A Bird's-Eye View of Uniform Spanning Trees and Forests

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Abstract. We survey the field of uniform spanning forest measures on infinite graphs, which are weak limits of uniform spanning tree measures from finite subgraphs. These limits can be taken with free or wired boundary conditions. Among other results, Pemantle (1991) proved that in $\mathbb{Z}^d$, the free and wired spanning forests coincide and that they give a single tree iff $d \le 4$. The theory has developed considerably since then and found further connections to random walks, potential theory, harmonic Dirichlet functions, invariant percolation and amenability. A crucial new tool is an algorithm invented by Wilson (1996) to generate random spanning trees. Uniform spanning forests also yield insights into loop-erased walks and harmonic measure from infinity.

§1. Introduction.

We begin with a brief history of this fertile and fascinating field. In subsequent sections, we will more carefully define and develop most of what we recount here. More details for much of the material surveyed here can be found in Benjamini, Lyons, Peres, and Schramm (1997), hereinafter referred to as BLPS (1997). One may also consult the book by Lyons and Peres (1997) for additional background.

A **spanning tree** of a (connected) graph is a subgraph that is connected, contains every vertex of the whole graph, and contains no cycle: see Fig. 1 for an example. The subject of random spanning trees of a graph goes back to Kirchhoff (1847), who showed its relation to electrical networks. One of these relations gives the probability that a uniformly chosen spanning tree will contain a given edge in terms of electrical current in the graph: see (3.1).

It turns out that generating spanning trees at random according to the uniform measure is of interest to computer scientists, who have developed various algorithms over the

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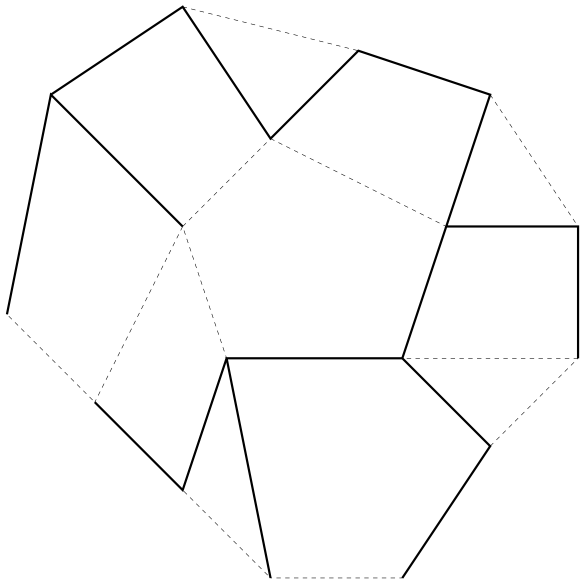

Figure 1. A spanning tree in a graph.

years for random generation of spanning trees. In particular, this is closely connected to generating a random state from any Markov chain. See Propp and Wilson (1996) for more on this issue.

Early algorithms for generating a random spanning tree used the Matrix-Tree Theorem, which counts the number of spanning trees in a graph via a determinant. A better algorithm than these early ones, especially for probabilists, was introduced by Aldous (1990) and Broder (1989). It says that if you start a simple random walk at *any* vertex of a finite graph G and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G. This beautiful algorithm is quite efficient and useful for theoretical analysis, yet Wilson (1996) found an even better one that we'll describe later.

The study of the analogue of a uniform spanning tree on an infinite graph was begun by Pemantle (1991) at the suggestion of the author. Suppose that G is an infinite graph. Let G_n be finite subgraphs with $G_1 \subset G_2 \subset G_3 \subset \cdots$ and $\bigcup G_n = G$. In this case, we say that G is **exhausted** by the sequence $\langle G_n \rangle$. Pemantle showed that the weak limit of the uniform spanning tree measures on G_n exists. (In other words, if μ_n denotes the uniform spanning tree measure on G_n and B, B' are finite sets of edges, then $\lim_n \mu_n(B \subset T, B' \cap T = \emptyset)$ exists, where T denotes a random spanning tree.) This limit is now called the **free uniform spanning forest*** on G, denoted FUSF. Considerations of electrical networks play the

* In graph theory, "spanning forest" usually means a maximal subgraph without cycles, i.e., a spanning

dominant role in Pemantle's proof. If G is itself a tree, then this measure is trivial, namely, it is concentrated on $\{G\}$. However, Häggström (1995) drew attention to another limit, namely, the weak limit of the uniform spanning tree measures on G_n^W, where G_n^W is the graph G_n with its boundary (i.e., the vertices of G_n that are adjacent to some vertex of $G \setminus G_n$) identified ("wired") to a single vertex. In fact, Pemantle implicitly proved the existence of this limit as well on any graph; it is now called the **wired uniform spanning forest**, denoted WUSF.

Pemantle (1991) discovered the following interesting properties, among others:

- The free and the wired uniform spanning forest measures are the same on all euclidean lattices $\mathbb{Z}^d$.
- Amazingly, on $\mathbb{Z}^d$, the uniform spanning forest is a single tree a.s. if $d \leq 4$; but when $d \geq 5$, there are infinitely many trees a.s.
- If $2 \leq d \leq 4$, then the uniform spanning tree on $\mathbb{Z}^d$ has a single end a.s. (as defined in Section 6); when $d \geq 5$, each of the infinitely many trees a.s. has at most two ends. (Just how many ends they have will be seen later.)

One of Pemantle's main tools was the Aldous-Broder algorithm. In addition, the study of loop-erased random walks due to Lawler (1991) was crucial. Using Wilson's (1996) algorithm in place of the one due to Aldous and Broder, as well as other tools, BLPS (1997) redeveloped the fundamentals of the theory, originally due to Pemantle (1991) and to Burton and Pemantle (1993). As the theory has become easier to apply and the scope has broadened, new results have emerged.

For example, the general question of when the free and wired measures are the same turns out to be quite interesting: BLPS (1997) showed that the measures are the same iff there are no nonconstant harmonic Dirichlet functions on G.

Burton and Pemantle (1993) found a beautiful generalization of Kirchhoff's theorem relating to electrical networks; their theorem gives the probability that a given set of edges belongs to a uniform spanning tree in terms of an electrical determinant. Burton and Pemantle (1993) also found the entropy of the uniform spanning forest measure on $\mathbb{Z}^d$; coincidentally, it turned out to be exactly the same as the entropy of an apparently entirely different dynamical system studied by Lind, Schmidt, and Ward (1990). This remained a mystery until R. Solomyak (1997) discovered that the Matrix-Tree Theorem is responsible. In the case $d = 2$, Burton and Pemantle (1993) showed that this is also the entropy of yet another system, that of domino tilings (perfect matchings) of the square lattice $\mathbb{Z}^2$; indeed,

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tree in each connected component. We mean, instead, a subgraph without cycles that contains every vertex.

there is a direct connection (due to Temperley) between spanning trees and domino tilings in planar graphs.

As we will see, uniform spanning forests have interesting connections not only to random walks, algorithms, domino tilings, and electrical networks, but also to planar duality, amenability, percolation, and hyperbolic spaces. Because of this, there is still an enormous number of fascinating open questions. We will present some of them in the last section of this survey.

Finally, we mention that in an entirely different context, that of euclidean functionals (such as length of the minimal spanning tree), it has also been recently discovered that comparison to a "wired" version is extremely useful: see Redmond and Yukich (1994) and Yukich (1996a,b), for example.

§2. Some Examples.

We begin with some pictures. First, consider Fig. 2. Examination of Fig. 2 reveals two spanning trees: one in white, the other in black. To be more precise, if G is a (possibly infinite) proper (connected) planar graph, meaning that G is drawn in the plane so that edges do not overlap and there are only finitely many vertices in every bounded region, then define its **planar dual** graph $G^\dagger$ to have one vertex in each face of G and one edge $e^\dagger$ for each edge e of G, with $e^\dagger$ joining the vertices in the faces on each side of e. See Fig. 3.

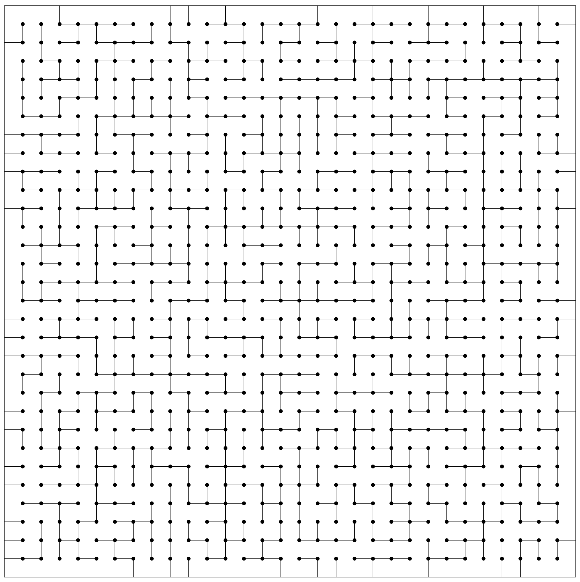

Figure 2. A uniformly chosen wired spanning tree on a subgraph of $\mathbb{Z}^2$, drawn by Wilson (see Propp and Wilson (1996)).

