Singularity of Some Random Continued Fractions

by Russell Lyons

Abstract. We prove singularity of some distributions of random continued fractions that correspond to iterated function systems with overlap and a parabolic point. These arose while studying the conductance of Galton-Watson trees.

§1. Introduction.

We are interested in the distributions of certain random continued fractions. They arose while studying the following question about Galton-Watson trees: Suppose that $p_k \in [0, 1)$ for $k \geq 1$ with $\sum_k p_k = 1$. Let T be the random genealogical tree resulting from the associated Galton-Watson process beginning with one individual, ρ, where each individual has k children with probability p_k. Adjoin a parent Δ to ρ. Then the probability γ that simple random walk starting at Δ will return to Δ is equal to the effective conductance of $T \cup \{\Delta\}$ from Δ to infinity when each edge has unit conductance (see, e.g., Doyle and Snell (1984) or Lyons and Peres (1998)). This quantity γ enters, for example, in calculations of the Hausdorff dimension of harmonic measure on the boundary of T (see Lyons, Pemantle and Peres (1995)). Thus, it is of interest to calculate the distribution of γ. If F_γ denotes its cumulative distribution function, then F_γ satisfies the equation

$$F(s) = \begin{cases} \sum_k p_k F^{*k}\left(\dfrac{s}{1-s}\right), & \text{if } s \in (0,1); \\ 0, & \text{if } s \leq 0; \\ 1, & \text{if } s \geq 1\ . \end{cases} \tag{1.1}$$

Moreover, the functional equation (1.1) has exactly two solutions, F_γ and the Heaviside function $\mathbf{1}_{[0,\infty)}$ (see Lyons, Pemantle and Peres (1997)). Only a numerical computation of F_γ is known. In all cases in which this calculation was carried out, it appeared that the

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distribution was absolutely continuous with respect to Lebesgue measure with bounded density. Furthermore, unpublished work of the author and K. Zumbrun has shown that one can obtain *a posteriori* bounds on the modulus of continuity of F_γ from such computer calculations. Therefore, it seems reasonable to conjecture, as in Lyons, Pemantle and Peres (1997), that the distribution of γ is always absolutely continuous, at least when $p_k = 0$ for all sufficiently large k.

O. Häggström (personal communication, 1995) observed that one can write

$$\gamma = [1, \gamma_1, 1, \gamma_2, 1, \gamma_3, \ldots] := \cfrac{1}{1 + \cfrac{1}{\gamma_1 + \cfrac{1}{1 + \cdots}}},$$

where γ_i are i.i.d. In the special case $p_1 = p_2 = 1/2$, the random variables γ_i have the c.d.f. $(\mathbf{1}_{[0,\infty)} + F_\gamma)/2$. Note that $[1, x_1, 1, x_2, \ldots, 1, x_n]$ is monotonic increasing in each x_i; this shows convergence of the infinite continued fraction. Thus, in attempting to prove the above conjecture on absolute continuity, we were led to consider the simpler random continued fractions

$$[1, X_1, 1, X_2, 1, X_3, \ldots],$$

where X_i are i.i.d. with the same distribution as

$$X = \begin{cases} 0 & \text{with probability } 1/2, \\ \alpha & \text{with probability } 1/2, \end{cases}$$

where $\alpha \in (0, \infty)$ is a given real number. That is, instead of trying to solve a fixed point problem, we are attempting to understand the nature of the map taking a distribution of X to the distribution of $[1, X_1, 1, X_2, 1, X_3, \ldots]$ for a very simple X. Let μ_α be the distribution of this random continued fraction.

Our main result is the following. Define α_c to be the solution of $\lambda_\alpha = \frac{1}{2}\log 2$, where λ_α is defined in (2.6) below.

THEOREM 1.1. *If $\alpha > 1/2$, then μ_α is supported on a Cantor set of Hausdorff dimension < 1. If $\alpha \le 1/2$, then the support of μ_α is an interval. If $\alpha > \alpha_c$, then μ_α is singular with respect to Lebesgue measure and is concentrated on a set of Hausdorff dimension < 1; we have $\alpha_c \in (0.2688, 0.2689)$. Whenever $\alpha > 0$, the measure μ_α is continuous.*

The third sentence is the most novel one.

