Probabilistic Aspects of Infinite Trees and Some Applications

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Abstract. We begin by using flows to assign a positive real number, called the branching number, to an arbitrary (irregular) infinite locally finite tree. The branching number represents the "average" number of branches per vertex and is the exponential of the dimension of (the boundary of) the tree, as introduced by Furstenberg. There are many senses in which this branching number is an average. We discuss some based variously on electrical networks, random walks, percolation, or tree-indexed (branching) random walks. A refinement of the notion of branching number uses ideas of potential theory. This creates quite precise connections among probabilistic processes on trees.

For applications, we consider the structure of the family trees of branching processes, the Hausdorff dimension and capacities of possibly random fractals, and random walks on Cayley graphs of infinite but finitely generated groups.

Résumé. Nous utilisons d'abord les flots pour associer un réel positif appelé le nombre de branchement à un arbre arbitraire (non régulier) infini, localement fini. Le nombre de branchement représente le nombre "moyen" de branches par sommet et c'est l'exponentielle de la dimension de (la frontière de) l'arbre, déjà introduit par Furstenberg. Ce nombre de branchement est une moyenne en de nombreux sens. Nous en discutons certains, issus des réseaux électriques, des marches aléatoires, de la percolation ou des marches aléatoires (avec branchement) indexées par un arbre. Un raffinement de la notion de nombre de branchement utilise les idées de la théorie du potentiel. Ceci crée des connections tout à fait précises entre les processus probabilistes sur les arbres.

Comme applications, nous considérons la structure des arbres généalogiques des processus de branchement, la dimension de Hausdorff et les capacités de fractals éventuellement aléatoires, et les marches aléatoires sur des graphes de Cayley de groupes infinis finiment engendrés.

1991 Mathematics Subject Classification. Primary 60J15, 60J80, 60K35, 05C05. Secondary 60D05, 82A43, 05C80, 60B15.
Key words and phrases. Galton-Watson, branching processes, Hausdorff dimension, capacity, Cayley graph, finitely generated group, electrical networks, random walks, percolation, fractals.
This talk was given at the Colloque Arbres in Versailles, May 1995.

§**1. Branching Number.** How do we assign an average branching number to an arbitrary infinite locally finite tree? If the tree is a binary tree, as in Figure 1, then clearly the answer will be "2". But in the general case, since the tree is infinite, no straight average is available. We must take some kind of limit or use some other procedure.

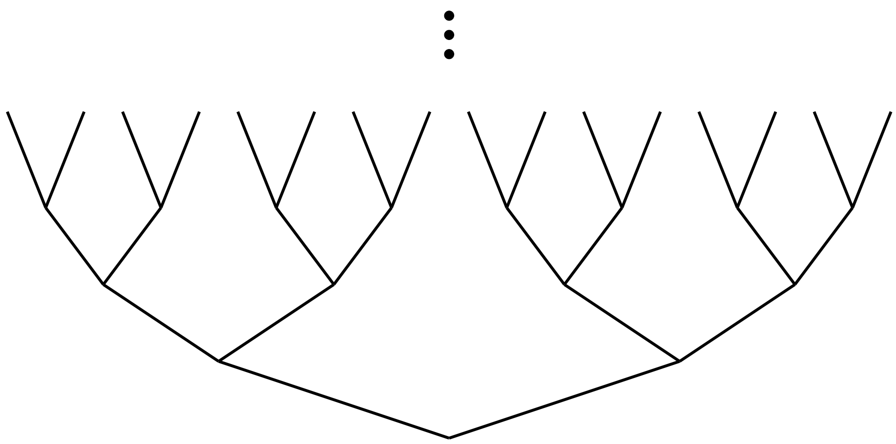

Figure 1. The binary tree.