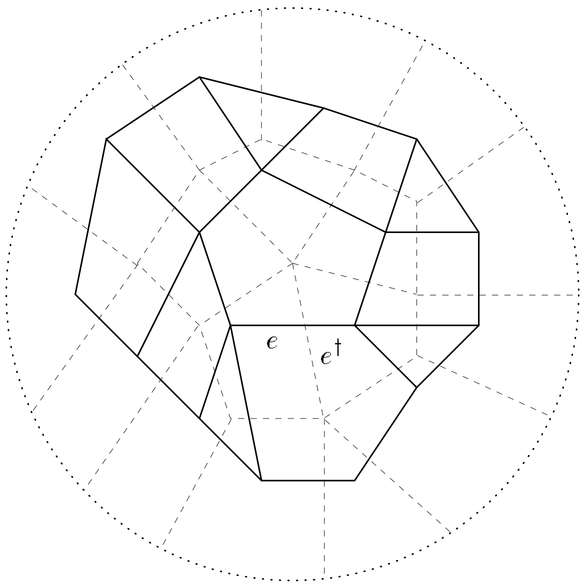

Figure 3. Planar dual graphs. The outer vertex is drawn as a dotted circle for convenience.

Thus, in Fig. 2, we may regard G to be a portion of the square lattice, with a vertex at each white square, and $G^\dagger$ a portion of the square lattice drawn in black, but with the outer boundary of the grid identified to a single vertex.

Another way of saying that the white part forms a spanning tree is to say that if we make two openings on the outer boundary, as in Fig. 4, then there is one and only one way to traverse the resulting maze without backtracking, and it is possible to get lost anywhere.

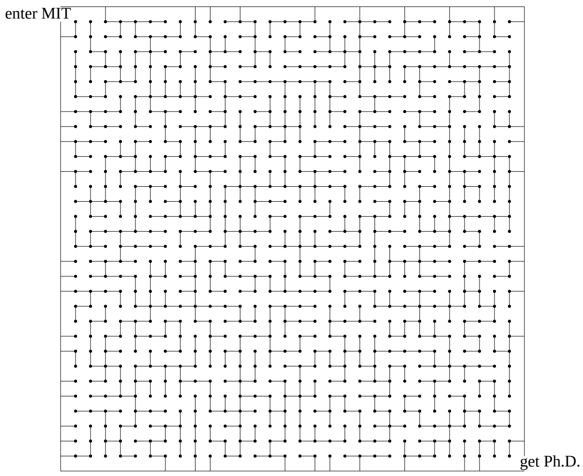

Figure 4. A maze resulting from Fig. 2, drawn by Wilson (see Propp and Wilson (1996)).

Furthermore, that the white part forms a spanning tree is equivalent to the black part forming a spanning tree: any cycle in one would disconnect the other, while one is disconnected only by a cycle in the other. This holds for any pair of planar dual graphs, G and $G^\dagger$: Spanning trees T of G are in one-to-one correspondence with "complementary" spanning trees $T^\dagger$ of $G^\dagger$ via

$$e \in T \iff e^\dagger \notin T^\dagger \,. \tag{2.1}$$

Now there are about 10^{500} spanning trees on the 31×31 portion of the square lattice shown in Fig. 2: this is an example of the entropy calculation of Burton and Pemantle (1993). This number is huge; yet the one in Fig. 2 was chosen at random according exactly to the uniform measure (or as closely as a computer can simulate randomness). How is this possible? As we mentioned in the introduction, generation of random spanning trees has drawn the attention of computer scientists. The method used for drawing Fig. 2 is the new one due to Wilson (1996). It is not only faster than all previous methods, but also turns out to be better from the theoretical viewpoint of analyzing the uniform probability measure on spanning trees and, especially, their extensions to forests on infinite graphs. We will describe Wilson's method in Section 4.

In order for the reader to have another picture in mind that is as close to the square lattice $\mathbb{Z}^2$ as possible, yet possesses a hyperbolic geometry, we present Fig. 5. This planar graph in the hyperbolic disc is also self-dual, but is transient for simple random walk. The dichotomy between transience and recurrence, as we will see in Section 5, is crucial for understanding the behavior of spanning trees and their limits on infinite graphs.

However, we begin with a very simple finite graph on which to begin our analysis and make some preliminary probabilistic observations. Namely, consider the ladder graph of Fig. 6. Among all spanning trees of this graph, what proportion contain the bottom rung (edge)? In other words, if we were to choose at random a spanning tree, what is the chance that it would contain the bottom rung? We have illustrated the entire probability spaces for the smallest ladder graphs in Fig. 7.

As shown, the probabilities in these cases are $1/1$, $3/4$, and $11/15$. The next one is $41/56$. Is the pattern clear? One thing that is fairly evident is that these numbers are decreasing, but hardly changing. It turns out that they come from every other term of the continued fraction expansion of $\sqrt{3} - 1 = 0.73^+$ and, in particular, converge to $\sqrt{3} - 1$. In the limit, then, the probability of using the bottom rung is $\sqrt{3} - 1$ and, even before taking the limit, this gives an excellent approximation to the probability. How can we easily calculate such numbers? Of course, in this case, there is a rather easy recursion to set up and solve, but we will illustrate the more general method of electric networks. In fact, this method will show us why these probabilities are decreasing even before we

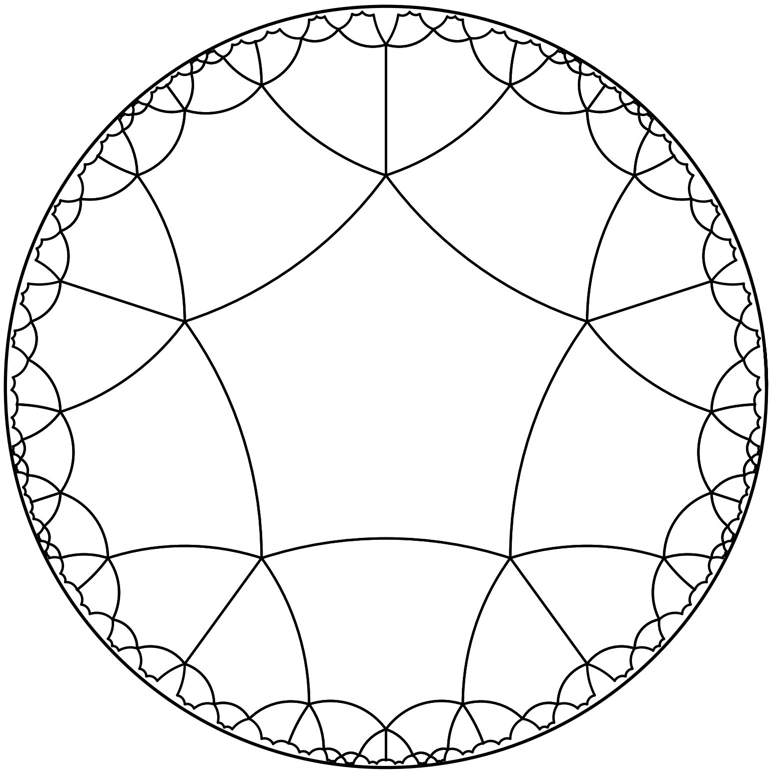

Figure 5. A self-dual graph in the hyperbolic disc.

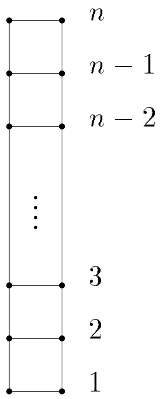

Figure 6. A ladder graph.

calculate them.

Suppose that each edge of our graph is an electric conductor of unit conductance. Hook up a battery between the endpoints of any edge e, say the bottom rung (Fig. 8). Kirchhoff (1847) showed that the proportion of current that flows directly along e is then equal to the probability that e belongs to a randomly chosen spanning tree!

Let's review a little electric network theory before proceeding. (One source for more details is the book by Doyle and Snell (1984).) A battery produces a voltage difference across its two terminals, in this case, the endpoints of e. Let's say that the voltage of one

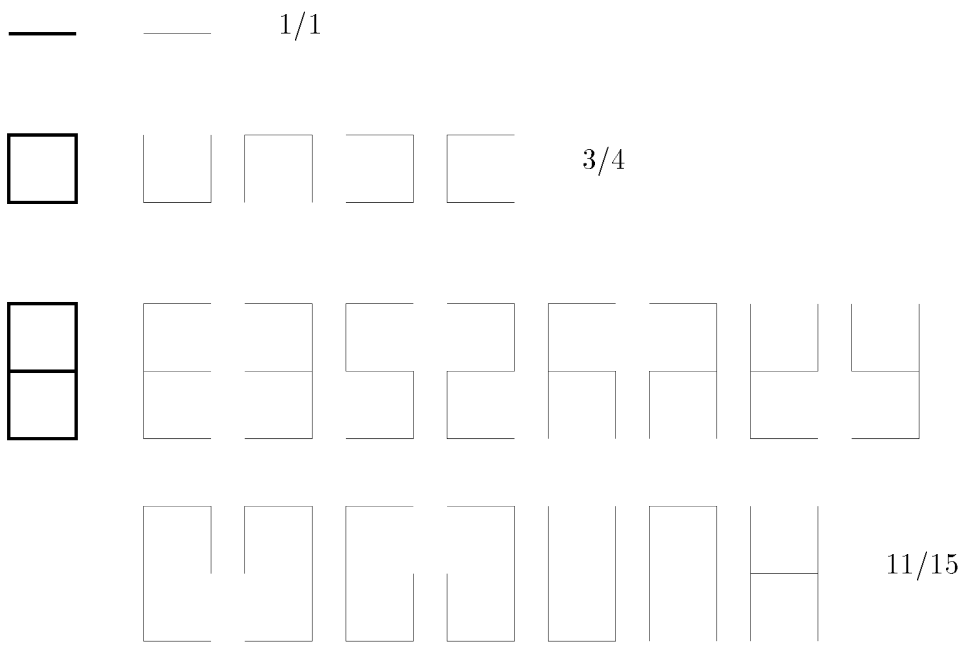

Figure 7. The ladder graphs of heights 0, 1, and 2, together with their spanning trees.