Define

$$T_\alpha(x) := \frac{x + \alpha}{1 + x + \alpha}.$$

Thus, $[1, x_1, 1, x_2, \ldots, 1, x_n] = T_{x_1} \circ T_{x_2} \circ \cdots \circ T_{x_n}(0)$. Note that T_0 is not a strict contraction. The support of μ_α is contained in $[0, M_\alpha]$ and includes the endpoints, where $T_\alpha M_\alpha = M_\alpha > 0$. If we solve this equation for M_α, we get $M_\alpha^2 + \alpha M_\alpha - \alpha = 0$, so that

$$M_\alpha = \frac{-\alpha + \sqrt{\alpha^2 + 4\alpha}}{2} .$$

In particular, $M_{1/2} = 1/2$ and $T_\alpha 0 > T_0 M_\alpha$ iff $\alpha > 1/2$. Since μ_α is supported on $T_0[0, M_\alpha] \cup T_\alpha[0, M_\alpha]$, it follows by iteration that $\operatorname{supp} \mu_\alpha$ is a Cantor set for $\alpha > 1/2$ and equals $[0, M_\alpha]$ for $\alpha \le 1/2$. Thus, Theorem 1.1 asserts that in some interval of α where there is overlap of $T_0[0, M_\alpha]$ and $T_\alpha[0, M_\alpha]$, the measure μ_α is singular. This contrasts with the case of two linear maps, $x \mapsto \alpha x$ and $x \mapsto \alpha(1 + x)$, where one has absolute continuity for almost all α in the overlap region (Solomyak 1995, Peres and Solomyak 1996). However, based on some numerical evidence, we believe that there is also an interval where μ_α is absolutely continuous:

CONJECTURE 1.2. *For all α sufficiently small, μ_α is absolutely continuous with respect to Lebesgue measure.*

After seeing a preprint of this work, Simon, Solomyak, and Urbańksi (1998) made a great deal of progress on this conjecture. They proved that for Lebesgue-a.e. $\alpha \in (0.215, \alpha_c)$, the measure μ_α is absolutely continuous. In particular, this shows that the threshold α_c in Theorem 1.1 is sharp.

Other work on random continued fractions includes Bernadac (1993, 1995), Bhattacharya and Goswami (1998), Chamayou and Letac (1991), Chassaing, Letac, and Mora (1984), Kaijser (1983), Letac (1986), Letac and Seshadri (1995), Pitcher and Foster (1974), and Pincus (1983, 1985, 1994). In particular, Pincus (1983) asks about singularity of distributions like μ_α when the images of two linear fractional maps overlap, which is our main contribution. Our analysis of the Hausdorff dimension is similar to that appearing on pp. 166–168 of Bougerol and Lacroix (1985), where they use Lyapunov exponent techniques to recover a result of Kinney and Pitcher (1965/1966).

Finally, returning to our original problem about Galton-Watson trees, we mention another related problem: Show that $\sum_{k \ge 1} Z_k^{-1}$ has an absolutely continuous distribution, where Z_k is the size of the kth generation of a Galton-Watson process. This arises as the resistance of the shorted Galton-Watson tree. In other words, if a graph is formed from a genealogical Galton-Watson tree by identifying all vertices in level k for each k separately, then $\left(\sum_{k \ge 1} Z_k^{-1}\right)^{-1}$ is the effective conductance from the original progenitor to infinity. Again, computation supports the conjecture that the distribution is absolutely continuous with a bounded density, at least when the offspring distribution is bounded.

In Section 2, we prove Theorem 1.1. In Section 3, we discuss the possibility of L^2 or L^∞ densities for $\alpha < \alpha_c$.

§2. Proof of Theorem 1.1.

By Theorem 6.5 of Urbański (1996), the Hausdorff dimension of $\operatorname{supp} \mu_\alpha$ is less than 1 when there is no overlap, i.e., when $\alpha > 1/2$. (According to Remark 6.6 in that paper, we also have that the Hausdorff dimension of $\operatorname{supp} \mu_\alpha$ equals its upper Minkowski dimension.)