One simple idea is as follows. Let S_n be the set of vertices at distance n from the root, 0. Define the **growth rate** of the tree to be

$$\operatorname{gr} T := \liminf_{n\to\infty} |S_n|^{1/n} .$$

This certainly will give the number "2" to the binary tree. However, notice that it barely accounts for the structure of the tree: only $|S_n|$ matters, not how the vertices at different levels are connected to each other.

Thus, we shall use a different approach. Consider the tree as a network of pipes and imagine water entering the network at the root. However much water enters a pipe leaves at the other end and splits up among the outgoing pipes (edges). Consider the following sort of restriction: Given $\lambda \geq 1$, suppose that the amount of water that can flow through an edge at distance n from 0 is only λ^{-n}. If λ is too big, then perhaps no water can flow. In fact, consider the binary tree. A moment's thought shows that water can still flow throughout the tree provided that $\lambda \leq 2$, but that as soon as $\lambda > 2$, then no water at all can flow. Obviously, this critical value of 2 for λ is the same as the branching number of the binary tree. So let us make a general definition: the **branching number** of a tree T is the supremum of those λ that admit a positive amount of water to flow through T; denote this critical value of λ by $\operatorname{br} T$. As we shall see, this definition is the exponential of what Furstenberg (1970) called the "dimension" of a tree, which is the Hausdorff dimension of its boundary.

It is not hard to check that $\operatorname{br} T$ is related to $\operatorname{gr} T$ by

$$\operatorname{br} T \leq \operatorname{gr} T\,. \tag{1}$$

Often, as in the case of the binary tree, equality holds here. However, there are many examples of strict inequality.

EXAMPLE: If T is a tree such that vertices at even distances from 0 have 2 children while the rest have 3 children, then $\operatorname{br} T = \operatorname{gr} T = \sqrt{6}$.

EXAMPLE: Let T be a tree embedded in the upper half plane with 0 at the origin. List S_n in clockwise order as $\langle v_1^n, \ldots, v_{2^n}^n \rangle$. Let v_k^n have 1 child if $k \leq 2^{n-1}$ and 3 children otherwise; see Figure 2. Then only one ray (infinite path that doesn't backtrack) consists of vertices having more than one child. This means that water cannot flow down any other ray when $\lambda > 1$, which means it cannot flow down this ray either. Hence $\operatorname{br} T = 1$, yet $\operatorname{gr} T = 2$.

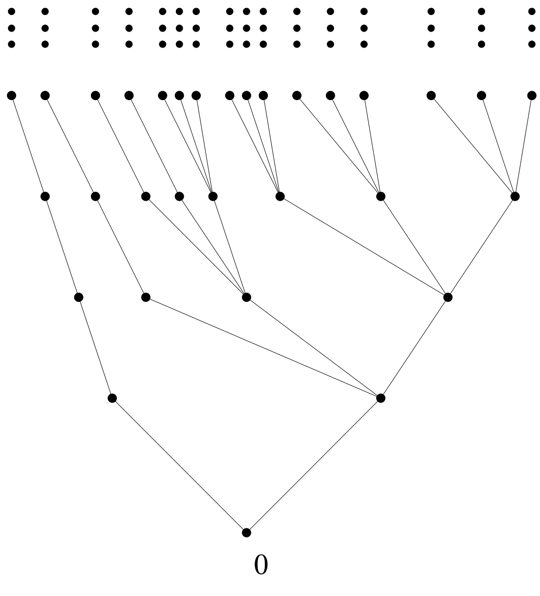

Figure 2. A tree with branching number 1 and growth rate 2.

EXAMPLE: If T_1 and T_2 are trees, form a new tree $T_1 \vee T_2$ from disjoint copies of T_1 and T_2 by joining their roots to a new point taken as the root of $T_1 \vee T_2$ (Figure 3). Then

$$\operatorname{br}(T_1 \vee T_2) = \operatorname{br} T_1 \vee \operatorname{br} T_2$$

since water can flow in the join $T_1 \vee T_2$ iff water can flow in one of the trees.