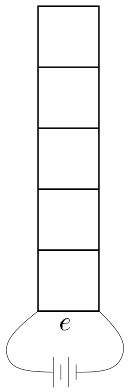

Figure 8. A battery is hooked up between the endpoints of e.

terminal is 0 and the other is V_0. The battery also establishes voltages $V(x)$ at all the other vertices x of the network. Current then flows in the various edges in the direction of decreasing voltage. More specifically, Ohm's Law says that if x is adjacent to y, then the current that flows on the edge $[x, y]$ in the direction from x to y equals $V(x) - V(y)$ times the conductance $C(x, y)$ of the edge $[x, y]$. We are initially starting with each edge having unit conductance, as we said, but we will consider modifications that lead to other conductances. At any vertex other than the terminals, the net current flow is 0 (charge cannot build up anywhere). Consonant with Ohm's Law, we define the effective conductance of the whole network between the two terminals of the battery as I/V_0, where I is the total amount of current flowing from one terminal to the other. Moreover, suppose that the network can be decomposed into two subnetworks that intersect only at a pair of vertices. Then the total current as well as the currents flowing in one subnetwork will not change if the other subnetwork is replaced by a single wire with conductance equal to the effective conductance of the replaced subnetwork. Furthermore, the total amount flowing through each of the subnetworks is proportional to their effective conductances.

Coming back to the ladder graph and its bottom rung, e, we see that current flows in two ways: some flows directly across e and some flows through the rest of the network. From the preceding paragraph, we see that the greater the effective conductance of the rest of the network, the less (proportionally) will flow directly across e. It is intuitively clear (and justified by Rayleigh's monotonicity principle) that the higher the ladder, the greater the effective conductance of the ladder minus the bottom rung, hence, by Kirchhoff's theorem, the less the chance that a random spanning tree contains the bottom rung. This confirms our observations. Now let's continue to the actual calculation of the probabilities.

Let C_n denote the effective conductance of the ladder minus the bottom rung when the ladder has n rungs. We can calculate C_{n+1} in terms of C_n by using the familiar series-parallel laws: namely, conductances in parallel add, while resistances in series add. Here, resistance is the reciprocal of conductance; edges are said to be in parallel when they have the same endpoints and in series when they share exactly one endpoint. As Fig. 9 shows, we obtain

$$C_{n+1} = \frac{1}{2 + \frac{1}{1+C_n}} .$$

According to Kirchhoff's theorem, it follows that the probability of the bottom rung being included is

$$\frac{1}{1+C_n} = \frac{1}{1 + \frac{1}{2+\frac{1}{1+C_{n-1}}}} = \frac{1}{1 + \frac{1}{2+\frac{1}{1+\frac{1}{2+\cdots}}}} ,$$

as claimed.

§3. Marginal Probabilities.

We have seen that Kirchhoff gave a useful and remarkable formula for the one-edge marginal distribution of a random spanning tree. What about the other marginals? In particular, given distinct edges $e_1, \ldots, e_k$, what is $\mathbf{P}[e_1, \ldots, e_k \in T]$, where T denotes a random spanning tree? This too has an electrical answer: Let $Y(e, f)$ be the amount of current flowing along the edge f when a battery is hooked up between the endpoints of e of such voltage that in the network as a whole, unit current flows from the tail of e to the head of e. (If $e = [u, v]$ is oriented from u to v, we call u the **tail** of e and v the **head** of e.) In this notation, Kirchhoff's theorem says that

$$\mathbf{P}[e \in T] = Y(e, e) . \tag{3.1}$$

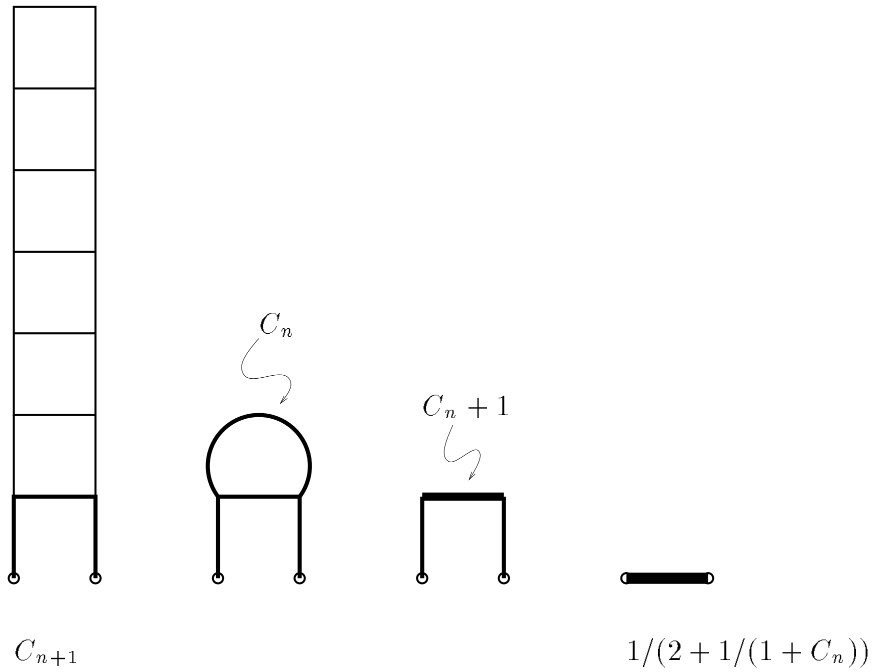

Figure 9. Calculating the effective conductance of a subnetwork.

Burton and Pemantle (1993) gave the following beautiful generalization:

$$\mathbf{P}[e_1, \ldots, e_k \in T] = \det\big[Y(e_i, e_j)\big]_{1 \le i,j \le k} . \tag{3.2}$$

This is called the Transfer Current Theorem. The case $k = 2$ appeared already in Brooks, Smith, Stone, and Tutte (1940). We sketch the ideas of the proof in BLPS (1997): Write

$$\begin{aligned}\mathbf{P}[e_1, \ldots, e_k \in T] = \mathbf{P}[e_1 \in T]\mathbf{P}[e_2 \in T \mid e_1 \in T]\mathbf{P}[e_3 \in T \mid e_1, e_2 \in T] \cdots \\ \cdots \mathbf{P}[e_k \in T \mid e_1, \ldots, e_{k-1} \in T] .\end{aligned} \tag{3.3}$$

Let $\mathbf{I}_i$ denote the current vector $Y(e_i, \bullet)$ in the Hilbert space $\ell^2(\mathsf{E})$, where E is the set of edges of the graph. It turns out that $Y(e_i, e_j)$ is the inner product of $\mathbf{I}_i$ and $\mathbf{I}_j$. Thus, the matrix in (3.2) is a Gram matrix. As such, its determinant is the square of the volume of the parallelepiped spanned by the vectors $\mathbf{I}_1, \ldots, \mathbf{I}_k$. The volume can be computed by multiplying successive altitudes. It turns out that the square of the altitude of $\mathbf{I}_i$ on the space spanned by $\mathbf{I}_1, \ldots, \mathbf{I}_{i-1}$ is exactly $\mathbf{P}[e_i \in T \mid e_1, \ldots, e_{i-1} \in T]$, so that (3.2) follows from (3.3).

§4. Generating a Random Spanning Tree.

Now that we've learned a little about the marginal probabilities of random spanning trees, let's go for the whole distribution. As we said, early algorithms for generating a random spanning tree used the Matrix-Tree Theorem, which says that the total number of spanning trees of a graph equals the determinant of the following matrix, $[a_{i,j}]_{1\le i,j\le n}$: Let the vertices of the graph be $v_0, \ldots, v_n$. For $1 \le i, j \le n$ only, let $a_{i,j}$ be -1 if v_i and v_j are adjacent, the degree of v_i if $i = j$, and 0 otherwise. (This is the same as the combinatorial Laplacian after the row and column corresponding to v_0 are removed.)