Since μ_α is a stationary measure for the Markov chain on $[0, M_\alpha]$ that has transitions $s \mapsto T_X s$, it follows that μ_α is continuous: By stationarity, for any $s \in [0, M_\alpha]$, we have

$$\mu_\alpha(\{s\}) = \frac{1}{2}\mu_\alpha(\{T_0^{-1}s\}) + \frac{1}{2}\mu_\alpha(\{T_\alpha^{-1}s\})\,.$$

In particular, if s is such that $\mu_\alpha(\{s\})$ is maximal, then $\mu_\alpha(\{T_0^{-1}s\}) = \mu_\alpha(\{s\})$. Likewise, $\mu_\alpha(\{T_0^{-n}s\}) = \mu_\alpha(\{s\})$. Since μ_α is finite, it follows that $\mu_\alpha(\{s\}) = 0$.

Define random variables A_n, B_n, C_n, D_n by

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} := \begin{pmatrix} 1 & X_1 \\ 1 & 1+X_1 \end{pmatrix} \cdots \begin{pmatrix} 1 & X_n \\ 1 & 1+X_n \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s+Xt \\ s+t+Xt \end{pmatrix}$$

and

$$\frac{s+Xt}{s+t+Xt} = \cfrac{1}{1+\cfrac{1}{X+\cfrac{s}{t}}}\,,$$

we have that

$$\frac{A_n s + B_n}{C_n s + D_n} = [1, X_1, 1, X_2, \ldots, 1, X_n + s]\,,$$

as is well known. We also write

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} := \begin{pmatrix} 1 & x_1 \\ 1 & 1+x_1 \end{pmatrix} \cdots \begin{pmatrix} 1 & x_n \\ 1 & 1+x_n \end{pmatrix}$$

for $x_i = 0, \alpha$.

Let Y be a random variable with distribution μ_α independent of all X_i. Then

$$\mu_\alpha = \mathcal{L}([1, X_1, \ldots, 1, X_n + Y])$$

for any n, where $\mathcal{L}(\bullet)$ is the law, i.e., the distribution, of a random variable $\bullet$. Now

$$\mathcal{L}([1, x_1, \ldots, 1, x_n + Y])$$

is a probability measure supported on the interval

$$[[1, x_1, \ldots, 1, x_n], [1, x_1, \ldots, 1, x_n + M_\alpha]] = \left[\frac{b_n}{d_n}, \frac{a_n M_\alpha + b_n}{c_n M_\alpha + d_n}\right].$$

The length of this interval is $M_\alpha/[d_n(c_n M_\alpha + d_n)]$ since

$$a_n d_n - b_n c_n = \det\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = 1.$$

Also,

$$\mu_\alpha\Big([[1, x_1, \ldots, 1, x_n], [1, x_1, \ldots, 1, x_n + M_\alpha]]\Big) \geq 2^{-n}, \tag{2.1}$$

with equality iff $\alpha \geq 1/2$. By Billingsley's Theorem (Falconer (1997), p. 171), μ_α is concentrated on a set of Hausdorff dimension

$$\mu_\alpha\text{-ess}\sup_x \ \liminf_{\epsilon \to 0} \frac{\log 1/\mu_\alpha\big((x-\epsilon, x+\epsilon)\big)}{\log 1/(2\epsilon)}, \tag{2.2}$$

which by (2.1) is at most

$$\lim_{n\to\infty} \frac{\log 2^n}{\log \dfrac{D_n(C_n M_\alpha + D_n)}{M_\alpha}} = \frac{\log 2}{2\lambda_\alpha}, \tag{2.3}$$

where λ_α is the (top) Lyapunov exponent of the random matrix $\begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix}$; see (2.4) for the definition of λ_α and Bougerol and Lacroix (1985), Cor. VI.2.3, for this property of λ_α. [Simon, Solomyak and Urbański (1998) have now proved that the quantity in (2.2) is in fact equal to that in (2.3) for Lebesgue-a.e. $\alpha \in (\alpha_c, 1/2)$.] In particular, μ_α is singular if $\lambda_\alpha > \frac{1}{2}\log 2$. Note that λ_α is strictly increasing in α since $\begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix}$ is stochastically increasing in α. Thus, if $\alpha > \alpha_c$, then μ_α is singular and concentrated on a set of Hausdorff dimension less than 1. It remains to estimate α_c.