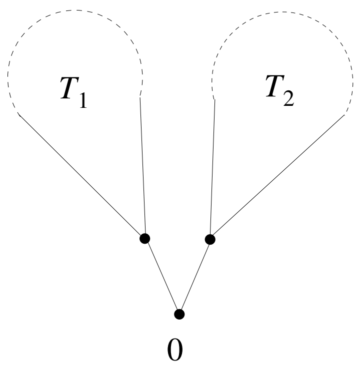

Figure 3. Joining two trees.

EXAMPLE: We will put two trees together such that $\mathrm{br}\,(T_1 \vee T_2) = 1$ but $\mathrm{gr}\,(T_1 \vee T_2) > 1$. Let $n_k \uparrow \infty$. Let T_1 (resp., T_2) be a tree such that v has 1 child (resp., 2 children) for $n_{2k} \le |v| \le n_{2k+1}$ and 2 (resp., 1) otherwise. If n_k increases sufficiently rapidly, then $\mathrm{br}\,T_1 = \mathrm{br}\,T_2 = 1$, so $\mathrm{br}\,(T_1 \vee T_2) = 1$. But if n_k increases sufficiently rapidly, then $\mathrm{gr}\,(T_1 \vee T_2) = \sqrt{2}$.

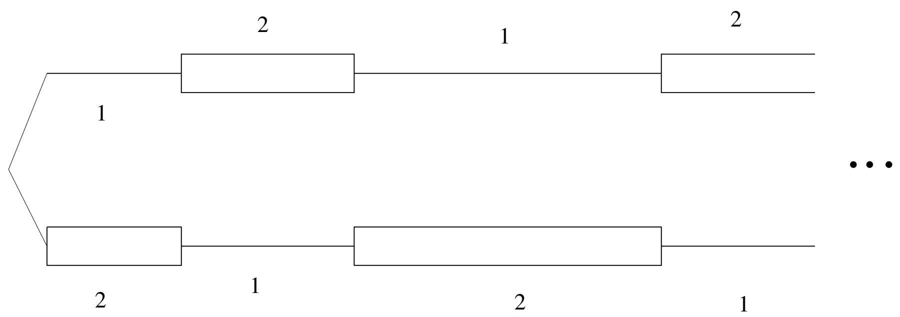

Figure 4. A schematic representation of a tree with branching number 1 and growth rate $\sqrt{2}$.

While $\mathrm{gr}\,T$ is easy to compute, $\mathrm{br}\,T$ may not be. Fortunately, Furstenberg (1967) gave a useful condition sufficient for equality in (1): Given a vertex v in T, let T^v denote the subtree of T formed by the descendants of v. This tree is rooted at v.

THEOREM 2. *If for all vertices $v \in T$, there is an isomorphism of T^v as a rooted tree to a subtree of T rooted at 0, then* $\mathrm{br}\,T = \mathrm{gr}\,T$.

We call trees satisfying the hypothesis of this theorem **subperiodic**. As we shall see, subperiodic trees arise naturally, which accounts for the importance of Theorem 2.

§**2. Electric Current.** We can ask another flow question on trees, namely: If λ^{-n} is the conductance of edges at distance n from the root of T and a battery is connected between the root and "infinity", will current flow? Of course, what we mean is that we establish a unit potential between the root and level N of T, let $N \to \infty$, and see whether the limiting current is positive. If so, the tree is said to have positive effective conductance and finite effective resistance.

EXAMPLE: Consider the binary tree. By symmetry, all the vertices at a given distance from 0 have the same potential, so they may be identified without changing any voltages or currents. This gives a new graph whose vertices may be identified with $\mathbb{N}$, while there are 2^n edges joining $n-1$ to n. These edges are in parallel, so they may be replaced by a single edge whose conductance is their sum, $(2/\lambda)^n$. Now we have edges in series, so the effective resistance is the sum of the edge resistances, $\sum_n (\lambda/2)^n$. This is finite iff $\lambda < 2$. Thus, current flows in the infinite binary tree iff $\lambda < 2$. Note the slight difference to water flow: when $\lambda = 2$, water can still flow on the binary tree.

