is only about Wigner matrices, $N \times N$ matrices such that

$$\mathbb{E}(X_{ij}) = 0, \mathbb{E}(X_{ij}^2) = \frac{1}{N}, \text{ higher moments are finite but arbitrary.}$$

The developed techniques also apply to varrying variances, covariance matrices, mean-field models of sparse random graphs.

Some statistics concerning point processes.

- **Correlation functions.** For a point process $\chi = \sum_{i=1}^{N} \delta_{\lambda_i}$:

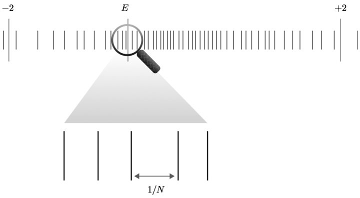
$$\rho_k^{(N)}(x_1, \dots, x_k) = \lim_{\varepsilon \to 0} \varepsilon^{-k}\, \mathbb{P}\left(\chi(x_i, x_i + \varepsilon) = 1, 1 \leq i \leq k\right).$$

Gaudin, Dyson, Mehta (GUE for example) : for any $E \in (-2, 2)$,

$$\rho_k^{(N)}\left(E + \frac{u_1}{N\varrho(E)}, \dots, E + \frac{u_k}{N\varrho(E)}\right) \underset{N\to\infty}{\longrightarrow} \det_{k\times k} \frac{\sin(\pi(u_i - u_j))}{\pi(u_i - u_j)}.$$

- **Counting numbers.**

$$\mathbb{P}\left(|\lambda_i - E| \geq \frac{\alpha}{N\pi\varrho(E)}\right) \underset{N\to\infty}{\longrightarrow} E(0, \alpha).$$



Jimbo-Miwa-Mori-Sato (GUE for example) : $E(0, \alpha)$ exists, it is independent of E and satisfies a Painlevé equation.

The Wigner-Dyson-Mehta conjecture. Correlation functions of symmetric Wigner matrices (resp. Hermitian, symplectic) converge to the limiting GOE (resp. GUE, GSE).

Pointwise convergence of correlation functions cannot hold.

Recently universality was proved under various forms.

Fixed (averaged) energy universality. For any $k \geq 1$, smooth $F : \mathbb{R}^k \to \mathbb{R}$, for arbitrarily small ε and $s = N^{-1+\varepsilon}$,

$$\lim_{N\to\infty} \frac{1}{\varrho(E)^k} \int_E^{E+s} \frac{\mathrm{d}x}{s} \int \mathrm{d}\mathbf{v} F(\mathbf{v}) \rho_k^{(N)} \left(x + \frac{\mathbf{v}}{N\varrho(E)} \right) \mathrm{d}\mathbf{v} \\ = \int \mathrm{d}\mathbf{v} F(\mathbf{v}) \rho_k^{(\mathrm{GOE})} (\mathbf{v}) \, .$$

Johansson (2001)	Hermitian class, fixed E, Gaussian divisible entries
Erdős Schlein Péché Ramirez Yau (2009)	Hermitian class, fixed E Entries with density
Tao Vu (2009)	Hermitian class, fixed E Entries with 3rd moment=0
Erdős Schlein Yau (2010)	Any class, averaged E

For symmetric matrices, Tao and Vu's four moments theorem states that universality holds (including for symmetric matrices) if the Wigner matrix first four moments are 0,1,0,3.

This does not include Jimbo, Miwa, Mori, Sato relations for gaps in the spectrum of Bernoulli matrices, for example.

Assume $\lambda_1 \leq \cdots \leq \lambda_N$ and define $\int_{-2}^{\gamma_i} \mathrm{d}\varrho = \frac{i}{N}$.

Key input for all recent results : **rigidity of eigenvalues** (Erdős Yau Yin) : $|\lambda_k - \gamma_k| \leq N^{-1+\varepsilon}$ in the bulk. Is this the optimal rigidity ? Are fluctuations in the bulk Gaussian ?

The true size of fluctuations is suggested by the following theorem.

Theorem (Gustavsson, O'Rourke)

Consider GOE/GUE/GSE. Let k_0 a bulk index and $k_{i+1} \sim k_i + N^{\theta_i}$, $0 < \theta_i < 1$. Then the normalized eigenvalues fluctuations

$$X_i = \frac{\lambda_{k_i} - \gamma_{k_i}}{\frac{\sqrt{\log N}}{N}} \sqrt{\beta(4 - \gamma_{k_i}^2)}$$

converge to a Gaussian vector with covariance

$$\Lambda_{ij} = 1 - \max\{\theta_k, i \leq k < j\}.$$

In particlar, $\lambda_i - \gamma_i$ has Gaussian fluctuations of size $\frac{\sqrt{\log N}}{N}$.

Extension to Wigner matrices hold under the four moment matching assumption (O'Rourke, Dallaporta, Vu).

Proof : determinantal point processes a la Costin-Lebowitz (GUE) + decimation relations (GOE, GSE).