Parametrize projective one-space $\mathbb{P}^1$ minus the horizontal direction by the vectors $\mathbb{R} \times \{1\}$. Leaving out the horizontal is of no consequence since $\begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix}$ maps it to another direction regardless of X. The random map $v \mapsto \begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix} v$ induces a random map $(s,1) \mapsto (T_X s, 1)$ on $\mathbb{R} \times \{1\}$ (thought of as $\mathbb{P}^1$). The stationary measure for this Markov chain on $\mathbb{R} \times \{1\}$ then becomes just μ_α itself.

Now λ_α is the expected change in the log norm on $\mathbb{R}^2$:

$$\lambda_\alpha = \int \mathbf{E} \log \left[\| \begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix} (s,1) \| / \|(s,1)\| \right] d\mu_\alpha(s)\,. \tag{2.4}$$

If we use the norm

$$\|(x,y)\|_\epsilon := \epsilon x + (1-\epsilon) y$$

for any choice of $\epsilon \in (0,1)$, then we get

$$\lambda_\alpha = \int_{s=0}^{M_\alpha} \left\{ \frac{1}{2} \log \frac{\epsilon s + (1-\epsilon)(1+s)}{\epsilon s + (1-\epsilon)} + \frac{1}{2} \log \frac{\epsilon(s+\alpha) + (1-\epsilon)(1+s+\alpha)}{\epsilon s + (1-\epsilon)} \right\} d\mu_\alpha(s)\,. \tag{2.5}$$

Since this holds for all $\epsilon \in (0,1)$, it also holds for $\epsilon = 0$ by the Bounded Convergence Theorem:

$$\lambda_\alpha = \frac{1}{2} \int_{s=0}^{M_\alpha} \log[(1+s)(1+s+\alpha)]\, d\mu_\alpha(s)\,. \tag{2.6}$$

Define K_α to be the operator on c.d.f.'s on $[0, M_\alpha]$ corresponding to $\nu \mapsto (\frac{1}{2}T_0 + \frac{1}{2}T_\alpha)\nu$. Thus,

$$(K_\alpha F)(s) = \frac{1}{2} F(T_0^{-1} s) + \frac{1}{2} F(T_\alpha^{-1} s) = \frac{1}{2} F(T_0^{-1} s) + \frac{1}{2} F(T_0^{-1} s - \alpha)\,.$$

Set $F_0(s) := s/M_\alpha$ $(0 \le s \le M_\alpha)$ and $F_{n+1} := K_\alpha F_n$. Then F_n converges to the c.d.f. $F_{(\alpha)}$ of μ_α because $\big|[[1, x_1, \ldots, 1, x_n], [1, x_1, \ldots, 1, x_n + M_\alpha]]\big| \le c_n^{-1} \le 1/n$. Since K_α is a monotone operator (i.e., if $F \le G$, then $K_\alpha F \le K_\alpha G$), we have that $F_{(\alpha)} \le F_{n+1} \le F_n$ whenever $F_1 \le F_0$. If $\alpha < 1/2$, then F_1 is linear from 0 to $T_\alpha 0$, from $T_\alpha 0$ to $T_0 M_\alpha$, and from $T_0 M_\alpha$ to M_α, with maximum slope in the middle portion. If $\alpha \ge 1/2$, then F_1 is linear from 0 to $T_0 M_\alpha$, constant from $T_0 M_\alpha$ to $T_\alpha 0$, and linear from $T_\alpha 0$ to M_α. Thus, in both cases, it follows that $F_1 \le F_0$ iff $F_1(T_0 M_\alpha) \le F_0(T_0 M_\alpha)$. Now

$$F_0(T_0 M_\alpha) = \frac{T_0 M_\alpha}{M_\alpha} = \frac{1}{1 + M_\alpha}$$

and

$$F_1(T_0 M_\alpha) = \frac{1}{2}[F_0(M_\alpha) + F_0(M_\alpha - \alpha)] = 1 - \frac{\alpha}{2M_\alpha}\,.$$

Therefore, $F_1 \le F_0$ iff $\alpha \ge 1/6$. (We remark that $M_{1/6} = 1/3$.)