In general, there will be a critical value of λ below which current flows and above which it does not. It turns out that this critical value is the same as that for water flow (Lyons 1990):

THEOREM 3. *If $\lambda < \operatorname{br} T$, then electrical current flows, but if $\lambda > \operatorname{br} T$, then it does not.*

§**3. Random Walks.** There is a well-known and easily established correspondence between electrical networks and random walks that holds for all graphs. Namely, given a finite connected graph G with conductances assigned to the edges, we consider the random walk which can go from a vertex only to an adjacent vertex and whose transition probabilities from a vertex are proportional to the conductances along the edges to be taken. That is, if v is a vertex with neighbors $u_1, \ldots, u_d$ and the conductance of the edge (v, u_i) is C_i, then $p(v, u_j) = C_j / \sum_{i=1}^{d} C_i$. Now consider two fixed vertices a_0 and a_1 of G. Suppose that a battery is connected across them so that the voltage at a_i equals i $(i = 0, 1)$. Then certain currents will flow along the edges and establish certain voltages at the other vertices in accordance with Kirchhoff's law and Ohm's law. The following proposition provides the basic connection between random walks and electrical networks:

PROPOSITION 4. *For any vertex v, the voltage at v equals the probability that the corresponding random walk visits a_1 before it visits a_0 when it starts at v.*

In fact, the proof of this proposition is simple: there is a discrete Laplacian (a difference operator) for which both the voltage and the probability mentioned are harmonic functions of v. The two functions clearly have the same values at a_i (the boundary points) and the uniqueness principle holds for this Laplacian, whence the functions agree at all vertices v. A superb elementary exposition of this correspondence is given by Doyle and Snell (1984).

What does this say about our trees? Given N, identify all the vertices of level N, i.e., S_N, to one vertex, a_1. Use the root as a_0. Then according to Proposition 4, the voltage at v is the probability that the random walk visits level N before it visits the root when it starts from v. When $N \to \infty$, the limiting voltages are all 0 iff the limiting probabilities are all 0, which is the same thing as saying that on the infinite tree, the probability of visiting the root from any vertex is 1, i.e., the random walk is recurrent. Since no current flows across edges whose endpoints have the same voltage, we see that no electrical current flows iff the random walk is recurrent. Denoting the dependence of the random walk on the parameter λ by RW_λ, we may translate Theorem 3 into the following theorem (Lyons 1990):

Theorem 5. *If $\lambda < \operatorname{br} T$, then RW_λ is transient, while if $\lambda > \operatorname{br} T$, then RW_λ is recurrent.*

Is this intuitive? Consider the relative weights at a vertex other than the root with, say, d children (see Figure 5). The edge leading back toward the root is λ times as likely to be taken as each other edge. Thus, if we consider only the distance from 0, which increases or decreases at each step of the random walk, a balance between increasing and decreasing occurs when $\lambda = d$. If d were constant, it is easy to see that indeed d would be the critical value separating transience from recurrence. What Theorem 5 says is that this same heuristic can be used, provided we substitute the "average" $\operatorname{br} T$ for d.

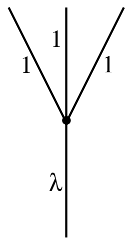

Figure 5. The relative weights at a vertex. The tree is growing upwards.

§4. **Percolation.** Suppose that we remove edges at random from T. To be specific, keep each edge with some fixed probability p and make these decisions independently for different edges. This random process is called **percolation**. By Kolmogorov's 0-1 law, the probability that an infinite connected component remains in the tree is either 0 or 1. On the other hand, this probability is monotonic in p, whence there is a critical value $p_c(T)$

where it changes from 0 to 1. It is also clear that the "bigger" the tree, the more likely it is that there will be an infinite component for a given p. That is, the "bigger" the tree, the smaller the critical value p_c. Thus, p_c is vaguely inversely related to a notion of average branching number. Actually, this vague heuristic is precise (Lyons 1990):

THEOREM 6. *For any tree,* $p_c(T) = 1/\operatorname{br} T$.