This is a **Log-correlated field**, as explained below.
The main difficulty proving fixed energy universality is the very slow decay of correlations :

$$\langle N\lambda_i, N\lambda_j \rangle \sim \log\left(\frac{N}{1+|i-j|}\right).$$

Each eigenvalue is localized on a very small window, almost regular spacing : smooth density and translation invariance are not necessarily intuitive for Wigner matrices.

Log-correlated random fields appear in the study of random surfaces, Liouville quantum gravity, models of turbulence...

Theorem (with Erdös, Yau, Yin)

Fixed energy universality holds for Wigner matrices from all symmetry classes. Individual eigenvalues fluctuate as a Log-correlated Gaussian field.

The **Dyson Brownian Motion** is an essential interpolation tool in our proof ($\mathrm{d}H_t = \frac{\mathrm{d}B_t}{\sqrt{N}} - \frac{1}{2}H_t\mathrm{d}t$) , as in the Erdős Schlein Yau approach to universality, which can be summarized as follows :

$$\begin{array}{ccc} & & H_0 \\ & & \updownarrow \\ \widetilde{H}_0 & \xrightarrow{\text{(DBM)}} & \widetilde{H}_t \end{array}$$

$\xrightarrow{\text{(DBM)}}$: for $t = N^{-1+\varepsilon}$, the eigenvaues of $\widetilde{H}_t$ satisfy averaged universality.

$\updownarrow$: Density argument. For any $t \ll 1$, there exists $\widetilde{H}_0$ such that H_0 and $\widetilde{H}_t$ have the same statistics on the microscopic scale.

What makes the Hermitian universality easier ? The $\xrightarrow{\text{(DBM)}}$ is replaced by The Harish-Chandra-Itzykson-Zuber integral formula : correlation functions of $\widetilde{H}_t$ are explicit only for $\beta = 2$.

A few curiosities about the proof of fixed energy universality.

(i) A game coupling three Dyson Brownian Motions.

(ii) Homogenization allows to obtain microscopic statistics from mesoscopic ones.

(iii) Need of a higher order type of Hilbert transform. Emergence of new explicit kernels for any Bernstein-Szegő measure. These include Wigner, Marchenko-Pastur, Kesten-McKay.

(iv) The relaxing time of DBM depends on the Fourier support of the test function : the step $\overset{\text{(DBM)}}{\longrightarrow}$ becomes the following.

$$\widetilde{F}(\boldsymbol{\lambda}, \Delta) = \sum_{i_1,\dots,i_k=1}^{N} F\left(\{N(\lambda_{i_j} - E) + \Delta, 1 \leq j \leq k\}\right)$$

Fact

If $\mathrm{supp}\hat{F} \subset \mathrm{B}(0, 1/\sqrt{\tau})$, then for $t = N^{-\tau}$,

$$\mathbb{E}\,\widetilde{F}(\boldsymbol{\lambda}_t, 0) = \mathbb{E}\,\widetilde{F}(\boldsymbol{\lambda}^{(\mathrm{GOE})}, 0).$$

First step : coupling two Dyson Brownian Motions. Let $\mathbf{x}(0)$ be the eigenvalues of $\widetilde{H}_0$ and $\mathbf{y}(0), \mathbf{z}(0)$ those of two indepndent GOE.

$$\mathrm{d}x_i/\mathrm{d}y_i/\mathrm{d}z_i = \sqrt{\frac{2}{N}}\mathrm{d}B_i(t) + \frac{1}{N}\left(\sum_{j\neq i}\frac{1}{x_i/y_i/z_i - x_j/y_j/z_j} - \frac{1}{2}x_i/y_i/z_i\right)\mathrm{d}t$$

Let $\delta_\ell(t) = e^{t/2}(x_\ell(t) - y_\ell(t))$. Then we get the parabolic equation

$$\partial_t\delta_\ell(t) = \sum_{k\neq\ell}\mathcal{B}_{k\ell}(t)\left(\delta_k(t) - \delta_\ell(t)\right), \mathcal{B}_{k\ell}(t) = \frac{1}{N(x_k(t) - x_\ell(t))(y_k(t) - y_\ell(t))}.$$

By the de Giorgi-Nash-Moser method, Caffarelli-Chan-Vasseur and Erdős-Yau, this PDE is Hölder-continuous for $t > N^{-1+\varepsilon}$, i.e. $\delta_\ell(t) = \delta_{\ell+1}(t) + \mathrm{O}(N^{-1-\varepsilon})$, i.e. gap universality :

$$x_{\ell+1}(t) - x_\ell(t) = y_{\ell+1}(t) - y_\ell(t) + \mathrm{O}(N^{-1-\varepsilon}).$$

This is not enough for fixed energy universality.

Second step : homogenization. The continuum-space analogue of our parabolic equation is

$$\partial_t f_t(x) = (\mathcal{K} f_t)(x) := \int_{-2}^{2} \frac{f_t(y) - f_t(x)}{(x-y)^2} \varrho(y) \mathrm{d}y.$$

$\mathcal{K}$ is some type of higher order Hilbert transform.

