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Percolation Perturbations in Potential Theory and Random Walks

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Abstract. We show that on a Cayley graph of a nonamenable group, a.s. the infinite clusters of Bernoulli percolation are transient for simple random walk, that simple random walk on these clusters has positive speed, and that these clusters admit bounded harmonic functions. A principal new finding on which these results are based is that such clusters admit invariant random subgraphs with positive isoperimetric constant.

We also show that percolation clusters in any amenable Cayley graph a.s. admit no nonconstant harmonic Dirichlet functions. Conversely, on a Cayley graph admitting nonconstant harmonic Dirichlet functions, a.s. the infinite clusters of p-Bernoulli percolation also have nonconstant harmonic Dirichlet functions when p is sufficiently close to 1. Many conjectures and questions are presented.

§1. Introduction.

The question of whether various potential-theoretic properties of graphs and manifolds are preserved under perturbations or approximations has been studied for more than a decade. For example, invariance under quasi-isometries of transience (i.e., the existence of nonconstant positive superharmonic functions) or of existence of harmonic functions in certain classes has been studied by Kanai (1986), T. Lyons (1987), Saloff-Coste (1992), Soardi (1993), Benjamini and Schramm (1996a), Thm. 3.5, and Holopainen and Soardi (1997).

In this paper, we study perturbations of graphs that are more radical than quasi-isometries and that are random. Namely, edges are deleted at random to form a percolation subgraph ω and the behavior of **simple random walk** $\langle X(t)\rangle$ on ω is examined (where each neighbor of $X(t)$ in ω is equally likely to be $X(t+1)$).

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We recall some definitions. Given a graph $G = \big(V(G), E(G)\big)$ and $p \in [0,1]$, the random subgraph ω_p formed by deleting each edge independently with probability $1-p$ is called **p-Bernoulli bond percolation**. The **critical probability** $p_c(G)$ is the infimum over all $p \in [0,1]$ such that there is positive probability for the existence of an infinite connected component in ω_p. The connected components of ω_p are also called **clusters**. In the case that G is an amenable Cayley graph, Burton and Keane (1989) show that p-Bernoulli percolation has a.s. at most one infinite cluster. For background on percolation, especially in $\mathbb{Z}^d$, see Grimmett (1989). Following earlier work of Grimmett and Newman (1990) on the direct product of a regular tree and $\mathbb{Z}$, a general study of percolation on discrete groups was initiated in Benjamini and Schramm (1996b). One phenomenon that was conjectured there to be general was a converse to the Burton and Keane result, namely, that on any nonamenable group, for some p, there are a.s. infinitely many infinite clusters in ω_p. This led to the definition

$$p_u(G) := \inf \Big\{ p : \mathbf{P}[\omega_p \text{ has exactly one infinite cluster}] = 1 \Big\} .$$

Thus, $p_u(G) = p_c(G)$ when G is an amenable Cayley graph. Häggström and Peres (1998) show that on a Cayley graph G, for *every* $p > p_u$, there is exactly one infinite cluster a.s. in p-Bernoulli percolation. It is known that $p_u < 1$ in many cases besides amenable groups, e.g., finitely presented groups with one end (Babson and Benjamini 1999) and Kazhdan groups (Lyons and Schramm 1998).

The **uniqueness phase** of Bernoulli percolation is the range of p where there is precisely one infinite cluster a.s.; the **nonuniqueness phase** is the range of p where there is more than one infinite cluster a.s.

The unique infinite percolation cluster of supercritical Bernoulli percolation on a graph, if there is such, can be viewed as a random perturbation of the graph. It is then natural to ask which properties of the graph are inherited by such a percolation cluster.

After presenting further definitions and reviewing some background in Section 2, we begin by studying in Section 3 purely geometric aspects of percolation clusters, namely, how the isoperimetric constant $\iota_E(G)$ (see Section 2) behaves under percolation. If ω is a random configuration of Bernoulli percolation on a Cayley graph G, then, of course, $\iota_E(\omega) = 0$ a.s. However,

THEOREM 1.1. *If G is a nonamenable Cayley graph and ω is a random configuration of Bernoulli percolation on G, then a.s. every infinite cluster of ω contains a subgraph ω' with $\iota_E(\omega') > 0$.*

(See Theorem 3.9.) In fact, we show in Theorem 3.10 that one can further require ω' to be a tree. Note that ω is a random variable, representing the configuration; in the sequel, we shall often say, however, that ω is a "percolation".

The theorem raises the following question about percolation in $\mathbb{Z}^d$. Recall that the **isoperimetric dimension** of a graph G is the supremum of all s such that

$$\inf\left\{\frac{|\partial_E V_1|^s}{|V_1|^{s-1}} : V_1 \subset V(G),\ 0 < |V_1| < \infty\right\} > 0\,.$$

QUESTION 1.2. Let ω be supercritical Bernoulli percolation in $\mathbb{Z}^d$. Is it true that for every $\epsilon > 0$, a.s. ω contains a subgraph with isoperimetric dimension at least $d - \epsilon$? Does it contain a subgraph with isoperimetric dimension d?

It is well known that if G is nonamenable, then the speed of simple random walk $\langle X(t)\rangle$ on G is positive, i.e.,

$$\lim_{t\to\infty} \frac{\mathrm{dist}_G\big(X(0), X(t)\big)}{t} > 0 \quad \text{a.s.}\,,$$

which results from the fact that the spectral radius is less than 1 (Kesten 1959); see Section 2. We prove the following extension in Section 4 as a consequence of (a more precise version of) Theorem 1.1:

THEOREM 1.3. *Let G be a nonamenable Cayley graph and ω be Bernoulli percolation on G. Let $\langle X(t)\rangle$ be simple random walk on ω. Given that the cluster K of $X(0)$ is infinite, we have a.s. that the speed of X is positive.*

(See Theorem 4.4.)

We conjecture the following generalization:

CONJECTURE 1.4. *If G is a Cayley graph on which simple random walk has positive speed, then a.s., simple random walk on each infinite cluster of p-Bernoulli percolation has positive speed.*

There are (nontransitive) graphs on which simple random walk has zero speed, but for which Bernoulli percolation a.s. produces clusters where simple random walk has positive speed. An example is a binary tree with a copy of $\mathbb{Z}$ attached to every vertex.

CONJECTURE 1.5. *If G is a Cayley graph on which simple random walk has zero speed, then a.s., simple random walk on every cluster of Bernoulli percolation also has zero speed.*

Any possible counterexample would have to be amenable and of exponential growth by Lemma 4.6 below.

The speed of random walk is related to other probabilistic behavior through the following theorem due to the work of Avez (1974), Derriennic (1980), Kaimanovich and Vershik (1983), and Varopoulos (1985). Recall that a function $F : V(G) \to \mathbb{R}$ is called **harmonic** if $F(x) = \sum_{y \sim x} F(y)/\deg_G x$ for all $x \in V$.

THEOREM 1.6. (SPEED, ENTROPY, AND BOUNDED HARMONIC FUNCTIONS) *The following conditions are equivalent for a given Cayley graph:*

(i) *the speed of simple random walk is zero;*

(ii) *the asymptotic entropy of simple random walk is zero;*

(iii) *there are no nonconstant bounded harmonic functions.*

Furthermore, Kaimanovich (1990) extended the equivalence of (ii) and (iii) to many random walks in a random environment (RWRE) that have a stationary measure; the extension to include (i) is easy (Lemma 4.6). Since simple random walk restricted to percolation clusters has an equivalent invariant measure (see Lemma 4.1), our conjectures and results about the speed of random walk have some alternative formulations in terms of entropy and bounded harmonic functions.

Transience holds more generally, of course, than positive speed. Transience of the infinite clusters of Bernoulli percolation in $\mathbb{Z}^d$, $d > 2$, was established in Grimmett, Kesten and Zhang (1993) (see Benjamini, Pemantle and Peres (1998) for a different proof). De Masi, Ferrari, Goldstein and Wick (1989) proved an invariance principle for simple random walk on the supercritical cluster in $\mathbb{Z}^2$.

CONJECTURE 1.7. *If G is a transient Cayley graph, then a.s. every infinite cluster of Bernoulli percolation on G is transient.*

The nonamenable case follows from Theorem 1.3; a slight extension is:

THEOREM 1.8. *Let G be a Cayley graph such that the ball of radius n has cardinality at least ζ^n for all n, where $\zeta > 1$. Then a.s., every infinite cluster of p-Bernoulli percolation on G is transient when $p > 1/\zeta$.*

(See Theorem 4.9.)

In Section 5, we study the existence of nonconstant harmonic Dirichlet functions on the percolation clusters of Cayley graphs. A function $F : V(G) \to \mathbb{R}$ is called **Dirichlet** if $\sum_x \sum_{y \sim x} |F(x) - F(y)|^2 < \infty$. Recall that $\mathcal{O}_{\mathbf{HD}}$ denotes the class of graphs that do not admit any nonconstant harmonic Dirichlet functions. As we shall see below, the picture in the nonamenable situation is rather involved and our understanding is far from complete. In the amenable case, we have

THEOREM 1.9. *If G is an amenable Cayley graph and ω is Bernoulli percolation, then a.s. every cluster of ω is in $\mathcal{O}_{\mathbf{HD}}$.*

(See Theorem 5.5.)