Let us look more closely at the intuition behind this. If a vertex v has d children, then the expected number of children after percolation is dp. If dp is "usually" less than 1, one would not expect that an infinite component would remain, while if dp is "usually" greater than 1, then one might guess that an infinite component would be present. Theorem 6 says that this intuition is precise when one replaces d by $\operatorname{br} T$.

Note that there is an infinite component with probability 1 iff the component of the root is infinite with positive probability.

§**5. Random Trees.** Percolation on a fixed tree produces random trees by random pruning, but there is a way to grow trees randomly due to Bienaymé in 1845. Given probabilities p_k adding to 1 ($k = 0, 1, 2, \ldots$), we begin with one individual and let it reproduce according to these probabilities, i.e., it has k children with probability p_k. Each of these children (if there are any) then reproduce independently with the same law, and so on forever or until some generation goes extinct. The family trees produced by such a process are called **(Bienaymé)-Galton-Watson trees**. A fundamental theorem in the subject is that extinction is a.s. iff $m \le 1$ and $p_1 < 1$, where $m := \sum_k kp_k$ is the mean number of offspring. This provides further justification for the intuition we sketched behind Theorem 6. It also raises a natural question: Given that a Galton-Watson tree is nonextinct (infinite), what is its branching number? All the intuition suggests that it is m a.s., and indeed it is. This was first proved by Hawkes (1981) and Falconer (1986). But here is the idea of a very simple proof (Lyons 1990).

According to Theorem 6, to determine $\operatorname{br} T$, we may determine $p_c(T)$. Thus, let T grow according to a Galton-Watson process, then perform percolation on T, i.e., keep edges with probability p. We are interested in the component of the root. Looked at as a random tree in itself, this component appears simply as some other Galton-Watson tree; its mean is mp by independence of the growing and the "pruning" (percolation). Hence, the component of the root is infinite w.p.p. iff $mp > 1$. This says that $p_c = 1/m$ a.s. on nonextinction, i.e., $\operatorname{br} T = m$.

§**6. Branching Random Walk.** Given a tree T, assign i.i.d. real-valued displacements to the edges. Let S_v be the sum of the displacements on the path from 0 to v. Denote the distance from v to 0 by $|v|$. We call a path $v_0, v_1, v_2, \ldots$ starting from the root ($v_0 = 0$) a **ray** when v_{i+1} is a child of v_i for every i. Denote by ξ_n the nth point of a ray ξ. Define the **boundary** of the tree, denoted ∂T, as the set of all rays. We may imagine that the rays represent individuals doing random walks on $\mathbb{R}$. Their paths are identical until certain splitting occurs, after which they are independent (and more splitting occurs). This is called either **tree-indexed random walk** or **branching random walk**.

A central question is how fast this branching random walk goes to infinity. Of course, there are many ways to define speed in this context. Some of them are represented by the following quantities:

$$
\begin{aligned}
A(T) &:= \lim_{n\to\infty} \sup_{|v|\geq n} \frac{S_v}{|v|}\,, \\
B(T) &:= \sup_{\xi\in\partial T} \liminf_{n\to\infty} \frac{S_{\xi_n}}{n}\,, \\
C(T) &:= \sup\left\{ \lim_{n\to\infty} \frac{S_{\xi_n}}{n}\,;\ \xi\in\partial T,\ \lim_{n\to\infty}\frac{S_{\xi_n}}{n} \text{ exists}\right\}.
\end{aligned}
$$