Now if $F \le G$ and h is an increasing function, then $\int h\, dF \ge \int h\, dG$. Since $s \mapsto \log[(1+s)(1+s+\alpha)]$ is increasing, this means that

$$\lambda_\alpha \ge \frac{1}{2} \int \log[(1+s)(1+s+\alpha)]\, dF_n(s)$$

for all n, giving us a lower bound for each n; in particular, as computation shows, $\lambda_\alpha > \frac{1}{2}\log 2$ for $\alpha = 0.2689$. Therefore, $\alpha_c < 0.2689$.

We can get an upper bound on λ_α by a similar method. We have that (2.5) holds for $\epsilon = 1$ by the Bounded and Monotone Convergence Theorems (use monotonicity in a neighborhood of $s = 0$ and boundedness elsewhere):

$$\lambda_\alpha = \frac{1}{2}\int_{s=0}^{M_\alpha} \log\left(1 + \frac{\alpha}{s}\right) d\mu_\alpha(s)\,.$$

(Note that for best numerical estimates, though without error bounds, one should choose a norm such that the expected log change is most nearly constant, rather than $\epsilon = 0, 1$ in (2.5). But the choices $\epsilon = 0, 1$ give us lower and upper bounds when $\alpha \geq 1/6$.)

Now if $F \leq G$ and h is a decreasing function, then $\int h\,dF \leq \int h\,dG$. Since $s \mapsto \log(1 + \alpha/s)$ is decreasing, we get the upper bound

$$\lambda_\alpha \leq \frac{1}{2}\int \log\left(1 + \frac{\alpha}{s}\right) dF_n(s)$$

for all n; in particular, $\lambda_\alpha < \frac{1}{2}\log 2$ for $\alpha \leq 0.2688$. This completes the estimate of α_c. ∎

§3. L^p Densities.

We next show that μ_α cannot be absolutely continuous with an L^2 density for $\alpha > \sqrt{6}/2 - 1 = 0.2247^+$. This result contrasts with the case of two equally likely linear maps (Solomyak 1995, Peres and Solomyak 1996), where all known instances of absolutely continuous measures have an L^2 density.

PROPOSITION 3.1. *If μ_α is absolutely continuous with an L^2 density, then $\alpha \leq \sqrt{6}/2 - 1$.*

Proof. Suppose that μ_α has a density f_α. Recall that Y has distribution μ_α. For $x_1, \ldots, x_n \in \{0, \alpha\}$, the probability measure $\mathcal{L}([1, x_1, \ldots, 1, x_n + Y])$ has a density g_n (depending on $x_1, \ldots, x_n$) supported on the interval

$$\left[\frac{b_n}{d_n}, \frac{a_n M_\alpha + b_n}{c_n M_\alpha + d_n}\right].$$

Therefore,

$$f_\alpha = \sum_{x_1,\ldots,x_n} 2^{-n} g_n\,.$$

Considering only the diagonal terms and using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned}
\int_0^{M_\alpha} f_\alpha(s)^2\,ds &\geq \sum_{x_1,\ldots,x_n} 2^{-2n} \int g_n(s)^2\,ds \\
&\geq \sum_{x_1,\ldots,x_n} 2^{-2n} \left| \left[\frac{b_n}{d_n}, \frac{a_n M_\alpha + b_n}{c_n M_\alpha + d_n} \right] \right|^{-1} \\
&= \sum_{x_1,\ldots,x_n} 2^{-2n} \frac{d_n(c_n M_\alpha + d_n)}{M_\alpha} \\
&= 2^{-n} \mathbf{E}\left[\frac{D_n(C_n M_\alpha + D_n)}{M_\alpha} \right] \\
&\geq 2^{-n} \mathbf{E}[D_n^2]/M_\alpha\,.
\end{aligned}$$