Medolla and Soardi (1995) proved that amenable transitive graphs are in $\mathcal{O}_{\mathbf{HD}}$ (see Remark 7.5 of Benjamini, Lyons, Peres, and Schramm (1998) for a short proof via uniform spanning forests). Soardi (1993) proved that $\mathcal{O}_{\mathbf{HD}}$ is invariant under quasi-isometries. (See Lyons and Peres (1998) for a simple proof due to Schramm.)

Our proof of Theorem 1.9 uses the uniform spanning forest measures and their connection to harmonic Dirichlet functions as presented in BLPS (1998). Such an ingredient can be motivated as follows: In order to study the influence of geometric properties on potential-theoretic behavior, it is useful to have a geometric representation of the potential-theoretic objects. The uniform spanning forest achieves this by representing the analytic space of harmonic Dirichlet functions by a random geometric object. The relevant definitions and properties are given in Section 5.

A positive answer to the following question would extend Theorem 1.9:

QUESTION 1.10. Let G be a Cayley graph, and suppose that $G \in \mathcal{O}_{\mathbf{HD}}$. Let ω be Bernoulli percolation on G in the uniqueness phase. Does it follow that a.s. the infinite cluster of ω is in $\mathcal{O}_{\mathbf{HD}}$? We do not know the answer even in the case that G is the direct product of a tree and $\mathbb{Z}$ or is a lattice in hyperbolic space $\mathbb{H}^d$ $(d \geqslant 3)$.

In the nonuniqueness phase of Bernoulli percolation on a (nonamenable) Cayley graph G, the infinite clusters are not in $\mathcal{O}_{\mathbf{HD}}$ (Corollary 4.7).

In the other direction, for graphs admitting nonconstant harmonic Dirichlet functions, we believe:

CONJECTURE 1.11. *Let G be a Cayley graph, $G \notin \mathcal{O}_{\mathbf{HD}}$. Then a.s. all infinite clusters of p-Bernoulli percolation are not in $\mathcal{O}_{\mathbf{HD}}$.*

We can prove this for p sufficiently large:

THEOREM 1.12. *Let G be a Cayley graph. If $G \notin \mathcal{O}_{\mathbf{HD}}$, then there is some $p_0 < 1$ such that for every $p \geqslant p_0$, almost surely no infinite cluster of p-Bernoulli percolation is in $\mathcal{O}_{\mathbf{HD}}$.*

(See Theorem 5.7.)

The last section of our paper discusses questions concerning the speed of simple random walk on a graph and a variant of the isoperimetric constant, called **anchored expansion**. The anchored expansion constant might be useful in studying the speed of simple random walk on infinite percolation clusters and other graphs as well.

Although in the above theorems, only Cayley graphs are mentioned, we work in the greater generality of transitive graphs. Similarly, we discuss percolation processes that are much more general than Bernoulli percolation.

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§2. Notation and Background.

Graph terminology, isoperimetric constant and ends. We use the letter G to denote a graph and Γ to denote a closed subgroup of the automorphism group $\mathrm{Aut}(G)$ of G. The vertices and edges of G will be denoted $V(G)$ and $E(G)$, respectively. When there is an edge in G joining vertices u, v, we write $u \sim v$. The **degree** $\deg v = \deg_G v$ of a vertex $v \in V(G)$ is the number of edges incident with it. A **tree** is a connected graph with no cycles. A **forest** is a graph whose connected components are trees. The **distance** between two vertices $v, u \in V(G)$ is denoted by $\mathrm{dist}(v, u) = \mathrm{dist}_G(v, u)$, and is the least number of edges of a path in G connecting v and u.

For a set of vertices $V_1 \subset V(G)$, let $\partial_E V_1$ denote the set of edges in $E(G)$ that have one endpoint in V_1 and one endpoint in $V(G) - V_1$. The graph G is **amenable** if there is a sequence of finite vertex subsets $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots \subset V(G)$, such that $\bigcup_n V_n = V(G)$ and $|\partial_E V_n|/|V_n| \to 0$ as $n \to \infty$. Here and in the sequel, $|A|$ denotes the cardinality of a set A. The (edge) **isoperimetric constant** of a graph G, also known as the **Cheeger constant**, is defined by

$$\iota_E(G) := \inf\bigl\{|\partial_E V_0|/|V_0| : \emptyset \neq V_0 \subset V(G),\ |V_0| < \infty\bigr\}\,.$$

An infinite set of vertices $V_0 \subset V(G)$ is **end-convergent** if for every finite $K \subset V(G)$, there is a component of $G - K$ that contains all but finitely many vertices of V_0. Two end-convergent sets V_0, V_1 are **equivalent** if $V_0 \cup V_1$ is end-convergent. An **end** of G is an equivalence class of end-convergent sets.

Spectral radius, speed and entropy. Given $v, u \in V(G)$, let $p_t(v, u)$ be the probability that simple random walk starting at v will be at u at time t. The **spectral radius** $\rho(G)$ is defined by

$$\rho(G) := \limsup_{t\to\infty} p_t(v, u)^{1/t}$$

and does not depend on the choice of v and u. Dodziuk's (1984) discrete version of Cheeger's inequality states that if G has bounded degrees and $\iota_E(G) > 0$, then $\rho(G) < 1$.

The **speed** of a random walk X starting at $o \in V(G)$ is

$$\Lambda = \Lambda(X) := \lim_{t\to\infty} \mathrm{dist}\big(o, X(t)\big)/t$$

when the limit exists. The lim inf speed is defined by

$$\Lambda^- = \Lambda^-(X) := \liminf_{t\to\infty} \mathrm{dist}\big(o, X(t)\big)/t\,.$$

If G has bounded degrees and $\rho(G) < 1$, then there are constants $\zeta > 0$ and $\beta < 1$ such that

$$\mathbf{P}\Big[\mathrm{dist}\big(o, X(t)\big) < \zeta t\Big] \leqslant \beta^t\,,$$

because the probability that the random walk is inside the ball of radius ζt about o is bounded by the number of vertices in the ball times the probability that X is at the most likely vertex. Consequently, in this situation, $\Lambda^- > 0$ a.s.

The **entropy** of a probability measure μ on a finite or countable set A is defined to be

$$H(\mu) := \sum_{x\in A} -\mu(x)\log\mu(x)\,.$$

Let μ_t denote the distribution of the location $X(t)$ of a random walk X at time t. If $\lim_t H(\mu_t)/t$ exists, it is called the **asymptotic entropy** of the random walk. If $\langle X(t)\rangle$ is simple random walk on a Cayley graph, then the asymptotic entropy exists. In fact, the Subadditive Ergodic Theorem ensures the existence of $\lim_t -t^{-1}\log\mu_t\big(X(t)\big)$ a.s. and in L^1; see Derriennic (1980). The same is true for stationary RWRE, as observed by Kaimanovich (1990). Similar reasoning applies to graphs with a transitive unimodular automorphism group (Kaimanovich and Woess 1998).

For further information about random walks, spectral radius, harmonic functions, etc., see Kaimanovich and Vershik (1983) and Woess (1994).

Automorphism groups, unimodularity, and the Mass-Transport Principle. Let $\Gamma \subset \mathrm{Aut}(G)$ be a subgroup of automorphisms of G with the topology of pointwise convergence. We say that Γ is (vertex) **transitive** if for every $v, u \in V(G)$, there is a $\gamma \in \Gamma$ with $\gamma u = v$. The graph G is **transitive** if $\mathrm{Aut}(G)$ is transitive. Recall that every closed subgroup $\Gamma \subset \mathrm{Aut}(G)$ has a unique (up to a constant scaling factor) Borel measure that, for every $\gamma \in \Gamma$, is invariant under left multiplication by γ; this measure is called (left) **Haar measure**. The group Γ is **unimodular** if Haar measure is also invariant under right multiplication.

Most of our theorems concern percolation that is invariant under a transitive unimodular closed subgroup of $\mathrm{Aut}(G)$. For example, when G is the (right) Cayley graph of Γ and Γ acts by left multiplication, then $\Gamma \subset \mathrm{Aut}(G)$ is (obviously) closed, unimodular, and transitive. If G is an amenable graph, then every closed transitive subgroup of $\mathrm{Aut}(G)$ is unimodular (Soardi and Woess 1990).

Several illustrations of the significance of unimodularity can be found in Benjamini, Lyons, Peres, and Schramm (1999). The most important one seems to be that when $\Gamma \subset \mathrm{Aut}(G)$ is unimodular, the Mass-Transport Principle takes the following simple form:

THEOREM 2.1. (MASS-TRANSPORT PRINCIPLE) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let $o \in V(G)$ be an arbitrary base point. Suppose that $\phi : V(G) \times V(G) \to [0, \infty]$ is invariant under the diagonal action of Γ. Then*

$$\sum_{v \in V(G)} \phi(o, v) = \sum_{v \in V(G)} \phi(v, o) . \tag{2.1}$$

See BLPS (1999) for a discussion of this principle and for a proof. In fact, Γ is unimodular iff (2.1) holds for every such ϕ. Hence, (2.1) can be taken as a definition of unimodularity.