Then clearly

$$A(T) \geq B(T) \geq C(T)\,.$$

Let X be a random variable with the law of the displacements. It has a "rate function"

$$\mu(s) := \inf_{t\leq 0} \mathbf{E}\left[e^{t(s-X)}\right]\,,$$

with an "inverse"

$$\mu_1(a) := \sup\{s\,;\ \mu(s) < a\}\,.$$

A special case of a fundamental theorem of Biggins (1976), often rediscovered, identifies the speed for Galton-Watson trees:

THEOREM 7. *If T is a Galton-Watson tree with mean m, then*

$$A(T) = B(T) = C(T) = \mu_1(1/m)$$

a.s. given nonextinction.

This can be partially generalized to any tree (Lyons and Pemantle 1992):

$$B(T) = C(T) = \mu_1(1/\operatorname{br} T) \quad \text{a.s.}$$

It is not generally true that $A(T)$ has the same value, however.

§7. **Hausdorff Dimension.** Consider again any finite connected graph with two distinguished vertices a_0 and a_1. This time, the edges e have assigned positive numbers $C(e)$ that represent the maximum amount of water that can flow through the edge (in either direction). How much water can flow into a_0 and out of a_1? Consider any set W of edges that separates a_0 from a_1, i.e., the removal of all edges in W would leave a_0 and a_1 in different components. Such a set W is called a **cutset**. Since all the water must flow through W, an upper bound for the maximum flow from a_0 to a_1 is $\sum_{e\in W} C(e)$. The beautiful Max-flow Min-cut Theorem of Ford and Fulkerson (1962) says that these are the only constraints: the maximum flow equals $\inf_{W \text{ a cutset}} \sum_{e\in W} C(e)$.

Applying this theorem to our tree situation, we see that the maximum flow from the root to infinity is

$$\inf\left\{\sum_{v\in\Pi} \lambda^{-|v|}\,;\ \Pi \text{ cuts the root from infinity}\right\}.$$

Here, we identify a set of vertices Π with their preceding edges when considering cutsets.

Now there is a natural metric on ∂T: if $\xi, \eta \in \partial T$ have exactly n edges in common, define their distance to be $d(\xi, \eta) := e^{-n}$. Thus, if $v \in T$ has more than one child with infinitely many descendants, the set of rays going through v,

$$B_v := \{\xi \in \partial T\,;\ \xi_{|v|} = v\}\,,$$

has diameter $|B_v| = e^{-|v|}$. We call a collection $\mathcal{C}$ of subsets of ∂T a **cover** if

$$\bigcup_{B\in\mathcal{C}} B = \partial T\,.$$

Note that Π is a cutset iff $\{B_v\,;\ v \in \Pi\}$ is a cover. The **Hausdorff dimension** of ∂T is defined to be

$$\dim T := \sup\left\{\alpha\,;\ \inf_{\mathcal{C} \text{ a cover}} \sum_{B\in\mathcal{C}} |B|^\alpha > 0\right\}.$$

This is, in fact, already familiar to us, since

$$\begin{aligned}
\operatorname{br} T &= \sup\{\lambda\,;\ \text{water can flow through pipe capacities } \lambda^{-|v|}\} \\
&= \sup\left\{\lambda\,;\ \inf_{\Pi \text{ a cutset}} \sum_{v\in\Pi} \lambda^{-|v|} > 0\right\} \\
&= \exp\sup\left\{\alpha\,;\ \inf_{\Pi \text{ a cutset}} \sum_{v\in\Pi} e^{-\alpha|v|} > 0\right\} \\
&= \exp\sup\left\{\alpha\,;\ \inf_{\mathcal{C} \text{ a cover}} \sum_{B\in\mathcal{C}} |B|^\alpha > 0\right\} \\
&= \exp\dim\partial T\,.
\end{aligned}$$