Write

$$\mathbf{M}_n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}.$$

Then

$$D_n^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} (\mathbf{M}_n \otimes \mathbf{M}_n) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

whence

$$\mathbf{E}[D_n^2] = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{E}[\mathbf{M}_n \otimes \mathbf{M}_n] \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By independence, we have

$$\mathbf{E}\big[\mathbf{M}_n \otimes \mathbf{M}_n\big] = R^n\,,$$

where

$$\begin{aligned}
R &:= \mathbf{E}\left[\begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix} \otimes \begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix} \right] \\
&= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & \alpha & \alpha & \alpha^2 \\ 1 & 1+\alpha & \alpha & \alpha+\alpha^2 \\ 1 & \alpha & 1+\alpha & \alpha+\alpha^2 \\ 1 & 1+\alpha & 1+\alpha & (1+\alpha)^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \alpha/2 & \alpha/2 & \alpha^2/2 \\ 1 & 1+\alpha/2 & \alpha/2 & \alpha/2+\alpha^2/2 \\ 1 & \alpha/2 & 1+\alpha/2 & \alpha/2+\alpha^2/2 \\ 1 & 1+\alpha/2 & 1+\alpha/2 & 1+\alpha+\alpha^2/2 \end{pmatrix}.
\end{aligned}$$

The characteristic polynomial of R is

$$t \mapsto t^4 - (4 + 2\alpha + \alpha^2/2)t^3 + (6 + 4\alpha + \alpha^2/2)t^2 - (4 + 2\alpha)t + 1\,,$$

which has its largest root > 2 iff $\alpha > \sqrt{6}/2 - 1$. Clearly, the Perron-Frobenius eigenvector of R has all its coordinates strictly positive, so that

$$2^{-n}\mathbf{E}[D_n^2] = 2^{-n}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} R^n \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \to \infty$$

as $n \to \infty$ if $\alpha > \sqrt{6}/2 - 1$. Hence μ_α cannot have a density in L^2 for $\alpha > \sqrt{6}/2 - 1$. ∎

Similar methods show that if $\alpha > 3\sqrt{2} - 4 = 0.24^+$, then μ_α cannot have a density in $L^{3/2}$. In the other direction, we can extend the method to show the following proposition:

PROPOSITION 3.2. *If μ_α is absolutely continuous with a density in L^p for every $p < \infty$, then $\alpha \le (3\sqrt{2} - 4)/2 = 0.1213^+$. In particular, this is the case if μ_α has a density in L^∞.*

Proof. Let p be a positive integer and

$$R_p(\alpha) := \mathbf{E}\Big[\mathbf{M}_1^{\otimes 2(p-1)}\Big]\,,$$

where $\mathbf{M}_1^{\otimes r}$ denotes the r-th tensor power of $\mathbf{M}_1 := \begin{pmatrix} 1 & X \\ 1 & 1+X \end{pmatrix}$. Write $r := 2(p-1)$. The method of proof of Proposition 3.1 shows that μ_α cannot have a density in L^p for $\alpha > \alpha_p$, where α_p is the value of α such that the largest eigenvalue γ_p of R_p is equal to $2^{(p-1)/2}$. Let the eigenvectors of $\mathbf{M}_1$ be $\mathbf{u}_1$ and $\mathbf{u}_2$. Then the eigenvectors of $\mathbf{M}_1^{\otimes r}$ are $\mathbf{u}_{i_1} \otimes \cdots \otimes \mathbf{u}_{i_r}$, where all i_j are either 1 or 2. Thus, the largest eigenvalue of $\mathbf{M}_1^{\otimes r}$ is δ^r, where δ is the largest eigenvalue of $\mathbf{M}_1$. It follows that $\gamma_p^{1/r}$ tends to the largest eigenvalue of $\begin{pmatrix} 1 & \alpha \\ 1 & 1+\alpha \end{pmatrix}$. It is then an easy matter to show that $\alpha_p \to (3\sqrt{2} - 4)/2$. ∎

REMARK 3.3. Numerical evidence suggests that in fact μ_α may not have a bounded density until α is less than about 0.05 (very roughly). Furthermore, the density seems to gain increasing smoothness the smaller α becomes.

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