Percolation terminology. A **bond percolation** ω on G is a random subset of $E(G)$. For a more precise definition, given a set A, let 2^A be the collection of all subsets $\eta \subset A$, equipped with the σ-field generated by the events $\{a \in \eta\}$, where $a \in A$. A bond percolation ω on G is then a random variable whose distribution is a probability measure $\mathbf{P}$ on $2^{E(G)}$. Similarly, a **site percolation** is given by a probability measure on $2^{V(G)}$, while a (mixed) **percolation** is given by a probability measure on $2^{V(G) \cup E(G)}$ that is supported on subgraphs of G. If ω is a bond percolation, then $\hat{\omega} := V(G) \cup \omega$ is the associated mixed percolation. In this case, we shall often not distinguish between ω and $\hat{\omega}$, and think of ω as a subgraph of G. Similarly, if ω is a site percolation, there is an associated mixed percolation $\hat{\omega} := \omega \cup \big(E(G) \cap (\omega \times \omega)\big)$, and we shall often not bother to distinguish between ω and $\hat{\omega}$.

Let $p \in [0, 1]$. Then the distribution of p-**Bernoulli bond percolation** ω on G is the product measure on $2^{E(G)}$ that satisfies $\mathbf{P}[e \in \omega] = p$ for all $e \in E(G)$. Similarly, one defines p-**Bernoulli site percolation** on $2^{V(G)}$.

If $v \in V(G)$ and ω is a percolation on G, the **component** (or **cluster**) $K(v)$ of v in ω is the set of vertices in $V(G)$ that can be connected to v by paths contained in ω.

Suppose that Γ is an automorphism group of a graph G. A percolation on G is Γ-**invariant** if its distribution $\mathbf{P}$ is invariant under each automorphism in Γ.

Insertion tolerance and component indistinguishability. Given a set Z, an element $z \in Z$, and a subset $\omega \in 2^Z$, let $\Pi_z \omega := \omega \cup \{z\}$. A probability measure $\mathbf{P}$ on 2^Z is **insertion tolerant** if $\mathbf{P}[\Pi_z \mathcal{A}] > 0$ for every $z \in Z$ and every measurable $\mathcal{A} \subset 2^Z$ satisfying $\mathbf{P}[\mathcal{A}] > 0$. For example, p-Bernoulli bond percolation is insertion tolerant when $p > 0$.

Let G be graph and Γ a closed transitive subgroup of $\mathrm{Aut}(G)$. Let ω be a Γ-invariant bond percolation. We say that ω has **indistinguishable components** if for every measurable $\mathcal{A} \subset 2^{V(G)} \times 2^{E(G)}$ that is invariant under the diagonal action of Γ, almost surely, for all infinite components C of ω, we have $(C, \omega) \in \mathcal{A}$, or for all infinite components C, we have $(C, \omega) \notin \mathcal{A}$. (That is, whether $(C, \omega) \in \mathcal{A}$ does not depend on C, but may depend on ω.)

The following is from Lyons and Schramm (1998):

THEOREM 2.2. (COMPONENT INDISTINGUISHABILITY) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Every Γ-invariant, insertion-tolerant, bond percolation process on G has indistinguishable components.*

For example, this shows that in p-Bernoulli bond percolation on a Cayley graph of a nonamenable group, almost surely, either all infinite clusters are transient, or all clusters are recurrent. In fact, as indicated by Theorem 1.3, a.s. all infinite clusters are transient.

Similar statements hold for site and mixed percolations.

§3. Geometry of Perturbations of Nonamenable Graphs.

Let G be an infinite graph and K a finite subgraph of G. Set

$$\alpha_K := \frac{1}{|V(K)|} \sum_{v \in V(K)} \deg_K v \,,$$

where $\deg_K v$ is the degree of v in K. Define

$$\alpha(G) := \sup\bigl\{\alpha_K : K \subset G \text{ is finite}\bigr\} .$$

Note that when all vertices in G have degree d, the isoperimetric constant of G satisfies

$$\iota_E(G) = d - \alpha(G) .$$

Let T be a regular tree and $o \in V(T)$ be some base point. Häggström (1997) has shown that when ω is an automorphism-invariant percolation on T and $\mathbf{E}[\deg_\omega o] \geqslant \alpha(T)$, there are infinite clusters in ω with positive probability. In BLPS (1999), it was shown that the same result applies to transitive graphs with a unimodular automorphism group. In Theorem 3.2, we extend this result and show that with the same assumptions but with a strict inequality $\mathbf{E}[\deg_\omega o] > \alpha(G)$, with positive probability, there is a subgraph in ω with $\iota_E > 0$. This will be used to prove

THEOREM 3.1. (UNIQUENESS GIVES A SUBGRAPH WITH $\iota_E > 0$) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and suppose that $\iota_E(G) > 0$. Let ω be a Γ-invariant percolation in G that has a.s. exactly one infinite component. Then (on a larger probability space) there is a percolation $\omega' \subset \omega$ such that $\omega' \neq \emptyset$ and $\iota_E(\omega') > 0$ a.s. Moreover, the distribution of the pair (ω', ω) is Γ-invariant.*

In the following, if $K \subset G$ is a subgraph and $v \in V(G)$ is not in K, then we set $\deg_K v := 0$.

As we indicated, the proof of Theorem 3.1 is based on the following more quantitative result.

THEOREM 3.2. (HIGH MARGINALS GIVE A SUBGRAPH WITH $\iota_E > 0$) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω be a Γ-invariant (nonempty) percolation in G. Let $h > 0$ and suppose that*

$$\mathbf{E}[\deg_\omega o \mid o \in \omega] > \bigl(\alpha(G) + 2h\bigr) . \tag{3.1}$$

Then there is (on a larger probability space) a percolation $\omega' \subset \omega$ such that $\omega' \neq \emptyset$ and $\iota_E(\omega') \geqslant h$ with positive probability. Moreover, the distribution of the pair (ω', ω) is Γ-invariant.

Proof. Given any subgraph ω of G, we define percolations ω_n on ω inductively as follows. Set $\omega_0 := \omega$. Suppose that ω_n has been defined. Let β_n be a $(1/2)$-Bernoulli site percolation on G, independent of $\omega_0, \ldots, \omega_n$. Let γ_n be the union of the finite components K of $\beta_n \cap \omega_n$ that satisfy

$$\frac{|\partial_{E(\omega_n)} K|}{|K|} < h ,$$

where $\partial_{E(\omega_n)} K$ denotes the set of edges of ω_n connecting K to its complement. Now set $\omega_{n+1} := \omega_n - \gamma_n$. Finally, define

$$\omega' := \bigcap_{n=0}^{\infty} \omega_n .$$

For future use, write $\Xi(h, \omega) := \omega'$.

Most of the proof will be devoted to showing that $\omega' \neq \emptyset$ with positive probability, but first we verify that $\iota_E(\omega') \geqslant h$. Indeed, let W be a finite nonempty set of vertices in G, let F be the set of all edges of G incident with W, and let $F_0 \subset F$. Suppose that $|F_0|/|W| < h$. To verify that $\iota_E(\omega') \geqslant h$ a.s., it is enough to show that the probability that $W \subset \omega'$ and $\omega' \cap F = F_0$ is zero. If $\omega_n \cap F = F_0$ for some n, then a.s. there is some $m > n$ such that W is a component of β_m. Now either $W \not\subset \omega_m$, in which case $W \not\subset \omega'$, or $W \subset \omega_m$, in which case W is not contained in ω_{m+1}, hence not in ω'. On the other hand, if $\omega_n \cap F \neq F_0$ for every n, then also $\omega' \cap F \neq F_0$. Consequently $\iota_E(\omega') \geqslant h$ a.s.

Now set

$$D_n := \mathbf{E} \deg_{\omega_n} o \,, \qquad D_\infty := \mathbf{E} \deg_{\omega'} o \,,$$
$$\theta_n := \mathbf{P}[o \in \omega_n] \,, \qquad \theta_\infty := \mathbf{P}[o \in \omega'] \,.$$

Our goal is to prove the inequality

$$D_{n+1} \geqslant D_n - (\theta_n - \theta_{n+1})\big(\alpha(G) + 2h\big) \,. \tag{3.2}$$

This will be achieved through use of the Mass-Transport Principle. Observe that

$$\theta_n - \theta_{n+1} = \mathbf{P}[o \in \gamma_n] \,.$$

Fix n and define the random function $m : V(G) \times V(G) \to [0, \infty)$ as follows. For every vertex $v \in V(G)$, let $K(v)$ be the component of v in γ_n, which we take to be $\emptyset$ if $v \notin \gamma_n$. Let $v, u \in V$. If $u \notin \gamma_n$, set $m(v, u) := 0$. If $v \in K(u)$, let $m(v, u) := \deg_{\omega_n} v / |K(u)|$. Otherwise, let $m(v, u)$ be $|K(u)|^{-1}$ times the number of edges in ω_n that connect v to a vertex in $K(u)$. Note that v and u need not be adjacent in order that $m(v, u) \neq 0$. Clearly, $\mathbf{E}m(v, u)$ is invariant under the diagonal action of Γ on $V(G) \times V(G)$. Consequently, the Mass-Transport Principle implies that

$$\sum_{v \in V(G)} \mathbf{E}m(o, v) = \sum_{v \in V(G)} \mathbf{E}m(v, o) \,.$$

A straightforward calculation shows that

$$\sum_{v \in V(G)} m(o, v) = \deg_{\omega_n} o - \deg_{\omega_{n+1}} o \,,$$

while, if $o \in \gamma_n$, we have that $\sum_{v \in V(G)} m(v, o)$ is equal to twice the number of edges of ω_n incident with $K(o)$, divided by $|K(o)|$. The number of edges of G with both endpoints in $K(o)$ is at most $\alpha(G)|K(o)|/2$, and, by construction, $|\partial_{\omega_n} K(o)| < h|K(o)|$. Hence

$$\sum_{v \in V(G)} m(v, o) < \alpha(G) + 2h \tag{3.3}$$

when $o \in \gamma_n$ and $\sum_{v \in V(G)} m(v,o) = 0$ otherwise. Therefore,

$$
\begin{aligned}
D_n - D_{n+1} = \mathbf{E}[\deg_{\omega_n} o - \deg_{\omega_{n+1}} o] &= \sum_{v \in V(G)} \mathbf{E} m(v,o) \\
&\leqslant (\alpha(G) + 2h)\mathbf{P}[o \in \gamma_n] = (\alpha(G) + 2h)(\theta_n - \theta_{n+1}),
\end{aligned}
$$

which is the same as (3.2).