§8. Capacity. An important tool in analyzing electrical networks is that of energy. Thompson's principle says that, given a finite graph and two distinguished vertices a_0, a_1, the unit current flow from a_0 to a_1 is the unit flow from a_0 to a_1 that minimizes energy, where, if the conductances are $C(e)$, the **energy** of a flow θ is defined to be $\sum_{e \text{ an edge}} \theta(e)^2/C(e)$. (It turns out that the energy of the unit current flow is equal to the effective resistance from a_0 to a_1.) Thus,

$$\begin{aligned} &\text{electrical current flows from the root of an infinite tree} \\ \iff& \\ &\text{there is a flow with finite energy.} \end{aligned} \tag{8}$$

Now on a tree, a unit flow can be identified with a function θ on the vertices of T that is 1 at the root and has the property that for all vertices v,

$$\theta(v) = \sum_i \theta(u_i)\,,$$

where u_i are the children of v. The energy of a flow is then

$$\sum_{v \in T} \theta(v)^2 \lambda^{|v|}\,,$$

whence

$$\operatorname{br} T = \sup \left\{ \lambda\,;\ \text{there exists a unit flow } \theta \quad \sum_{v \in T} \theta(v)^2 \lambda^{|v|} < \infty \right\}. \tag{9}$$

We can also identify unit flows on T with Borel probability measures μ on ∂T via

$$\mu(B_v) = \theta(v)\,.$$

A bit of algebra shows that (9) is equivalent to

$$\operatorname{br} T = \exp \sup \left\{ \alpha\,;\ \exists \text{ a probability measure } \mu \text{ on } \partial T \quad \iint \frac{d\mu(\xi)\,d\mu(\eta)}{d(\xi,\eta)^\alpha} < \infty \right\}.$$

Define **capacity** to be the reciprocal of the minimum energy:

$$\operatorname{cap}_\alpha(\partial T)^{-1} := \inf \left\{ \iint \frac{d\mu(\xi)\,d\mu(\eta)}{d(\xi,\eta)^\alpha}\,;\ \mu \text{ a probability measure on } \partial T \right\}.$$

Then statement (8) says that

$$\text{random walk with parameter } \lambda = e^\alpha \text{ is transient} \iff \operatorname{cap}_\alpha(\partial T) > 0\,. \tag{10}$$

It follows from Theorem 5 that

$$\text{the critical value of } \alpha \text{ for positivity of } \operatorname{cap}_\alpha(\partial T) \text{ is } \dim \partial T. \tag{11}$$

A refinement of Theorem 6 is (Lyons 1992):

THEOREM 12. *Percolation with parameter* $p = e^{-\alpha}$ *yields an infinite component a.s. iff* $\mathrm{cap}_\alpha(\partial T) > 0$. *Moreover,*

$$\mathrm{cap}_\alpha(\partial T) \le \mathbf{P}[\textit{the component of the root is infinite}] \le 2\,\mathrm{cap}_\alpha(\partial T)\,.$$

The case of the first part of this theorem where all the degrees are uniformly bounded was shown first by Fan (1989, 1990).

One way to use this theorem is to combine it with (10); this allows us to translate problems freely between the domains of random walks and percolation (Lyons 1992). The theorem actually holds in a much wider context (on trees) than that mentioned here: Fan allowed the probabilities p to vary depending on the generation and Lyons allowed the probabilities as well as the tree to be completely arbitrary.

§9. Embedding Trees Into Euclidean Space. The results described above, especially those concerning percolation, can be translated to give results on closed sets in Euclidean space. We shall describe only the simplest such correspondence here, which was the one that was part of Furstenberg's motivation in 1970. Namely, for a closed nonempty set $E \subseteq [0,1]$ and for any integer $b \ge 2$, consider the system of b-adic subintervals of $[0,1]$. Those whose intersection with E is non-empty will form the vertices of the associated tree. Two such intervals are connected by an edge iff one contains the other and the ratio of their lengths is b. The root of this tree is $[0,1]$. We denote the tree by $T_b(E)$. Were it not for the fact that certain numbers have two representations in base b, we could identify $\partial T_b(E)$ with E. Note that such an identification would be Hölder continuous in one direction (only).