Fact

Let f_0 be a smooth continuous-space extension of $\delta(0)$: $f_0(\gamma_\ell) = \delta_\ell(0)$. Then for any small $\tau > 0$ ($t = N^{-\tau}$) thre exists $\varepsilon > 0$ such that

$$\delta_\ell(t) = \left(e^{t\mathcal{K}} f_0\right)_\ell + \mathrm{O}(N^{-1-\varepsilon}).$$

Proof. Rigidity of the eigenvalues and the Duhamel formula.

Third step : the continuous-space kernel.

1. For the translation invariant equation

$$\partial_t g_t(x) = \int_{\mathbb{R}} \frac{g_t(y) - g_t(x)}{(x-y)^2} \mathrm{d}y,$$

   the fundamental solution is the Poisson kernel $p_t(x, y) = c\ \frac{t}{t^2+(x-y)^2}$.

2. For us, t will be close to 1, so the edge curvture cannot be neglected. Fortunately, $\mathcal{K}$ can be fully diagonalized and ($x = 2\cos\theta$, $y = 2\cos\phi$)

$$k_t(x, y) = \frac{c_t}{|e^{\mathrm{i}(\theta+\phi)} - e^{-t/2}|^2 |e^{\mathrm{i}(\theta-\phi)} - e^{-t/2}|^2}.$$

   Called the Mehler kernel by Biane in free probability context, not considered as a higher order Hilbert transform fundamental solution.

3. Explicit kernels can be obtained for all Bernstein-Szego measures,

$$\varrho(x) = \frac{c_{\alpha,\beta}(1-x^2)^{1/2}}{(\alpha^2 + (1-\beta^2)) + 2\alpha(1+\beta)x + 4\beta x^2}.$$

Fourth step : from mesoscopic to microscopic. Homogenization yields

$$\delta_\ell(t) = \int k_t(x,y) f_0(y) \varrho(y) \mathrm{d}y + \mathrm{O}(N^{-1-\varepsilon})$$

The LHS is microscopic-type of statistics, the RHS is mesoscopic. This yields, up to negligible error,

$$N x_\ell(t) = N y_\ell(t) - \Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{x}_0),$$

where $\Psi_t(\mathbf{x}_0) = \sum h(N^\tau(x_i(0) - E))$ for some smooth h. We wanted to prove

$$\mathbb{E}\, \widetilde{F}(\mathbf{x}_t, 0) = \mathbb{E}\, \widetilde{F}(\mathbf{z}_t, 0) + \mathrm{o}(1).$$

We reduced it to

$$\mathbb{E}\, \widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{x}_0)) = \mathbb{E}\, \widetilde{F}(\mathbf{y}_t, \Psi_t(\mathbf{y}_0) + \Psi_t(\mathbf{z}_0)) + \mathrm{o}(1).$$

The observables $\Psi_t(\mathbf{y}_0)$, $\Psi_t(\mathbf{x}_0)$ and $\Psi_t(\mathbf{z}_0)$ are mesoscopic and independent, while $\mathbf{x}_t$ and $\mathbf{z}_t$ are microscopic and dependent.

Fifth step and conclusion : CLT for GOE beyond the natural scale. Do $\Psi_t(\mathbf{x}_0)$ and $\Psi_t(\mathbf{y}_0)$ have the same distribution ? No, their variance depend on their fourth moment.

A stronger result holds : $\mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+c)$ does not depend on c.

We know that $\mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+\Psi_t(\mathbf{z}_0)+c) = \mathbb{E}\,\widetilde{F}(\mathbf{y}_t, -\Psi_t(\mathbf{y}_0)+\Psi_t(\mathbf{z}_0))$.

Exercise

Let X be a random variable. If $\mathbb{E}\,g(X+c)=0$ for all c, is it true that $g\equiv 0$?

Not always. But true if X is Gaussian (by Fourier).

Lemma

$\mathbb{E}\left(e^{\mathrm{i}\lambda\Psi_t(\mathbf{z}(0))}\right) = e^{-\frac{\lambda^2}{2}\tau\log N} + \mathrm{O}(N^{-1/100})$.

The proof uses algebraic ideas of Johansson and rigidity of β-ensembles (with Erdős and Yau).

By Parseval, proof when the support of $\hat{F}$ has size $1/\sqrt{\tau}$. This is why DBM needs to be run till time almost 1.

Homogenization of the Dyson Brownian Motion allows to access microscopic statistics from mesoscopic ones.

1. Universality at fixed energy.
2. Eigenvalues fluctuate like a Gaussian Log-correlated field.
3. Eigenvectors perturbations in a non-perturbative regime.

Some problems about microscopic statistics of random matrices :

1. Are extreme gaps and extreme deviations universal ?
2. Log-correlated field for β-ensembles ?

A major problem in the field now : beyond the mean-field case, universality for sparse+geometry-dependent models of random matrices, approaching random Schrödinger operators.