Induction and (3.2) give

$$
D_n \geqslant D_0 - \theta_0\big(\alpha(G) + 2h\big) + \theta_n\big(\alpha(G) + 2h\big);
$$

taking a limit as $n \to \infty$ yields the inequality

$$
D_\infty \geqslant D_0 - \theta_0\big(\alpha(G) + 2h\big) + \theta_\infty\big(\alpha(G) + 2h\big). \tag{3.4}
$$

This gives $D_\infty > 0$, because (3.1) is equivalent to $D_0 - \theta_0\big(\alpha(G) + 2h\big) > 0$. Consequently, $\omega' \neq \emptyset$ with positive probability. ∎

Remark 3.3. (The Density of ω') The following lower bound for θ_∞ is a consequence of (3.4) and the inequality $\theta_\infty \deg_G o \geqslant D_\infty$:

$$
\begin{aligned}
\mathbf{P}[o \in \omega'] = \theta_\infty &\geqslant \frac{D_0 - \big(\alpha(G) + 2h\big)\theta_0}{\deg_G o - \big(\alpha(G) + 2h\big)} \\
&= \mathbf{P}[o \in \omega] \left(1 - \frac{\deg_G o - \mathbf{E}[\deg_\omega o \mid o \in \omega]}{\iota_E(G) - 2h}\right).
\end{aligned} \tag{3.5}
$$

Remark 3.4. (A Weak Inequality Suffices) In fact, in place of (3.1), it is enough to assume the weak inequality $\mathbf{E}[\deg_\omega o \mid o \in \omega] \geqslant \big(\alpha(G) + 2h\big)$. The reason is that the inequality (3.3) is strict when $o \in \gamma_n$, which implies that (3.4) is strict when $D_\infty \neq D_0$.

Theorem 3.5. (Threshold for a Forest) *If ω is a forest a.s., then Theorem 3.2 is true when $\alpha(G)$ is replaced by 2.*

Proof. In a finite tree $K \subset G$, we have $\alpha_K < 2$. Hence the proof of Theorem 3.2 applies with 2 replacing $\alpha(G)$ everywhere. ∎

Proof of Theorem 3.1. Fix a base point $o \in V(G)$. Let ω_* be the infinite component of ω. Conditioned on ω, for every vertex $v \in V(G)$, let $\phi(v)$ be chosen uniformly among the vertices of ω_* closest to v, with all $\phi(v)$ independent given ω, and for edges $e = [v,u] \in E(G)$, let $\phi(e)$ be chosen uniformly among shortest paths in ω_* joining $\phi(v)$ and $\phi(u)$, with all $\phi(e)$ independent given all $\phi(v)$ and ω. For integers j, let η_j be the set of edges $e \in E(G)$

such that $\phi(e)$ is contained within a ball of radius j about one of the endpoints of e. Then $\eta_1 \subset \eta_2 \subset \cdots$ are Γ-invariant bond percolations on G with $\bigcup_j \eta_j = E(G)$. Consequently, $\mathbf{E} \deg_{\eta_j} o \to \deg_G o$ as $j \to \infty$. For each j, choose independently a random sample of $\Xi(h, \eta_j)$ and denote it ξ_j. By (3.5), we have that $\mathbf{P}[\xi_j \neq \emptyset] \to 1$. Let $J := \inf\{j : \xi_j \neq \emptyset\}$. Then $J < \infty$ a.s. Set $\omega' := \phi(\xi_J)$. Since $\iota_E(\xi_J) \geqslant h$ a.s., we have also $\iota_E\big(\phi(\xi_J)\big) > 0$ a.s. ∎

Suppose that G is transitive, $\iota_E(G) > 0$, and ω is, say, Bernoulli percolation on G that has a.s. more than one infinite component. Then Theorem 3.1 does not apply to ω. However, as observed by Burton and Keane (1989), insertion tolerance shows that there are a.s. components of ω with at least three ends. Hence the next theorem does apply.

THEOREM 3.6. (A FOREST WITH $\iota_E > 0$ INSIDE MANY-ENDED PERCOLATION) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and let ω be a Γ-invariant percolation on G. Suppose that a.s., there are components of ω with at least three ends. Then there is (on a larger probability space) a random forest $\mathfrak{F} \subset \omega$ with $\iota_E(\mathfrak{F}) > 0$, $\mathfrak{F} \neq \emptyset$ a.s., and the distribution of the pair $(\mathfrak{F}, \omega)$ is Γ-invariant.*

We shall need the following two lemmas from BLPS (1999). Recall that $K(x)$ denotes the component of x in ω.

LEMMA 3.7. (ENDS, p_c AND DEGREES) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω be a Γ-invariant percolation on G that has infinite components with positive probability. If*

(i) *some component of ω has at least three ends with positive probability,*

then

(ii) *some component of ω has $p_c < 1$ with positive probability and*

(iii) *for every vertex x, $\mathbf{E}\big[\deg_\omega x \bigm| |K(x)| = \infty\big] > 2$.*

If ω is a forest a.s., then the three conditions are equivalent.

LEMMA 3.8. (TRIMMING TO A FOREST) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω be a Γ-invariant percolation on G such that a.s. there is a component of ω with at least three ends. Then (on a larger probability space) there is a random forest $\mathfrak{F} \subset \omega$ such that the distribution of the pair $(\mathfrak{F}, \omega)$ is Γ-invariant and a.s. whenever a component K of ω has at least three ends, there is a component of $K \cap \mathfrak{F}$ that has infinitely many ends.*

Proof of Theorem 3.6. By Lemma 3.8, there is a random forest $\mathfrak{F}' \subset \omega$ with some components having infinitely many ends a.s. and the distribution of $(\mathfrak{F}', \omega)$ is Γ-invariant. Let

$\mathfrak{F}''$ be the union of the infinite components of $\mathfrak{F}'$. By Lemma 3.7, $\mathbf{E}[\deg_{\mathfrak{F}''} o \mid o \in \mathfrak{F}''] > 2$. Given ω, for each j, let ξ_j be an independent sample of $\Xi(1/j, \mathfrak{F}'')$. Put $\mathfrak{F} := \xi_J$, where $J := \inf\{j : \xi_j \neq \emptyset\} < \infty$ a.s.: Clearly, $J < \infty$ with positive probability. If the set $\mathcal{A}$ of ω where $J = \infty$ had positive probability, then we would obtain a contradiction to what has just been proved by noting that $\mathcal{A}$ is Γ-invariant and by conditioning on $\mathcal{A}$. ∎

We can now deduce the following extension of Theorem 1.1:

THEOREM 3.9. *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and suppose that $\iota_E(G) > 0$. Let ω be a Γ-invariant percolation on G that has infinite clusters a.s. Then in each of the following cases (on a larger probability space) there is a percolation $\omega' \subset \omega$ such that $\omega' \neq \emptyset$, $\iota_E(\omega') > 0$ a.s., and the distribution of the pair (ω', ω) is Γ-invariant:*

(i) *ω is Bernoulli percolation;*

(ii) *ω has a unique infinite cluster a.s.;*

(iii) *ω has a cluster with at least three ends a.s.;*

(iv) *$\mathbf{E}[\deg_\omega o \mid o \in \omega] > \alpha(G)$ and ω is ergodic.*

Proof. In Bernoulli percolation, if there is more than one infinite cluster, then there is a cluster with at least three ends by insertion tolerance and ergodicity. Consequently, (i) follows from (ii) and (iii). Parts (ii)–(iv) follow from Theorems 3.1, 3.6, and 3.2. ∎

Although it will not be needed in the sequel, we note that Theorem 3.1 can be strengthened as follows.