The earliest algorithms would compute the number of spanning trees containing a given edge, compute the total number of spanning trees in the graph, divide, and get the probability of putting that edge in the tree. After this random decision was made, this edge would be contracted if it were put in and deleted otherwise. Then the algorithm would proceed in the same way on the resulting smaller graph. As we said, better algorithms, especially for probabilists, were introduced by Aldous (1990) and Broder (1989) and then by Wilson (1996).

To describe Wilson's method, we define the important idea of loop erasure* of a path. If $\mathcal{P}$ is any finite path $\langle v_0, v_1, \ldots, v_l\rangle$ in G, we define the **loop erasure** of $\mathcal{P}$ by erasing cycles in $\mathcal{P}$ in the order they appear. That is, let $\mathcal{P}_0 := \mathcal{P}$. Proceed inductively to define $\mathcal{P}_k$ for $k \le l$ as follows. Given $\mathcal{P}_k = \langle u_0, \ldots, u_m\rangle$, we have two cases to consider:

(1) for some $j \le m$, we have $v_{k+1} = u_j \in \mathcal{P}_k$; in this case, let $\mathcal{P}_{k+1} := \langle u_0, \ldots, u_j\rangle$.

(2) otherwise, let $\mathcal{P}_{k+1} := \langle u_0, \ldots, u_j, v_{k+1}\rangle$.

The last path in this sequence, $\mathcal{P}_l$, is the loop erasure of $\mathcal{P}$. In case (1) above, the cycle $\langle u_j, u_{j+1}, \ldots, u_m, v_{k+1}\rangle$ also belongs to $\mathcal{P}$. We may alternatively consider this as a cycle of edges $\langle [u_j, u_{j+1}], \ldots, [u_m, v_{k+1}]\rangle$ that are removed from $\mathcal{P}$.

Now in order to generate a random spanning tree, first pick any vertex r to be the "root" of the tree. Then create a growing sequence of trees $T(i)$ $(i \ge 0)$ as follows. Let $T(0) := \{r\}$. Suppose that $T(i)$ is known. If $T(i)$ spans G, we are done. Otherwise, pick any vertex v that is not in $T(i)$ and start a simple random walk from v until it hits $T(i)$. Now create $T(i+1)$ by adding to $T(i)$ the loop erasure of this path from v to $T(i)$. The final tree in this growing sequence is the output of Wilson's algorithm (we simply forget the root). It has exactly the uniform distribution, no matter in what order we choose new starting vertices for loop-erased random walks!

* This ought to be called "cycle erasure", but the name is fixed.

How can this be? We give a brief sketch of the proof that Wilson's algorithm does indeed give the uniform distribution. However, our sketch will gloss over important subtleties. Now, when Wilson's algorithm finishes, we get a spanning tree T and a collection $C_1, \ldots, C_s$ of cycles erased by the algorithm. Consider the edges belonging to the tree or to any of the cycles (with multiplicity). The probability of the random walk having made the transitions it made along these edges is

$$\left(\prod_{v \neq r} \frac{1}{\deg v}\right)\left(\prod_{i=1}^{s} \prod_{u \in C_i} \frac{1}{\deg u}\right)$$

by the Markov property and the fact that all neighbors are equally likely transitions for simple random walk. Since we get two products, one involving only the tree and the other involving only the cycles, the tree is independent of the cycles and thus all trees have the same probability.

We now show how to use Wilson's algorithm to prove Kirchhoff's theorem. For those who are unfamiliar with the connection between electric networks and random walks, we present some of the basics first. (See Doyle and Snell (1984) for a superb elementary introduction.)

Consider the network shown in Fig. 10. Again, for simplicity, each edge has unit conductance. A battery joins two vertices, labelled 0 and 1 since it establishes voltages 0 and 1 at them, respectively. Note that the edge containing the battery is not part of the graph and, indeed, is shown only for physical intuition. Let x be any vertex. The fundamental relationship between electric networks and random walks is the equation

$$V(x) := \text{voltage at } x = \mathbf{P}_x[\text{visit 1 before 0}]\,. \tag{4.1}$$

Here, $\mathbf{P}_x$ means the probability measure for simple random walk on the graph started at x. (The random walk cannot pass through the battery.)

The reason that (4.1) holds, as we will see, is that both sides, when viewed as functions of x, are harmonic for $x \neq 0, 1$ and, clearly, have the same "boundary" values at 0 and 1*. Here, we say that a function is **harmonic** at x if its value at x is the average of the values of neighboring vertices. Furthermore, these discrete harmonic functions obey a uniqueness principle, just as in the continuous case. All this is easy to show:

Write $x \sim y$ to indicate that x and y are adjacent. As we saw, for $x \neq 0, 1$, the sum of the currents flowing out of x is 0:

$$0 = \sum_{y \sim x} [V(x) - V(y)] = (\deg x)\, V(x) - \sum_{y \sim x} V(y)\,,$$

* Thus, (4.1) is the discrete analogue of the solution of the Dirichlet problem via Brownian motion.

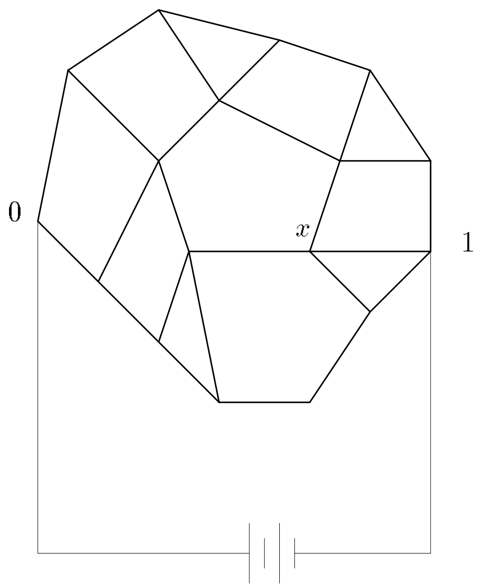

Figure 10. A battery connects vertices 0 and 1, establishing voltages 0 and 1, respectively.

i.e.,

$$V(x) = \frac{1}{\deg x} \sum_{y \sim x} V(y)\,.$$

This shows the harmonicity of the voltage function. For the hitting probabilities, we have (again, for $x \neq 0, 1$)

$$\begin{aligned} \mathbf{P}_x[\text{visit 1 before 0}] &= \sum_{y \sim x} \mathbf{P}_x[\text{first step is to } y] \cdot \mathbf{P}_y[\text{visit 1 before 0}] \\ &= \frac{1}{\deg x} \sum_{y \sim x} \mathbf{P}_y[\text{visit 1 before 0}]\,. \end{aligned}$$

Thus, the hitting probabilities are harmonic too.

Next, the Maximum Principle says that any function that is harmonic for $x \neq 0, 1$ and that achieves it maximum value at some $x \neq 0, 1$ is constant. (Here, as always, we are assuming that the network is connected.) Indeed, the values neighboring x must equal the value at x since the function is maximum there and is harmonic there. Thus, the maximum value propagates outward from x until we see that all values are the maximum.

From the Maximum Principle flows the Uniqueness Principle: Two functions that are harmonic for $x \neq 0, 1$ and that agree for $x = 0, 1$ are identically equal. The reasoning follows the usual course: subtract the two functions and apply the Maximum Principle.

This proves (4.1). Thus, we have a probabilistic interpretation of the voltage. There is also a probabilistic interpretation of the current: Use a battery of such voltage that a unit current flows from 1 to 0. Start a random walk at 1 and stop it when it reaches 0. Then

$$\mathbf{E}\big[\#\text{ traversals of }[x,y] - \#\text{ traversals of }[y,x]\big] = \text{amount of current flowing along }[x,y] \tag{4.2}$$

if $x \sim y$. It is not required that x, y be different from $0, 1$; indeed, we will use only the case when $x = 1$ and $y = 0$. We will not prove (4.2), but we use it now to establish Kirchhoff's theorem.

Let the edge e be $[x, y]$. To construct T, we use Wilson's algorithm rooted at y and started at x. (Recall that we can make these choices however we please.) As soon as this first random walk visits y, we will discover whether $e \in T$ without having to generate the rest of T; namely, $e \in T$ iff the random walk comes to y from x. Since for this random walk, $\big(\#\text{ traversals of }[x,y] - \#\text{ traversals of }[y,x]\big)$ is $1-0$ or $0-0$, the expectation in (4.2) is simply the probability that the random walk comes to y from x, and so equals $\mathbf{P}[e \in T]$.

§5. Infinite Graphs.

Wilson's method for generating spanning trees will also give a random spanning tree $T := \bigcup_{n=1}^{\infty} T(n)$ on any infinite but still recurrent graph G, provided, of course, we choose new starting vertices of the loop-erased random walks in such a way as to guarantee that T spans. How should we interpret the tree T? Suppose that G' is a finite subgraph of G and consider T_G and $T_{G'}$, the random spanning trees generated by Wilson's method on G and G', respectively. It is not hard to show that for any event B depending on only finitely many edges, we can make $\Big|\mathbf{P}[T_G \in B] - \mathbf{P}[T_{G'} \in B]\Big|$ arbitrarily small by choosing G' sufficiently large. Thus, it is reasonable to call the random spanning tree of G the **uniform spanning tree** of G. Also, this shows that Wilson's method on a recurrent graph generates a random spanning tree whose distribution again does not depend on the choice of root nor on the order in which we carry out the algorithm*.