THEOREM 3.10. (A FOREST IN THE UNIQUENESS REGIME WITH $\iota_E > 0$) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and suppose that $\iota_E(G) > 0$. Let ω be a Γ-invariant percolation on G that has a.s. exactly one infinite component. Then (on a larger probability space) there is a random forest $\mathfrak{F} \subset \omega$ with $\mathfrak{F} \neq \emptyset$ and $\iota_E(\mathfrak{F}) > 0$ a.s., and the distribution of the pair $(\mathfrak{F}, \omega)$ is Γ-invariant.*

Proof. Let $\omega' \subset \omega$ be as in Theorem 3.1. Since $\iota_E(\omega') > 0$, Theorem 13.7 from BLPS (1998) constructs a percolation $\omega'' \subset \omega'$ such that a.s. ω'' has all components with infinitely many ends and the distribution of (ω'', ω) is Γ-invariant. By Lemma 3.8, there is a forest $\mathfrak{F}' \subset \omega''$ such that some tree in $\mathfrak{F}'$ has infinitely many ends a.s. and the distribution of $(\mathfrak{F}', \omega)$ is Γ-invariant. Let $\mathfrak{F}''$ be the union of the infinite components of $\mathfrak{F}'$. By Lemma 3.7, $\mathbf{E}[\deg_{\mathfrak{F}''} o \mid o \in \mathfrak{F}''] > 2$. The proof is completed as for Theorem 3.6. ∎

Applying Theorem 3.10 to the case where $\omega = G$ a.s., we obtain an invariant random forest in G with $\iota_E(\mathfrak{F}) > 0$. This is related to the result of Benjamini and Schramm (1997) which says that every bounded-degree graph with $\iota_E > 0$ contains a tree T with $\iota_E(T) > 0$. In fact, the latter result can be used to extend our theory to the non-transitive setting as follows.

Corollary 3.11. (The Non-Transitive Case) *Let G be a graph of bounded degree with $\iota_E(G) > 0$. Then there is some $p_0 < 1$ such that p-Bernoulli bond percolation on G has a subgraph with $\iota_E > 0$ a.s. whenever $p > p_0$.*

Proof. By the result of Benjamini and Schramm (1997) mentioned above, there is a tree $T \subset G$ with $\iota_E(T) > 0$. Let T_3 be the 3-regular tree. There is a map ϕ that takes $V(T_3)$ into $V(T)$, takes every edge $e = [v, u] \in E(T_3)$ to a path of bounded length $\phi(e)$ in T joining $\phi(v)$ to $\phi(u)$, and when $e, e' \in E(T_3)$ are distinct, the corresponding paths $\phi(e), \phi(e')$ are edge-disjoint. Consequently, p-Bernoulli percolation on T can be pulled back via ϕ to a bond percolation ω on T_3 in which the events $\{e \in \omega\}$ $(e \in E(T_3))$ are mutually independent. Moreover, $\mathbf{P}[e \in \omega] \geqslant 1 - k(1-p)$, where k is the maximum length of a path $\phi(e')$, $e' \in E(T_3)$. Consequently, ω dominates $\big(1 - k(1-p)\big)$-Bernoulli bond percolation ω' on T_3. By Theorem 3.5, when $3\big(1 - k(1-p)\big) > 2$, there is with positive probability, and therefore a.s., a subgraph $\omega'' \subset \omega'$ with $\iota_E(\omega'') > 0$. Now $\phi(\omega'')$ is the required subgraph of ω. ∎

§4. Speed and Transience.

In this section, we prove that in many cases, simple random walk on the infinite components of invariant percolation on a nonamenable transitive graph G has positive speed.

Let ω be a percolation on G. It will be useful to consider **delayed simple random walk** $Z = Z^\omega$ on ω, defined as follows. Set $Z(0) := o$, where $o \in V(G)$ is some fixed base point. If $n \geqslant 0$, conditioned on $\langle Z(0), \ldots, Z(n)\rangle$ and ω, let $Z'(n+1)$ be chosen from $Z(n)$ and its neighbors in $E(G)$ with equal probability. Set $Z(n+1) := Z'(n+1)$ if the edge $[Z(n), Z'(n+1)]$ belongs to ω; otherwise, let $Z(n+1) := Z(n)$.

For any sequence $\langle z(0), z(1), \ldots\rangle$, let $\mathcal{S}z$ be the shifted sequence $\langle \mathcal{S}z(0), \mathcal{S}z(1), \ldots\rangle$ defined by $\mathcal{S}z(n) := z(n+1)$ and let $\mathcal{S}(z, \omega) := (\mathcal{S}z, \omega)$. For $\gamma \in \Gamma$, we set

$$\gamma(w, \omega) := (\gamma w, \gamma\omega)\,,$$

where $(\gamma w)(n) := \gamma\big(w(n)\big)$.

The following lemma is from Lyons and Schramm (1998); it generalizes similar lemmas in Lyons and Peres (1998) and in Häggström (1997).

Lemma 4.1. (Stationarity of Random Walk) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let $o \in V(G)$ be some base point. Let $(\mu, 2^{E(G)})$ be a Γ-invariant percolation process on G; let ω denote the percolation configuration. Let $\widehat{\mu}_o$ be the joint law of ω and delayed simple random walk on ω, as defined above. Then $\widehat{\mu}_o[\mathcal{A}] = \widehat{\mu}_o[\mathcal{S}\mathcal{A}]$ for every Γ-invariant $\mathcal{A} \subset V^{\mathbb{Z}} \times \Omega$. In other words, the restriction of $\widehat{\mu}_o$ to the Γ-invariant σ-field is $\mathcal{S}$-stationary.*

Let μ' be the measure on subgraphs $\omega \subset G$ whose Radon-Nikodym derivative with respect to μ is $\deg_\omega o / \mathbf{E}_\mu[\deg_\omega o]$. Let $\widehat{\mu}'_o$ be the joint law of ω and simple random walk (non-delayed) on ω starting at o. Then the restriction of $\widehat{\mu}'_o$ to the Γ-invariant σ-field is $\mathcal{S}$-stationary.

Lemma 4.2. (Speed Exists and is Not Random) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω be a Γ-invariant percolation on G. Then the speed Λ of delayed simple random walk on ω exists and is an ω-measurable random variable (possibly zero).*

If ω has indistinguishable components and is ergodic, then, conditioned on $|K(o)| = \infty$, Λ is equal a.s. to a constant.

The same statements hold for simple random walk in place of delayed simple random walk.

Proof. Let $f_n(\hat{Z}) := \mathrm{dist}_G(o, Z(n))$. Then

$$f_{n+m}(\hat{Z}) \leqslant f_n(\hat{Z}) + f_m(\mathcal{S}^n \hat{Z})$$

by the triangle inequality. Consequently, the Subadditive Ergodic Theorem shows that the speed

$$\Lambda = \Lambda(\hat{Z}) = \lim_{n\to\infty} f_n(\hat{Z})/n = \lim_{n\to\infty} \mathrm{dist}_G\big(o, Z(n)\big)/n$$

exists a.s.

To show that the speed $\Lambda(\hat{Z})$ depends only on ω and not on the path of the random walk a.s., define $F(\hat{Z})$ to be the variance, conditioned on ω, of the speed of an independent random walk starting from $Z(0)$. By Lévy's 0-1 Law, $F(\mathcal{S}^n \hat{Z})$ converges to zero a.s. But by stationarity, the distribution of $F(\mathcal{S}^n \hat{Z})$ is the same for all n. Hence, it is 0.

The statement concerning indistinguishable components is a consequence of the definition.

The same proof applies to simple random walk since the measures μ and μ' of Lemma 4.1 are mutually absolutely continuous. ∎

Our main tool to convert the geometric information of Section 3 to probabilistic information is the following:

THEOREM 4.3. (SPEED WHEN THERE IS A SUBGRAPH WITH $\iota_E > 0$) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let $\omega' \subset \omega$ be percolations on G such that the distribution of the pair (ω', ω) is Γ-invariant. Suppose that $\omega' \neq \emptyset$ and $\iota_E(\omega') > 0$ a.s. Then simple random walk on ω has positive speed a.s. on the event that $K(o) \cap \omega' \neq \emptyset$.*

Proof. Let V^* be the vertices of ω' that are in the ω-component of o, and let Z be delayed simple random walk on ω starting at o. Note that given ω, Z is reversible with uniform stationary distribution. Given ω and ω' with $o \in \omega'$, there is an induced walk Z^* on V^* defined as follows. Set $t_0 := 0$. Since the transformed $\hat{Z}$ is stationary (Lemma 4.1), the Poincaré recurrence theorem (see, e.g., Petersen (1983), p. 34) shows that conditioned on $o \in \omega'$, there is a.s. some first time $t_1 > 0$ such that $Z(t_1) \in \omega'$. (Strictly speaking, Lemma 4.1 does not apply when there is an extra "scenery" ω', but the lemma extends easily to this situation; see Lyons and Schramm (1998).) Inductively, for $k > 0$, let t_k be the first time $t > t_{k-1}$ such that $Z(t) \in \omega'$. Define $Z^*(k) := Z(t_k)$. Then given ω and ω' with $o \in \omega'$, Z^* is just the Markov chain Z induced on the states V^*. In particular, it is reversible with the same stationary distribution on V^*, i.e., uniform.

We claim that, given ω and ω' with $o \in \omega'$, the spectral radius $\rho(Z^*)$ is less than 1 a.s. Given two vertices $u^*, v^* \in V^*$, let $p^*(u^*, v^*)$ denote the transition probability of the Markov chain Z^*. Let G^* be the graph whose vertices are V^* and whose edges $[u^*, v^*]$ are those pairs with $p^*(u^*, v^*) > 0$. Note that there is some positive lower bound $c > 0$ for $p^*(u^*, v^*)$ whenever $[u^*, v^*] \in \omega'$. Consequently,

$$\inf\left\{\frac{1}{|K^*|} \sum_{e^* \in \partial_E K^*} p^*(e^*) :\ K^* \subset V(G^*) \text{ is finite}\right\} \geqslant c\iota_E(\omega') > 0\,.$$

Since the stationary distribution is uniform, this implies that $\rho(Z^*) < 1$ a.s. (see, e.g., Kaimanovich (1992)), as claimed.

Fix any $\rho_0 < 1$ such that $\rho(Z^*) < \rho_0$ with positive probability, and let $\mathcal{A}$ be the event that $o \in \omega'$ and $\rho(Z^*) < \rho_0$. Then, for some $\zeta < 1$ and all $v \in V(G)$,

$$\mathbf{P}\big[Z(t_k) = v \bigm| \omega, \omega'\big] \leqslant \zeta^k \quad \text{on } \mathcal{A}\,,$$

which gives

$$\mathbf{P}\big[Z(t_k) = v \bigm| \mathcal{A}\big] \leqslant \zeta^k\,. \tag{4.1}$$

Since the number of vertices $v \in V(G)$ with $\mathrm{dist}_G(o, v) < r$ is bounded by $(\deg_G o)^{r+1}$, by summing (4.1) over all such vertices, we get

$$\mathbf{P}\Big[\mathrm{dist}_G\big(o, Z(t_k)\big) < r \bigm| \mathcal{A}\Big] \leqslant \zeta^k (\deg_G o)^{r+1}.$$

Let β be such that $(\deg_G o)^\beta = 1/\zeta$, and choose $r := (\beta/2)k - 1$. Then

$$\mathbf{P}\Big[\mathrm{dist}_G\big(o, Z(t_k)\big) < \beta k/2 - 1 \;\big|\; \mathcal{A}\Big] \leqslant \zeta^{k/2}. \tag{4.2}$$

By the Borel-Cantelli lemma, it follows that

$$\liminf_{k\to\infty} \mathrm{dist}_G\big(o, Z(t_k)\big)/k \geqslant \beta/2$$

a.s. on $\mathcal{A}$. Also, the ergodic theorem ensures that $\lim t_k/k < \infty$ a.s., whence

$$\liminf_{k\to\infty} \mathrm{dist}_G\big(o, Z(t_k)\big)/t_k > 0$$

a.s. on $\mathcal{A}$ and hence a.s. when $K(o) \cap \omega' \neq \emptyset$. This shows that the speed of Z is positive a.s. when $K(o) \cap \omega' \neq \emptyset$ by Lemma 4.2. By the obvious coupling of delayed random walk and simple random walk, it follows that also the speed of simple random walk is positive a.s. when $K(o) \cap \omega' \neq \emptyset$. ∎

Theorem 4.4. (Speed) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and suppose that $\iota_E(G) > 0$. Let ω be a Γ-invariant percolation on G. Then simple random walk on some infinite cluster of ω has positive speed with positive probability in each of the following cases:*

(i) *ω is Bernoulli percolation that has infinite components a.s.;*

(ii) *ω has a unique infinite cluster a.s.;*

(iii) *ω has a cluster with at least three ends with positive probability;*

(iv) $\mathbf{E}[\deg_\omega o \mid o \in \omega] > \alpha(G)$.

Proof. This follows from Theorems 3.9 and 4.3. ∎

In case G is a tree, (iii) and (iv) of this theorem were established by Häggström (1997).

In case the percolation is ergodic and has indistinguishable components, like Bernoulli percolation, we have the stronger conclusion that simple random walk has positive speed on every infinite component a.s.

Remark 4.5. It does not suffice in Theorem 4.4 to drop in (i) the assumption that ω is Bernoulli. For example, if ω is the wired uniform spanning forest (see Section 5), then every component is a tree with one end (BLPS (1998)), whence is recurrent.

In order to derive additional consequences of Theorem 4.4, we now extend Theorem 1.6:

Lemma 4.6. *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \operatorname{Aut}(G)$. Let ω be a Γ-invariant bond percolation on G. The following are equivalent:*

(i) *the speed of simple random walk $X(t)$ on $K(o)$ is zero a.s. in the G-metric:*

$$\lim_{t\to\infty} \operatorname{dist}_G\big(X(t)\big)/t = 0\,;$$

(ii) *the speed of simple random walk $X(t)$ on $K(o)$ is zero a.s. in the ω-metric:*

$$\lim_{t\to\infty} \operatorname{dist}_\omega\big(X(t)\big)/t = 0\,;$$

(iii) *the asymptotic entropy of simple random walk on $K(o)$ is zero a.s.;*

(iv) *there are no nonconstant bounded harmonic functions on $K(o)$ a.s.*

Proof. Because of Lemma 4.1, the equivalence of (iii) and (iv) follows from Kaimanovich and Woess (1998). Clearly, (ii) implies (i). We show that (i) implies (iii) implies (ii).

Assume (i). Fix ω such that the speed on $K(o)$ is zero. Let μ_t^ω denote the law of $X(t)$ on $K(o)$. Let B_r denote the ball of radius $\lfloor r \rfloor$ in G centered at o. Given $\epsilon > 0$, choose t_0 large enough that for all $t \geqslant t_0$, we have $\mu_t^\omega(B_{t\epsilon}) \geqslant 1-\epsilon$. Let $D := \deg_G o$. Then for $t \geqslant t_0$, concavity of log gives the inequality

$$\sum_{x\in B_{t\epsilon}} -\mu_t^\omega(x)\log\mu_t^\omega(x) \leqslant \mu_t^\omega(B_{t\epsilon})\log\big(|B_{t\epsilon}|/\mu_t^\omega(B_{t\epsilon})\big) \leqslant \log\big(D^{t\epsilon}/(1-\epsilon)\big)\,.$$

Similarly,

$$\sum_{x\notin B_{t\epsilon}} -\mu_t^\omega(x)\log\mu_t^\omega(x) = \sum_{x\in B_t - B_{t\epsilon}} -\mu_t^\omega(x)\log\mu_t^\omega(x) \leqslant \epsilon\log(D^t/\epsilon)\,.$$

Since this holds for all $t \geqslant t_0$ and ϵ was arbitrary, (iii) follows.

Now assume that (ii) does not hold. Let $\mathcal{A}_\ell$ be the event that the speed is at least ℓ, and note that $\mathcal{A}_\ell$ is ω measurable, by Lemma 4.2. Then by the famous bound of Varopoulos (1985) and Carne (1985), we have

$$\lim_{t\to\infty} -\frac{1}{t}\log\mu_t^\omega\big(X(t)\big) \geqslant \ell^2/2$$

on $\mathcal{A}_\ell$. In other words, (iii) does not hold. ∎

Corollary 4.7. *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \operatorname{Aut}(G)$. Suppose that G is nonamenable. Let ω be any Γ-invariant, ergodic, insertion* *tolerant percolation that has more than one infinite component a.s. Then every infinite component of ω admits nonconstant bounded harmonic Dirichlet functions.*

Proof. By Theorem 2.2 and ergodicity, it suffices to establish the existence of nonconstant bounded harmonic Dirichlet functions on $K(o)$ with positive probability.

We know from insertion tolerance and ergodicity that there are infinitely many infinite components a.s. By insertion tolerance again, we also have that some, hence all, infinite components have at least three ends. By Theorem 4.4, all infinite components are transient. By insertion tolerance, it follows that with positive probability, $K(o)$ has a finite subset K whose removal breaks $K(o)$ into at least two transient components. In such a case, $K(o)$ has nonconstant bounded harmonic Dirichlet functions (e.g., the probability that a simple random walk starting at v eventually stays in a fixed transient component of $K(o) - K$ is such, as a function of v). See Soardi (1994), Theorems 4.20 and 3.73. This establishes our goal. ∎

In order to prove transience in certain amenable cases, we shall use:

LEMMA 4.8. (TRANSIENCE OF BIG TREES) *If T is any locally finite tree with $p_c(T) < 1$, then simple random walk is transient on T.*

Proof. By Lyons (1990), the branching number of T is $1/p_c(T)$ and this is the critical value for transience of biased random walk on T. Since this is larger than 1, it follows that, in particular, simple random walk is transient. ∎

For a Cayley graph G, let ζ_n be the number of elements of G at distance n from o. It is evident that $\langle \zeta_n \rangle$ is submultiplicative, whence the **growth rate** $\mathrm{gr}(G) := \lim \zeta_n^{1/n} = \inf \zeta_n^{1/n}$ exists.

THEOREM 4.9. (TRANSIENCE ABOVE THE RECIPROCAL GROWTH RATE) *Let G be a Cayley graph with $\mathrm{gr}(G) > 1$, and let $p \in (1/\mathrm{gr}(G), 1)$. Then simple random walk is transient on every infinite cluster of p-Bernoulli percolation a.s.*

Proof. Let ω be p-Bernoulli percolation. By ergodicity and indistinguishability of components, it suffices to prove transience of $K(o)$ with positive probability. As shown in Lyons (1995), there is a tree $T \subset G$ with $p_c(T) = 1/\mathrm{gr}(G)$. This means that the component ω' of o in $\omega \cap T$ has $p_c(\omega') < 1$ with positive probability, whence by Lemma 4.8, ω is transient with positive probability. By Rayleigh monotonicity, the same is true of the component of o in G. ∎

The following conjecture would imply Conjecture 1.5 (by taking $\omega := G$ and using Lemma 4.6):

Conjecture 4.10. (Monotonicity of Entropy) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω and ω' be two Γ-invariant percolations on G such that $\omega' \subseteq \omega$. Then the asymptotic entropy of delayed simple random walk on ω' is at most the asymptotic entropy of delayed simple random walk on ω.*

The following conjecture for finite graphs can be shown to imply Conjecture 4.10.