Consider the special case $G := \mathbb{Z}^2$ (with the usual edge set). The degree of every vertex as seen in T is random, but identically distributed. As a first look at this distribution, let us find its mean. Now, the expected degree of $(0,0)$ equals the sum over all edges incident to $(0,0)$ of the probability that the edge belongs to T. By symmetry, these probabilities are all the same, so it suffices to find one of them and then multiply by 4. By taking limits, one can show that Kirchhoff's theorem (3.1) still holds. Let a unit current enter the

* This can also be shown directly.

network at $(0,0)$ and exit at $(1,0)$. We want to know how much travels directly along the edge e with these two endpoints. Here is the engineer's well-known solution: To restore symmetry, first put in a unit current at $(0,0)$ and take it out at infinity. One-quarter will flow directly along e. Second, put in unit current at infinity and take it out at $(1,0)$. Again, one-quarter will flow along e. Superposing these two currents, we obtain the current we desire and we see that one-half flows along e. Thus, the expected degree is $4 \times (1/2) = 2$.

This solution is very beautiful, but it suffers from some defects (of which the engineers are aware). The most prominent defect is that one cannot put current in at a vertex and take it out at infinity; one can do this on a network iff simple random walk on the graph is transient. Nevertheless, the answer is correct. Here are two ways to see this.

Since $\mathbb{Z}^2$ is self-dual, the "complement" $T^\dagger$ of T on the dual graph has the same law as T. By definition, $\mathbf{P}[e \in T] = \mathbf{P}[e^\dagger \notin T^\dagger]$; since, however, T and $T^\dagger$ have the same distribution in this case, we also have $\mathbf{P}[e \in T] = \mathbf{P}[e^\dagger \in T^\dagger]$. Therefore, $\mathbf{P}[e \in T] = 1/2$, as desired.

The second way to get this result is considerably more general. Let G be a graph. For a subset $\mathsf{V}' \subset \mathsf{V}$, let $\partial\mathsf{V}'$ be the set of edges that join a vertex in V' to a vertex not in V'. We say that G is **amenable** if there is an exhaustion V_n of V with

$$\lim_{n\to\infty} |\partial\mathsf{V}_n|/|\mathsf{V}_n| = 0\,.$$

Certainly $\mathbb{Z}^2$ is amenable. In fact, it is easy to show that the Cayley graph of every finitely generated abelian group is amenable by considering the balls V_n consisting of all vertices within distance n of the identity.

Now suppose that G is an amenable infinite graph as witnessed by the exhaustion $\langle \mathsf{V}_n \rangle$. We claim that the "average" degree of vertices in *any deterministic* spanning tree of G is 2. That is, if $\deg_T(v)$ denotes the degree of v in a spanning tree T of G, then

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{v\in\mathsf{V}_n} \deg_T(v) = 2\,.$$

The reader may easily deduce this from the facts that the number of components of $T \cap G_n$ is at most $|\partial\mathsf{V}_n|$ and a tree with k vertices has $k-1$ edges, where G_n is the maximal subgraph of G containing V_n. Thus, from the Bounded Convergence Theorem, we deduce that when a spanning tree is chosen at random,

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{v\in\mathsf{V}_n} \mathbf{E}[\deg_T(v)] = 2\,.$$

In particular, if $G = (\mathsf{V}, \mathsf{E})$ is also **transitive**, such as $\mathbb{Z}^2$, meaning that for every pair of vertices u and v, there is a bijection of V with itself that preserves adjacency and takes

u to v, then every vertex has expected degree 2. (Compare Theorem 3.2 in Thomassen (1990).)

What else can we say about the distribution of the degree of a vertex in $\mathbb{Z}^2$? Again by taking limits, one may show that the Transfer Current Theorem (3.2) holds. Luckily, there are methods for calculating the entries of the transfer current matrix. One gets, for example, that the probability that the degree of $(0,0)$ is 2 is the interesting number

$$\frac{4}{\pi}\left(2-\frac{9}{\pi}+\frac{12}{\pi^2}\right)=0.45^- ;$$

see Burton and Pemantle (1993).

Is there an analogue of the uniform spanning tree on a transient graph? If so, what does it look like on $\mathbb{Z}^d$ for $d \geq 3$?

It turns out that there is: as we mentioned, Pemantle (1991) showed that on any graph G, if $\langle G_n \rangle$ is an exhaustion of G, the weak limit of the uniform spanning tree measures on G_n exists and is unique, regardless of the exhaustion. This is the free uniform spanning forest, FUSF. The existence of the limit is derived from Rayleigh's monotonicity principle. What does the limit look like? First of all, the random subgraph it produces cannot have any cycles: any cycle in G is part of G_n for all sufficiently large n, but T_{G_n} has no cycles a.s. Since there are only a countable number of possible cycles, there are none a.s. in the limit. However, we cannot argue in the same way that T_G is connected. Perhaps the connections in T_{G_n} between two fixed vertices get longer and longer as $n \to \infty$; their distributions may not be tight. Indeed, this can happen: Pemantle (1991) showed that one gets a tree on $\mathbb{Z}^d$ iff $d \leq 4$. We will explain what is behind this stunning result shortly.

As we saw, there is another possibility for taking limits. In disregarding the complement of G_n, we are disregarding the possibility that a spanning tree or forest of G may connect the boundary vertices of G_n in ways that would affect the possible connections within G_n itself. An alternative approach takes the opposite view and forces all connections outside of G_n: Let G_n^W be the graph obtained from G by identifying all the vertices in G_n that are adjacent to a vertex outside G_n to a single vertex. Let μ_n^W be the random spanning tree measure on G_n^W. Again, Rayleigh's monotonicity principle proves the existence and uniqueness of a limiting probability measure μ^W, the wired uniform spanning forest, WUSF. The term "wired" comes from thinking of G_n^W as having its boundary wired together. In statistical mechanics, especially the random cluster model (see, e.g., Grimmett 1995), measures on infinite configurations are also defined using limiting procedures similar to those employed here to define FUSF and WUSF. One needs to specify boundary conditions, just as we did. The terms "free" and "wired" have analogous uses there. On a recurrent graph, it is not hard to see that $\mathsf{FUSF} = \mathsf{WUSF}$.

It turns out that the wired uniform spanning forest can be constructed by an analogue of Wilson's method: Let G be a *transient* graph. Run a loop-erased random walk from any vertex for infinite time; this is defined by transience. Then run an independent random walk from another vertex until it encounters the first path or, if it doesn't, for infinite time. Add its loop erasure to the first path. Then run another independent random walk from any vertex not already included in the union of the first two paths until it encounters the previous paths or else for infinite time, add its loop erasure, and so on. Let T be the (infinite) union of the loop-erased paths. One may show that the resulting distribution on spanning forests T is independent of the order in which we choose starting vertices, as long as the order is such as to guarantee that all vertices are eventually chosen, i.e., so that we get a spanning forest. We refer to this method of generating a random spanning forest as **Wilson's method rooted at infinity**. The good news is that it gives the wired uniform spanning forest.

Because of Wilson's method rooted at infinity, it is much easier to analyze the wired spanning forest than the free. Indeed, it follows immediately that the wired spanning forest is a tree a.s. iff random walk from any vertex x and independent loop-erased random walk from any other vertex y will intersect with probability 1. Lawler (1991) provided detailed estimates concerning intersections of this type on $\mathbb{Z}^d$; in particular, we obtain Pemantle's theorem that the uniform spanning forest on $\mathbb{Z}^d$ is a tree iff $d \leq 4$. Actually, there is one more step we can take that makes Pemantle's theorem still easier: Lyons, Peres, and Schramm (1998) showed that on any graph, random walk from any vertex x and independent loop-erased random walk from any other vertex y will intersect with probability 1 iff random walk from any vertex x and independent random walk from any other vertex y will intersect with probability 1. The latter property is much easier to analyze and quite standard. Pemantle (1991) also showed that when $d \geq 5$, the uniform spanning forest has infinitely many trees a.s. For other graphs, the number of trees is either infinite a.s. or equal a.s. to the dimension of the space of bounded harmonic functions; see BLPS (1997) for the full story.

§6. Ends of Trees.

We now ask about the topology of the trees of a random spanning forest. Call an infinite path in a tree that starts at any vertex and does not backtrack a **ray**. Call two rays **equivalent** if they have infinitely many vertices in common. An equivalence class of rays is called an **end**. How many ends do the trees of a random spanning forest have?

Let's begin by thinking about the case of the wired uniform spanning forest on a regular tree of degree $d+1$. Choose a vertex, o. Begin Wilson's method rooted at infinity from o. We obtain a ray ξ from o to start our forest. Now o has d other neighbors, $x_1, \ldots, x_d$. By beginning random walks at each of them in turn, we see that the events $A_i := \{x_i \text{ connected to } o\}$ are independent. Furthermore, it is easy to calculate that the probability that random walk started at x_i ever hits o is $1/d$. This is the probability of A_i. On the event A_i, we add only the edge (o, x_i) to the forest and then we repeat the analysis from x_i. Thus, the tree containing o includes, apart from the ray ξ, a critical Galton-Watson tree with binomial offspring distribution $(d, 1/d)$. In addition, each vertex on ξ has another random subtree attached to it; its first generation has the binomial distribution $(d-1, 1/d)$, but subsequent generations yield Galton-Watson trees with binomial distribution $(d, 1/d)$. In particular, a.s. every tree added to ξ is finite. This means that the tree containing o has only one end, the equivalence class of ξ. This analysis is easily extended to form a complete description of the entire wired spanning forest. The resulting description is due to Häggström (1997), whose work predates Wilson's algorithm (despite the publication dates) and, thus, was not this easy.

Now in general, we see that if we begin Wilson's algorithm at a vertex o in a graph, it immediately generates one end of the tree containing o. In order for this tree to have more than one end, however, we need a succession of "coincidences" to occur, building up other ends by gradually adding on finite pieces. This is possible, but it suggests that usually the wired spanning forest has trees with only one end each. Indeed, this is very often the case.

First, consider the planar recurrent case, which has an amazingly simple analysis. Let G be a proper planar graph whose planar dual $G^\dagger$ is locally finite. Given a finite connected subgraph G_n of G with connected complement $G \setminus G_n$, let $G_n^\dagger$ be its planar dual. Notice that $G_n^\dagger$ can be regarded as a finite subgraph of $G^\dagger$, but with the outer boundary vertices identified to a single vertex. Spanning trees T of G_n are in one-to-one correspondence with "complementary" spanning trees $T^\dagger$ of $G_n^\dagger$ as in (2.1). Therefore, the FUSF of G is "dual" to the WUSF of $G^\dagger$; that is, the relation (2.1) transforms one to the other. In fact, we can couple (generate simultaneously) the two spanning forests by (2.1).

Theorem 6.1. *Suppose that G is a proper recurrent planar graph and $G^\dagger$ its planar dual.*

If $G^\dagger$ is locally finite and recurrent, then the uniform spanning tree on G has only one end a.s.

Proof. Because G is recurrent, its free and wired spanning forests coincide and give a single tree a.s. Likewise for $G^\dagger$. If T had at least two ends, then the bi-infinite path in T joining two of them would separate $T^\dagger$ into at least two trees. But since $T^\dagger$ is a single tree, this is impossible. ∎

Remark 6.2. Loop-erased random walk as a path indexed by $\mathbb{N}$ is obviously defined in any transient graph. It is not well defined on recurrent graphs in general, but this theorem provides a simple way to define loop-erased random walk from any vertex x on graphs G like $\mathbb{Z}^2$ that satisfy the hypotheses of Theorem 6.1: just take the ray from x in the spanning tree. A detailed analysis of this path on transitive graphs like $\mathbb{Z}^2$ is given by Lawler (1991).

Similar reasoning shows:

Proposition 6.3. *Suppose that G is a proper planar graph and $G^\dagger$ its planar dual. Assume that $G^\dagger$ is locally finite. If each tree of the* WUSF *of G has only one end a.s., then the* FUSF *of $G^\dagger$ has only one tree a.s. If, in addition, the* WUSF *of G has infinitely many trees a.s., then the tree of the* FUSF *of $G^\dagger$ has infinitely many ends a.s.*

Going beyond the planar setting, Pemantle (1991) showed that on $\mathbb{Z}^d$, the uniform spanning forest has one end when $d \le 4$ and at most two ends per tree when $d \ge 5$. This result was sharpened and extended in BLPS (1997):

Theorem 6.4. *If G is a transient Cayley graph, then* WUSF*-a.s., every tree has only one end.*

Combining this with Theorem 6.1, one can show that all Cayley graphs of groups that are not finite extensions of $\mathbb{Z}$ have the property that WUSF-a.s., every tree has only one end; see BLPS (1997).

By contrast, we have:

Proposition 6.5. *If G is a transitive graph with* WUSF $\neq$ FUSF*, then* FUSF*-a.s., there is a tree with infinitely many ends.*

§7. Equality of Free and Wired Spanning Forests.

Let us return to the question of when $\mathsf{FUSF} = \mathsf{WUSF}$. In all cases, there is a simple inequality between these two probability measures, namely,

$$\forall e \in \mathsf{E} \quad \mathsf{FUSF}(e \in T) \geq \mathsf{WUSF}(e \in T) \tag{7.1}$$

by Rayleigh's monotonicity principle. In fact, it turns out that FUSF stochastically dominates WUSF by a theorem of Feder and Mihail (1992), whence by Strassen's (1965) theorem, FUSF and WUSF can be **coupled** in the sense that there is a probability measure on the set

$$\Big\{(T(1), T(2))\,;\ T(i) \text{ is a spanning forest of } G \text{ and } T(1) \subseteq T(2)\Big\}$$

that projects in the first coordinate to WUSF and in the second to FUSF.

This allows us to deduce the following result (compare Häggström 1995):

PROPOSITION 7.1. *If* $\mathbf{E}[\deg_T(v)]$ *is the same under* FUSF *and* WUSF *for every* $v \in \mathsf{V}$ *or if equality holds in (7.1), then* $\mathsf{FUSF} = \mathsf{WUSF}$.

Proof. The coupling shows that $\mathbf{E}[\deg_T(v)]$ under FUSF is at least the expectation under WUSF, with equality for all $v \in \mathsf{V}$ iff $\mathsf{FUSF} = \mathsf{WUSF}$. Since equality in (7.1) implies equality of expected degrees, the second claim follows as well. ∎

We will now see that $\mathsf{FUSF} = \mathsf{WUSF}$ for many graphs, including Cayley graphs of abelian groups such as $\mathbb{Z}^d$.

Let G be an amenable infinite graph as witnessed by the sequence $\langle \mathsf{V}_n \rangle$ (see Section 5). Let T be any deterministic spanning forest all of whose components (trees) are infinite. If k_n denotes the number of trees of $T \cap G$, then $k_n = o(|\mathsf{V}_n|)$, where G_n is the maximal subgraph of G containing V_n. Thus, the average degree of vertices is 2:

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{v \in \mathsf{V}_n} \deg_T(v) = 2\,.$$

In particular, if T is random, then

$$\lim_{n\to\infty} |\mathsf{V}_n|^{-1} \sum_{v \in \mathsf{V}_n} \mathbf{E}[\deg_T(v)] = 2\,. \tag{7.2}$$

Thus, we obtain the following, essentially due to Häggström (1995):

COROLLARY 7.2. *On any transitive amenable graph,* $\mathsf{FUSF} = \mathsf{WUSF}$.

Proof. By transitivity and (7.2), $\mathbf{E}[\deg_T(v)] = 2$ for every $v \in \mathsf{V}$ and for both FUSF and WUSF. ∎

In order to continue our analysis, we reformulate the electrical laws for finite graphs in terms of finite-dimensional Hilbert space.

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite graph. For purposes of the electrical laws, we regard each edge $e \in \mathsf{E}$ as occurring with both possible orientations; denote the edge opposite to $e = [u, v]$ by $\check{e} := [v, u]$. Since we are interested in flows on E, it is natural to consider that what flows on e is the negative of what flows on $\check{e}$. Thus, define $\ell^2_-(\mathsf{E})$ to be the space of **antisymmetric** functions θ on E (i.e., $\theta(\check{e}) = -\theta(e)$ for each edge e) with inner product

$$(\theta, \theta') := \frac{1}{2} \sum_{e \in \mathsf{E}} \theta(e)\theta'(e) = \sum_{e \in \mathsf{E}_{1/2}} \theta(e)\theta'(e),$$

where $\mathsf{E}_{1/2} \subset \mathsf{E}$ is a set of edges containing exactly one of each pair $e, \check{e}$.

We may express Kirchhoff's laws as follows. Let $\chi^e := \mathbf{1}_e - \mathbf{1}_{\check{e}}$ denote the unit flow along e represented as an antisymmetric function in $\ell^2_-(\mathsf{E})$. A function $I \in \ell^2_-(\mathsf{E})$ is called a **current** if it satisfies the following two properties:

Kirchhoff's Node Law: For any vertex v,

$$\Big(\sum_{w \sim v} \chi^{[v,w]}, I\Big) = 0$$

except if a battery is connected at v, in which case the inner product is called the amount of current flowing into the network at v.

Kirchhoff's Cycle Law: If $e_1, e_2, \ldots, e_n$ is a directed cycle in G, then

$$\Big(\sum_{i=1}^{n} \chi^{e_i}, I\Big) = 0\,.$$

Now let $\bigstar$ denote the subspace in $\ell^2_-(\mathsf{E})$ spanned by the **stars** $\sum_{w \sim v} \chi^{[v,w]}$ $(v \in \mathsf{V})$ and let $\diamondsuit$ denote the subspace spanned by the **cycles** $\sum_{i=1}^{n} \chi^{e_i}$ ($e_1, \ldots, e_n$ forming a directed cycle in G). These subspaces are clearly orthogonal. Moreover, the sum of $\bigstar$ and $\diamondsuit$ is all of $\ell^2_-(\mathsf{E})$: suppose that $\theta \in \ell^2_-(\mathsf{E})$ is orthogonal to both $\bigstar$ and $\diamondsuit$. Since θ is orthogonal to $\diamondsuit$, there is a function F such that $\theta = \nabla F$, where $\nabla F([u, v]) := F(u) - F(v)$ is called the **gradient** of F. Since θ is orthogonal to $\bigstar$, the function F is harmonic at every vertex. Therefore F is constant on G, whence $\theta = 0$, as desired.