Conjecture 4.11. *Let G be a finite graph and $C : E(G) \to \mathbb{R}^+$. Consider the continuous-time (reversible) Markov chain $\langle X(t) \rangle$ on $V(G)$ whose transition rate from u to v is $C(u, v)$. Let $h_t(v, C)$ be the entropy of $X(t)$ when $X(0) = v$ and $h_t(C) := \sum_{v \in V(G)} h_t(v, C)$. Then for all t, given two functions C' and C with $C'(e) \leqslant C(e)$ for all $e \in E(G)$, we have $h_t(C') \leqslant h_t(C)$.*

Here is an equivalent formulation of this conjecture. Given a matrix B, let $H(B)$ be the sum of $-b_{i,j} \log b_{i,j}$ over all entries $b_{i,j}$ of the matrix. Let $\mathcal{A}_n$ be the space of $n \times n$ real symmetric matrices with non-negative off-diagonal terms and with each row summing to zero. Then a reformulation of Conjecture 4.11 is that $H(\exp A)$ is (weakly) monotone increasing in the off-diagonal entries of A, where A ranges in $\mathcal{A}_n$.

§5. Harmonic Dirichlet Functions.

In this section, we study the existence of nonconstant harmonic Dirichlet functions on percolation components.

We first describe the spanning forest measures we use. A **spanning tree** of a finite graph is a subgraph without cycles that is connected and includes every vertex of the graph. Motivated by some questions of R. Lyons, Pemantle (1991) showed that if an infinite graph G is exhausted by finite subgraphs G_n, then the uniform distributions on the spanning trees of G_n converge weakly to a measure supported on spanning forests* of G. We call this the **free uniform spanning forest** (FSF), since there is another natural construction where the exterior of G_n is identified to a single vertex ("wired") before passing to the limit. This second construction, which we call the **wired uniform spanning forest** (WSF), was implicit in Pemantle's paper and was made explicit by Häggström (1995). Both measures are concentrated on the set of forests, all of whose trees are infinite. See BLPS (1998) or Lyons (1998) for an exposition and more details. For convenience, we will use the symbols FSF and WSF also for the uniform measure on spanning trees of a finite graph. For the

* In graph theory, "spanning forest" usually means a maximal subgraph without cycles, i.e., a spanning tree in each connected component. We mean, instead, a subgraph without cycles that contains every vertex.

proof of Theorem 5.7, we shall have need of one more measure on infinite transient graphs, the **oriented wired spanning forest**, denoted OWSF. We refer to BLPS (1998) for its definition. For our purposes, it is enough to know that it is the same as WSF, except that each edge in the forest is oriented in such a way that there is exactly one outgoing edge from each vertex. All these measures, FSF, WSF, and OWSF, are invariant under $\mathrm{Aut}(G)$. We typically denote the random spanning forest by $\mathfrak{F}$.

The following three lemmas are taken from BLPS (1998):

LEMMA 5.1. ($\mathcal{O}_{\mathbf{HD}}$ CRITERION) *For any (connected) graph G, we have $\mathsf{FSF} = \mathsf{WSF}$ iff $G \in \mathcal{O}_{\mathbf{HD}}$.*

Write $\mathbf{E}^G_{\mathsf{WSF}}$, $\mathbf{E}^G_{\mathsf{FSF}}$ for expectation with respect to the random spanning forests on G.

LEMMA 5.2. (DOMINATION) *For any graph G, we have $\mathbf{E}^G_{\mathsf{FSF}}[\deg_{\mathfrak{F}} v] \geqslant \mathbf{E}^G_{\mathsf{WSF}}[\deg_{\mathfrak{F}} v]$ for every $v \in V$, with equality for every v iff $\mathsf{FSF} = \mathsf{WSF}$.*

LEMMA 5.3. (WSF-EXPECTED DEGREE) *In any infinite transitive graph G, the WSF-expected degree of every vertex is 2.*

The following lemma is from BLPS (1999):

LEMMA 5.4. (SMALL TREES AND EXPECTED DEGREE) *Let Γ be a closed unimodular subgroup of $\mathrm{Aut}(G)$ that acts transitively on G and let ω be the configuration of a Γ-invariant percolation on G. Fix a vertex o. Let F_o be the event that $K(o)$ is an infinite tree with finitely many ends, and let F'_o be the event that $K(o)$ is a finite tree.*

(i) *If $P[F_o] > 0$, then $\mathbf{E}\big[D(o) \bigm| F_o\big] = 2$.*
(ii) *If $P[F'_o] > 0$, then $\mathbf{E}\big[D(o) \bigm| F'_o\big] < 2$.*

Fix any base point $o \in V(G)$ and let $\mathcal{A}_o$ be the event that $K(o)$ is infinite. Let $\mathbf{E}$ refer to the probability measure of the percolation. Extending the notation above, we write $\mathbf{E}^\omega_{\mathsf{FSF}}$ and $\mathbf{E}^\omega_{\mathsf{WSF}}$ for expectation with respect to the free and wired spanning forest measures on ω (given ω).

THEOREM 5.5. ($\mathcal{O}_{\mathbf{HD}}$ STABILITY WHEN AMENABLE) *Let G be an amenable graph with a transitive automorphism group $\Gamma \subset \mathrm{Aut}(G)$ and ω a Γ-invariant percolation. Then a.s. every component of ω is in $\mathcal{O}_{\mathbf{HD}}$.*

Proof. We show that the measures FSF and WSF coincide on $K(o)$ a.s. given $\mathcal{A}_o$. By the argument in Burton and Keane (1989), a.s. no component in any invariant percolation on G can have more than 2 ends. Applying this to the percolations given by taking the FSF

or the WSF of each component (independently for each component) of ω in conjunction with Lemma 5.4, we obtain that

$$\mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{WSF}}[\deg_{\mathfrak{F}} o] \bigm| \mathcal{A}_o\big] = 2 = \mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] \bigm| \mathcal{A}_o\big] .$$

Hence, the result follows from Lemma 5.2. ∎

LEMMA 5.6. (EXPECTED DEGREE FOR RECURRENT TREES) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Let ω be a Γ-invariant random forest in G. Suppose that a.s. all components of ω are recurrent. Then $\mathbf{E}[\deg_\omega o] \leqslant 2$.*

Proof. This follows from Lemmas 3.7, 5.4, and 4.8. ∎

THEOREM 5.7. ($\neg\mathcal{O}_{\mathbf{HD}}$ STABILITY WHEN HIGH MARGINALS) *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. If $G \notin \mathcal{O}_{\mathbf{HD}}$, then there is some $p_0 < 1$ such that for every Γ-invariant bond percolation ω with $\inf_{e\in E} \mathbf{P}[e \in \omega] > p_0$, some component of ω is not in $\mathcal{O}_{\mathbf{HD}}$ with positive probability. If ω is ergodic and has indistinguishable components, then the same hypotheses imply that a.s., no infinite component of ω is in $\mathcal{O}_{\mathbf{HD}}$.*

Proof. We show that $p_0 := 2/\mathbf{E}^{G}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o]$ works.

First, $p_0 < 1$ by Lemmas 5.1, 5.2, and 5.3. Let $\mathcal{T}_o$ be the event that $K(o)$, the component of o in ω, is transient. We claim that

$$\mathbf{P}[\mathcal{T}_o] > 0 \tag{5.1}$$

and

$$\mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{WSF}}[\deg_{\mathfrak{F}} o] \bigm| \mathcal{T}_o\big] = 2 < \mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] \bigm| \mathcal{T}_o\big] . \tag{5.2}$$

This suffices for the first statement by Lemma 5.2. The second statement then follows by ergodicity.

Let $\mathfrak{F}'$ be the union of components of $\mathfrak{F}$ that are contained in recurrent components of ω. Lemma 5.6 implies that if $\mathbf{P}[\mathcal{T}_o] < 1$, then $\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o \mid o \in \mathfrak{F}'] \leqslant 2$, which means $\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o \mid \neg\mathcal{T}_o] \leqslant 2$. Consequently, (5.1) and the inequality in (5.2) will be established once we prove

$$\mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o]\big] > p_0 \mathbf{E}^{G}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] = 2 .$$

Now

$$\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] = \sum_{x\sim o} \mathbf{P}^{\omega}_{\mathsf{FSF}}\big[[o,x] \in \mathfrak{F}\big] = \sum_{\substack{x\sim o \\ [o,x]\in\omega}} \mathbf{P}^{\omega}_{\mathsf{FSF}}\big[[o,x] \in \mathfrak{F}\big] . \tag{5.3}$$

Let B_n be the ball of radius n centered at o in G. By Kirchhoff's theorem and Rayleigh's monotonicity principle (see Lyons and Peres (1998) or BLPS (1998)), for each n and each $e \in \omega$,

$$\mathbf{P}^{\omega \cap B_n}_{\mathsf{FSF}}[e \in \mathfrak{F}] \geqslant \mathbf{P}^{B_n}_{\mathsf{FSF}}[e \in \mathfrak{F}] .$$

Taking a limit as $n \to \infty$, we obtain

$$\mathbf{P}^{\omega}_{\mathsf{FSF}}[e \in \mathfrak{F}] \geqslant \mathbf{P}_{\mathsf{FSF}}[e \in \mathfrak{F}]$$

by the definition of the FSF, whence (5.3) gives

$$\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] \geqslant \sum_{\substack{x \sim o \\ [o,x] \in \omega}} \mathbf{P}^{G}_{\mathsf{FSF}}[e \in \mathfrak{F}] .$$

Taking expectation, we obtain

$$\begin{aligned} \mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o]\big] &\geqslant \mathbf{E}\left[\sum_{x \sim o} \mathbf{1}_{[o,x] \in \omega} \mathbf{P}^{G}_{\mathsf{FSF}}\big[[o,x] \in \mathfrak{F}\big]\right] \\ &= \sum_{x \sim o} \mathbf{P}\big[[o,x] \in \omega\big] \mathbf{P}^{G}_{\mathsf{FSF}}\big[[o,x] \in \mathfrak{F}\big] \\ &> \sum_{x \sim o} p_0 \mathbf{P}^{G}_{\mathsf{FSF}}\big[[o,x] \in \mathfrak{F}\big] \\ &= p_0 \mathbf{E}^{G}_{\mathsf{FSF}}[\deg_{\mathfrak{F}} o] , \end{aligned}$$

as desired.