Let I^e be the unit current from the tail of e to the head of e; this means that a unit total of current flows in the network, not that the flow directly across e is unit. According to Kirchhoff's Cycle Law, I^e is orthogonal to $\diamondsuit$, whence in $\bigstar$. Now $\chi^e - I^e$ is a sourceless flow, i.e., is orthogonal to $\bigstar$, and is thus an element of $\diamondsuit$. Therefore, the expression

$$\chi^e = I^e + (\chi^e - I^e)$$

is the orthogonal decomposition of χ^e relative to $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$. In particular,

$$I^e = P_\bigstar \chi^e ,$$

where $P_\bigstar$ denotes the orthogonal projection onto $\bigstar$.

Recall that $\mathsf{E}_{1/2} \subset \mathsf{E}$ is a set of edges containing exactly one of each pair $e, \check{e}$. What is the matrix of $P_\bigstar$ in the orthogonal basis $\{\chi^e \,;\; e \in \mathsf{E}_{1/2}\}$? We have

$$(P_\bigstar \chi^e, \chi^f) = (I^e, \chi^f) = I^e(f) = Y(e, f) .$$

In other words, the matrix coefficient at (e, f) is our familiar friend, the corresponding entry of the transfer current matrix! Since $P_\bigstar$ is self-adjoint, the transfer current matrix is symmetric*. Therefore $Y(e, f) = Y(f, e)$. (This is known as the reciprocity law.)

We now move to infinite graphs. There are two natural ways of defining currents between vertices of an infinite graph, corresponding to the two ways of defining uniform spanning forests. We will define these currents using Hilbert space; then we indicate how they correspond to limits of currents on finite subgraphs.

Considering now an infinite graph $G = (\mathsf{V}, \mathsf{E})$, we must redefine $\bigstar$ and $\diamondsuit$. Let $\bigstar$ denote the closure of the linear span of the stars and $\diamondsuit$ the closure of the linear span of the cycles of G. Since every star and every cycle are orthogonal, it is still true that $\bigstar \perp \diamondsuit$. However, it is no longer necessarily the case that $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$; in fact, we are about to see that this is equivalent to $\mathsf{FUSF} = \mathsf{WUSF}$ on G.

Define the space of **Dirichlet** functions

$$\mathbf{D} := \{F \,;\; \nabla F \in \ell^2_-(\mathsf{E})\} .$$

Our previous reasoning for finite graphs now shows that an element $\theta \in (\bigstar \oplus \diamondsuit)^\perp$, the orthocomplement of $\bigstar \oplus \diamondsuit$, is the gradient of a harmonic function $F \in \mathbf{D}$. Thus, if $\mathbf{HD}$ denotes the set of $F \in \mathbf{D}$ that are harmonic, we obtain the orthogonal decomposition

$$\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit \oplus \nabla \mathbf{HD} . \tag{7.3}$$

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* Warning: when the conductances are not unit, this property does not hold; instead, the so-called transfer impedance matrix is the matrix of $P_\bigstar$ in the appropriate basis and so is symmetric. Nevertheless, it is the transfer current matrix that gives the marginals for the weighted random spanning trees corresponding to non-unit conductances.

Let $P_{\diamondsuit}^{\perp}$ denote the orthogonal projection onto $\diamondsuit^{\perp}$. We define two possibly different currents,

$$I_F^e := P_{\diamondsuit}^{\perp}\chi^e \,,$$

the **free current** between the endpoints of e (also called the "limit current"), and

$$I_W^e := P_{\bigstar}\chi^e \,,$$

the **wired current** between the endpoints of e (also called the "minimal current"). The names for these currents are explained by the following two well-known propositions. (See, e.g., Soardi (1994), Cor. 3.17 and Thm. 3.25.)

PROPOSITION 7.3. *Let G be an infinite graph exhausted by subgraphs $\langle G_n \rangle$. Let e be an edge in G_1 and I_n be the unit current flow in G_n from the tail of e to the head of e. Then $\|I_n - I_F^e\| \to 0$ as $n \to \infty$.*

PROPOSITION 7.4. *Let G be an infinite graph exhausted by subgraphs $\langle G_n \rangle$. Let G_n^W be formed by identifying the boundary of G_n to a single vertex. Let e be an edge in G_1 and I_n be the unit current flow in G_n^W from the tail of e to the head of e. Then $\|I_n - I_W^e\| \to 0$ as $n \to \infty$.*

We now put this all together:

THEOREM 7.5. *For any graph, $\mathsf{FUSF} = \mathsf{WUSF}$ iff $I_W^e = I_F^e$ for every edge e iff $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$ iff $\mathbf{HD} = \mathbb{R}$.*

Proof. From Kirchhoff's theorem (3.1) and Propositions 7.3 and 7.4, we obtain the extensions of Kirchhoff's theorem to infinite graphs: $\mathsf{FUSF}(e \in T) = I_F^e(e)$ and $\mathsf{WUSF}(e \in T) = I_W^e(e)$. Now use Proposition 7.1 to deduce the first equivalence.

For the second equivalence, note that

$$I_F^e(e) = (I_F^e, \chi^e) = (P_{\diamondsuit}^{\perp}\chi^e, \chi^e) = (P_{\diamondsuit}^{\perp}\chi^e, P_{\diamondsuit}^{\perp}\chi^e) = \|P_{\diamondsuit}^{\perp}\chi^e\|^2 \,,$$

where we have used the fact that $P_{\diamondsuit}^{\perp}$ is its own square and self-adjoint. Likewise, $I_W^e(e) = \|P_{\bigstar}\chi^e\|^2$. Since $\diamondsuit^{\perp} \supseteq \bigstar$, it follows that $I_F^e(e) = I_W^e(e)$ iff $I_F^e = I_W^e$. Now $\ell^2_-(\mathsf{E}) = \bigstar \oplus \diamondsuit$ is equivalent to $P_{\bigstar} = P_{\diamondsuit}^{\perp}$. Since $\{\chi^e \,;\; e \in \mathsf{E}_{1/2}\}$ is a basis for $\ell^2_-(\mathsf{E})$, this is also equivalent to $P_{\bigstar}\chi^e = P_{\diamondsuit}^{\perp}\chi^e$ (i.e., $I_F^e = I_W^e$) for all edges e, as desired.

The third equivalence follows from (7.3). ∎

This result incorporates Doyle's (1988) theorem on uniqueness of currents. Combining it with Corollary 7.2, we see that every transitive amenable graph has no nonconstant harmonic Dirichlet functions. This result is due to Medolla and Soardi (1995). See Benjamini, Lyons and Schramm (1997) for more applications of Theorem 7.5 to the study of harmonic Dirichlet functions.

If one graph is "similar" to another, how similar must their spanning forest measures be? For example, if $\mathsf{FUSF} = \mathsf{WUSF}$ on one graph, must this be the case on the other? To make the notion of "similar" precise, given two graphs $G = (\mathsf{V}, \mathsf{E})$ and $G' = (\mathsf{V}', \mathsf{E}')$, call a function $\phi : \mathsf{V} \to \mathsf{V}'$ a **rough isometry** if there are positive constants a and b such that for all $u, v \in \mathsf{V}$,

$$a^{-1}d(u,v) - b \le d'(\phi(u), \phi(v)) \le ad(u,v) + b \tag{7.4}$$

and such that every vertex in G' is within distance b of the image of V. Here, d and d' denote the usual graph distances on G and G'. Being roughly isometric is an equivalence relation. Soardi (1993) proved:

THEOREM 7.6. *Let G and G' be roughly isometric graphs with bounded degrees. Then G has nonconstant harmonic Dirichlet functions iff G' does.*

For a short proof found by Schramm, see Lyons and Peres (1997).

Thus, if G and G' are roughly isometric graphs, then the free and wired spanning forests coincide on one iff they do on the other. However, this does not mean that the geometries of the forests cannot change. Indeed, rough isometries can change the wired spanning forest from a single tree to infinitely many trees: see the example of Theorem 3.5 in Benjamini and Schramm (1996a). On the other hand, changing the generators in a Cayley graph cannot have this effect; see BLPS (1997).

Pemantle (1991) showed that the tail σ-field of the uniform spanning forest is trivial on $\mathbb{Z}^d$; his proof shows that this is a consequence of the fact that $\mathsf{FUSF} = \mathsf{WUSF}$. R. Solomyak (1998) showed a directional tail triviality of the free spanning forest on the Cayley graphs of Fuchsian groups (with the standard generators) that are *not* co-compact. Finally, using ideas about projections in Hilbert space, BLPS (1997) showed that the tail σ-fields of both the free and the wired spanning forest on *every* graph is trivial.

§8. Size and Distance of Trees.

How "close" is the WUSF on a Cayley graph G to being a tree? That is, how close is the forest to being connected? One answer comes from asking how "many" edges need to be added to connect all the trees. For example, let T be the WUSF, $0 < \epsilon < 1$, and let T_ϵ be gotten by adding to T each edge of G independently with probability ϵ. Then BLPS (1997) showed that T_ϵ is connected a.s. for all $\epsilon > 0$ iff G is amenable. Of course, since all the trees in the WUSF have only one end, were we to subtract, rather than add, edges with probability ϵ, we would obtain only finite trees a.s., no matter what $\epsilon > 0$ we used. Comparing with Bernoulli percolation, we are led to view the WUSF as a "critical" model, like percolation at p_c, the critical value for Bernoulli percolation.

Another approach to measuring how far the WUSF is from being a tree is to ask how far the trees in it are from each other. The following beautiful extension of Pemantle's (1991) theorem is due to Benjamini, Kesten, Peres, and Schramm (1998). To express the result, we use the natural metric induced from the graph structure after randomly modifying the graph.

THEOREM 8.1. *Contract the edges of each tree in the uniform spanning forest on $\mathbb{Z}^d$. The diameter of the resulting graph is a.s. $\lfloor (d-1)/4 \rfloor$.*

§9. Planar Graphs and Hyperbolic Lattices.

Let G be a graph of (uniformly) bounded degree that is roughly isometric to $\mathbb{H}^d$, hyperbolic space of dimension d. There is an interesting phase transition between dimensions 2 and 3 in hyperbolic space: there are nonconstant harmonic Dirichlet functions on G iff $d = 2$. In fact, a general result for planar graphs of Benjamini and Schramm (1996a,b) says

LEMMA 9.1. *Suppose that G is a transient planar graph with bounded degree. Then* $\mathbf{HD} \neq \mathbb{R}$.

That graphs roughly isometric to $\mathbb{H}^d$ ($d \geq 2$) are transient follows from Kanai's (1986) theorem showing that transience is preserved under rough isometries. Thus, by Lemma 9.1, they have nonconstant harmonic Dirichlet functions when $d = 2$. However, the situation is reversed in higher dimensions:

LEMMA 9.2. *If G is roughly isometric to $\mathbb{H}^d$ for some $d \geq 3$ and has bounded degree, then* $\mathbf{HD} = \mathbb{R}$ *on G.*

Proof. For $d \geq 3$, there are no nonconstant harmonic Dirichlet functions in $\mathbb{H}^d$; see Sario *et al.* (1977). This transfers to G by a theorem of Holopainen and Soardi (1998). ∎

Taking stock, we arrive at the following surprising conclusions:

THEOREM 9.3. *If G is a self-dual proper planar Cayley graph roughly isometric to $\mathbb{H}^2$, then the* WUSF *of G has infinitely many trees a.s., each having one end a.s., while the* FUSF *of G has one tree a.s. with infinitely many ends a.s. If G is a Cayley graph roughly isometric to $\mathbb{H}^d$ for $d \geq 3$, then the* WUSF = FUSF *of G has infinitely many trees a.s., each having one end a.s.*

Proof. From Wilson's method rooted at infinity, it follows that there are infinitely many trees in the WUSF; see BLPS (1997). Use Kanai's (1986) theorem, Theorem 6.4, and Proposition 6.3 to finish the first part. Use Lemma 9.2 and Propositions 7.5 and 6.3 for the second part. ∎

An example of a self-dual planar Cayley graph roughly isometric to $\mathbb{H}^2$ was shown in Fig. 5.

Here is a summary of the phase transitions. It is quite surprising that the free spanning forest undergoes three phase transitions as the growth rate increases. (Volume of balls in hyperbolic space grows exponentially in the radius, but only polynomially in euclidean space.)

	$\mathbb{Z}^d$		$\mathbb{H}^d$	
d	$2-4$	≥ 5	2	≥ 3
FUSF: trees	1	∞	1	∞
ends	1	1	∞	1
WUSF: trees	1	∞	∞	∞
ends	1	1	1	1

§10. Open Questions.

There are an enormous number of open questions related to uniform spanning forests. The connections to different areas continue to expand; it seems to be an excellent field for a young probabilist to enter. We will give a sample of some interesting questions here.

Question. Let G be a proper transient planar graph with bounded degree and a bounded number of sides to its faces. Is the free spanning forest a single tree a.s.? If true, this would strengthen Lemma 9.1 of Benjamini and Schramm (1996a,b), which says that FUSF $\neq$ WUSF. It follows from their work that a.s. two independent random walks intersect only finitely many times, which implies that the WUSF has infinitely many trees a.s.

Question. (This question was originally asked independently by Benjamini and Propp; see Propp (1997).) One may consider the uniform spanning tree on $\mathbb{Z}^2$ embedded in $\mathbb{R}^2$. In fact, consider it on $\epsilon\mathbb{Z}^2$ in $\mathbb{R}^2$ and let $\epsilon \to 0$. Is there some sort of limit? Does it have some sort of conformal invariance? This could be expected on the basis of conformal invariance of simple random walk. It is also supported by some simulations (Schramm, personal communication). In addition, the uniform spanning tree is intimately tied to random domino tiling of $\mathbb{Z}^2$: see, e.g., Burton and Pemantle (1993). New work of Rick Kenyon (1997) shows that domino tiling has a strong sort of conformal invariance. Finally, critical Bernoulli percolation is believed to have conformal invariance in the limit (see, e.g., Langlands, Pouliot and Saint-Aubin (1994) and Benjamini and Schramm (1996c)) and the uniform spanning tree is a sort of critical model, as explained in Section 8.

Question. Tóth and Werner (1997), §11, explain the connection between certain self-repelling random walks on $\mathbb{Z}$ and a Markov chain on $\mathbb{Z}^2$ that never visits a state more than once. They consider coalescing paths of this Markov chain; these paths form the wired random spanning tree of $\mathbb{Z}^2$ that is associated to this chain by Wilson's method rooted at infinity. The dual of this tree is considered, as well as the lattice-filling curve that "threads" between the two trees; see Fig. 11 for the lattice-filling curve of a uniform spanning tree. Most of their paper is devoted to analyzing the continuous analogue of these objects. In particular, a stochastic differential equation describes the space-filling curve. Note that the lattice-filling curve completely determines the tree and its dual. Can one give a stochastic differential equation for the space-filling curve that presumably is the limit as $\epsilon \to 0$ of the lattice-filling curves threading between the uniform spanning tree and its dual on $\epsilon\mathbb{Z}^2$?

Question. Is each tree of the wired uniform spanning forest on any graph recurrent a.s.? For each tree, is $p_c = 1$ a.s.?

Question. If a graph has positive isoperimetric constant, must each tree in its wired uniform spanning forest have one end a.s.?

Question. Does each tree in the wired uniform spanning forest on Galton-Watson trees have one end a.s.?

Question. Is there a "natural" coupling of FUSF and WUSF? For example, if G is a Cayley graph, is there a coupling that is invariant under multiplication by elements of G?

Question. Under the hypotheses of Proposition 6.5, does every tree have infinitely many ends FUSF-a.s.?

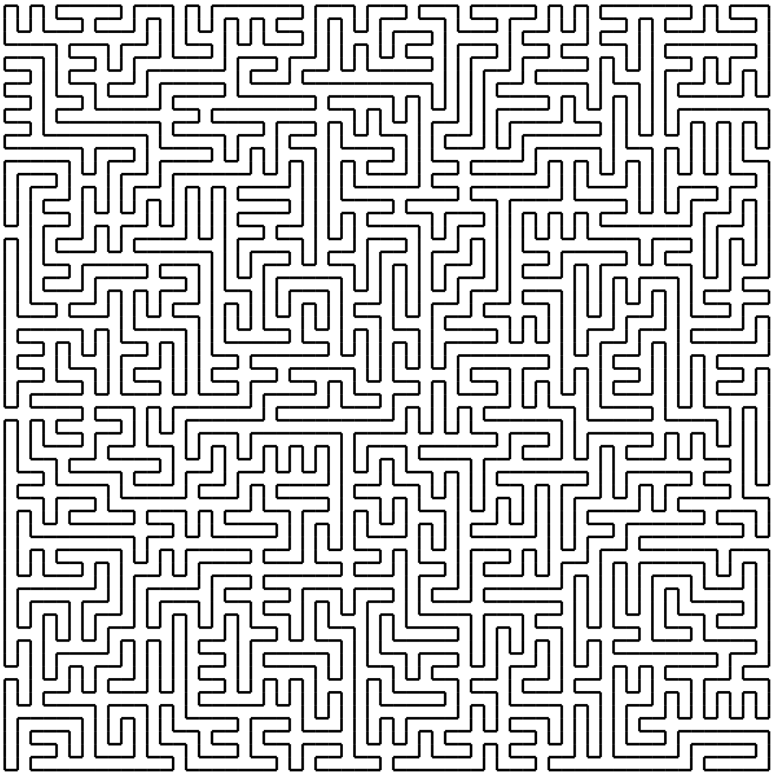

Figure 11. The inside of this lattice-filling curve drawn by Schramm is a uniform spanning tree on a square grid, while the outside is another on the dual grid (wired).

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