For the equality in (5.2), we use the oriented wired spanning forest, OWSF, on each transient component of ω, chosen independently on each component. Let $\varphi(x, y)$ be the probability that ($K(x)$ is transient and that) $[x, y]$ belongs to the oriented wired spanning forest of ω. Since OWSF is Γ-invariant, φ is invariant under the diagonal action of Γ, whence the Mass-Transport Principle says that

$$\sum_x \varphi(o, x) = \sum_x \varphi(x, o) .$$

The left-hand side is the expected outdegree of o, which is $\mathbf{P}[\mathcal{T}_o]$. Hence, the right-hand side, the expected in-degree of o, is also $\mathbf{P}[\mathcal{T}_o]$. This shows that $\mathbf{E}\big[\mathbf{E}^{\omega}_{\mathsf{WSF}}[\deg_{\mathfrak{F}} o] \bigm| \mathcal{T}_o\big] = 2$. ∎

Example 5.8. It does not suffice in Theorem 5.7 to assume merely that the components of ω are infinite. For example, if ω is given by the WSF, then every component is a tree with one end (BLPS (1998)), whence is recurrent and in $\mathcal{O}_{\mathbf{HD}}$. However, as stated in Conjecture 1.11, we believe that this *is* sufficient for Bernoulli percolation.

EXAMPLE 5.9. The hypothesis that Γ be unimodular cannot be omitted in Theorem 5.7. For example, let G be a regular tree of degree 3 and ξ be an end of G. Let Γ be the group of automorphisms of G that fix ξ. Let H_n, $n \in \mathbb{Z}$, be the horocycles with respect to ξ. (More precisely, fix a base point $o \in V$, let $\langle v_m \rangle$ be a sequence converging to ξ. Then a vertex v is in H_n iff $\text{dist}(v_m, v) - \text{dist}(v_m, o) = n$ for all but finitely many m.) To define ω, we first define a percolation η. Given any $p_0 < 1$, for each n independently, let all the edges joining H_n to H_{n+1} be in η with probability p_0. Each component of η is a finite tree a.s. For each component K of η, let $n(K)$ be the largest n such that $K \cap H_n \neq \emptyset$. Choose an edge joining $K \cap H_{n(K)}$ to $H_{n(K)+1}$ at random uniformly among all such edges and independently for each K; let η' be the set of the chosen edges (over all K). Now let $\omega := \eta \cup \eta'$. Each component of ω is a tree with exactly one end, so is recurrent and in $\mathcal{O}_{\mathbf{HD}}$. Yet $G \notin \mathcal{O}_{\mathbf{HD}}$.

QUESTION 5.10. Does Theorem 5.7 hold for Bernoulli percolation when the unimodularity assumption is omitted?

§6. Anchored Expansion and Stability.

Cheeger's inequality relates the isoperimetric constant, which is geometric, to the spectral radius, which governs the exponential decay of return probabilities of simple random walk or Brownian motion. We discuss a geometric constant that we hope can replace the isoperimetric constant in graphs and manifolds that are not uniformly expanding.

Consider, for example, the hyperbolic space $\mathbb{H}^n$ and perturb the metric on an extremely sparse sequence of balls with radii growing very slowly to infinity; for instance, pick the center of the n-th ball at distance e^n from a fixed origin, and let $\log n$ be its radius. If we modify the metric inside these balls so that it is flat on sub-balls of half the radius, we get a manifold with zero isoperimetric constant. Many properties of M (such as the existence of nonconstant bounded harmonic functions or the speed of Brownian motion), are, however, unchanged from $\mathbb{H}^n$.

DEFINITION 6.1. Fix some base point $o \in G$. Call

$$\iota_E^*(G) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial_E S|}{|S|} : o \in S \subset V(G),\ S \text{ is connected},\ n \leqslant |S| < \infty \right\}$$

the **anchored expansion constant** of G. Note that $\iota_E^*(G)$ is independent of the choice of the base point o and that $\iota_E^*(G) \geqslant \iota_E(G)$.

By attaching a sequence of paths of length $1, 2, \ldots$ at a very sparse sequence of vertices of a binary tree, we get an example of a graph G for which $\iota_E^*(G) > \iota_E(G) = 0$.

Isoperimetric quantities such as this, where the set S is required to contain a fixed vertex o, were introduced in greater generality by Thomassen (1992). He used the word "rooted" instead of "anchored".

Benjamini and Schramm (1996) show that $\iota_E(G)$ is related to $p_c(G)$; in fact, their proof of Thm. 2 shows that $p_c(G) \leqslant (1 + \iota_E^*(G))^{-1}$.

When the isoperimetric constant $\iota_E(G)$ of a bounded degree (not necessarily transitive) graph is positive, Dodziuk's (1984) discrete version of Cheeger's inequality gives an upper bound $\overline{\rho} < 1$ for the spectral radius $\rho(G)$, where $\overline{\rho}$ depends on $\iota_E(G)$ and the maximum degree in G. In turn, this implies that the lim inf speed Λ^- of simple random walk X starting at a base point $o \in V(G)$ is positive almost surely.

By analogy, we make the following

CONJECTURE 6.2. *Let G be a bounded degree graph with $\iota_E^*(G) > 0$. Then $\Lambda^- > 0$ with positive probability.*

It might even be the case that $\iota_E^*(G) > 0$ implies $\Lambda^- > 0$ a.s. In fact, after seeing an earlier draft of this paper, Virag (1998) proved that the limsup speed is positive when G is a tree of bounded degree with $\iota_E^*(G) > 0$. He also has shown that whenever $\iota_E^*(G) > 0$, there is a subgraph G' of G with $\iota_E(G') > 0$.

Thomassen (1992) has shown that if a graph satisfies a weaker type of anchored isoperimetric inequality, then it is transient; Conjecture 6.2 has a stronger hypothesis and a stronger conclusion.

The motivation for looking at $\iota_E^*(G)$ is that ι_E^* is more stable than ι_E under random perturbations of G. For example, let G be an infinite graph of bounded degree and pick a probability distribution $\mathbf{P}$ on the strictly positive integers. Replace each edge $e \in G$ by a path of length L_e, where L_e is distributed according to $\mathbf{P}$, and all L_e ($e \in E(G)$) are independent. Let $G^{\mathbf{P}}$ denote the random graph obtained in this way. If $\mathbf{P}$ has a bounded support, then $\iota_E(G) > 0$ implies $\iota_E\left(G^{\mathbf{P}}\right) > 0$, while if $\mathbf{P}$ has unbounded support then, almost surely, $\iota_E\left(G^{\mathbf{P}}\right) = 0$.

QUESTION 6.3. Does $\iota_E(G) > 0$ imply that $\iota_E^*\left(G^{\mathbf{P}}\right) > 0$ a.s. when $\mathbf{P}$ is the geometric distribution on the positive integers? What about other distributions $\mathbf{P}$ with finite mean? Can this be settled in the case where G is a regular tree?

It is not hard to construct $\mathbf{P}$ and G such that $\iota_E(G) > 0$ while $\iota^*_E\left(G^{\mathbf{P}}\right) = 0$ a.s. (Take G to be a binary tree and $\mathbf{P}$ to have a fat tail.)

Lyons, Pemantle, and Peres (1995) proved that simple random walk on a random perturbation of any regular tree, with $\mathbf{P}$ the geometric distribution, has positive speed almost surely. More generally, simple random walk has positive speed on every supercritical Galton-Watson tree a.s. given nonextinction.

QUESTION 6.4. Is $\iota^*_E(T) > 0$ a.s. for supercritical Galton-Watson trees given nonextinction?

QUESTION 6.5. If $\iota_E(G) > 0$ and ω is Bernoulli percolation on G, must every infinite component K of ω have $\iota^*_E(K) > 0$ a.s.?

After seeing an earlier draft of this paper, answers to the last three questions have largely been provided by Chen and Peres (1998). That is, Questions 6.3 and 6.4 were answered completely and Question 6.5 was answered affirmatively for p-Bernoulli percolation when p is sufficiently close to 1.

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