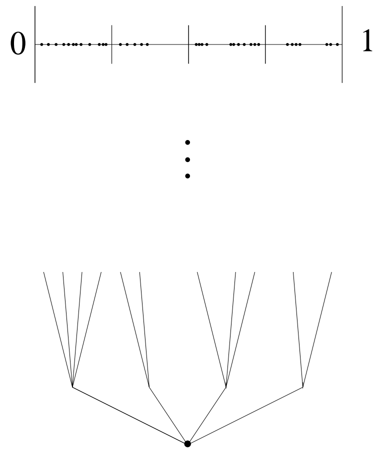

Figure 6. In this case $b = 4$.

Hausdorff dimension is defined for subsets of $[0,1]$ just as it was for ∂T:

$$\dim E := \sup \left\{ \alpha \, ; \inf_{\mathcal{C} \text{ a cover of } E} \sum_{B \in \mathcal{C}} |B|^\alpha > 0 \right\} ,$$

where $|B|$ denotes the (Euclidean) diameter of E. Covers of $\partial T_b(E)$ by sets of the form B_v correspond to covers of E by b-adic intervals. It easy to show that restricting to such covers does not change the computation of Hausdorff dimension, whence we may conclude that

$$\dim E = \frac{\dim \partial T_b(E)}{\log b} = \log_b(\operatorname{br} T_b(E)) .$$

Capacity is also defined as it was on the boundary of a tree:

$$(\operatorname{cap}_\alpha E)^{-1} := \inf \left\{ \int\!\!\int \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \, ; \, \mu \text{ a probability measure on } E \right\} .$$

It was shown by Benjamini and Peres (1992) that

$$(\operatorname{cap}_\alpha E)/3 \le \operatorname{cap}_{\alpha \log b}(\partial T_b(E)) \le b \operatorname{cap}_\alpha E . \tag{13}$$

This means that the percolation criterion Theorem 12 can be used in Euclidean space. In fact, inequalities such as (13) also hold for more general kernels than distance to a power and for higher-dimensional Euclidean spaces. Nice applications to the path of Brownian motion were found by Peres (1994) by replacing the path by an "intersection-equivalent" random fractal that is much easier to analyze, being an embedding of a Galton-Watson tree.

Also, the fact (11) translates into the analogous statement about subsets $E \subseteq [0,1]$; this is a classical theorem of Frostman (1935).

§10. **Cayley Graphs.** Suppose we investigate RW_λ on graphs other than trees. What do we mean? Fix a vertex 0 in a graph G. We always assume that G is connected and that there are only a finite number of edges incident to each vertex. If e is an edge whose endpoint furthest from 0 is at distance n, let the conductance of e be λ^{-n}. The argument we used for trees in Section 3 to see that transience is equivalent to positive current flow also applies to arbitrary graphs. In order to understand what the critical value $\lambda_c(G)$ separating transience from recurrence measures, consider the class of spherically symmetric graphs, where we call G **spherically symmetric** about 0 if for all pairs of vertices v, w at the same distance from 0, there is an isomorphism of G fixing 0 that takes v to w. Let $\tilde{M}_n$

be the number of edges that lead from a vertex at distance $n-1$ from 0 to a vertex at distance n. Then the critical value of λ is the growth rate of G:

$$\lambda_c(G) = \liminf_{n\to\infty} \tilde{M}_n^{1/n} .$$

In fact, we have the following more precise criterion for transience:

PROPOSITION 14. *If G is spherically symmetric about 0, then* RW_λ *is transient iff* $\sum_n \lambda^n/\tilde{M}_n < \infty$.

Proof. Since G is spherically symmetric, all vertices at a given distance from 0 have the same voltage. Therefore, they can be identified without changing any voltages (or currents). This yields the graph $\mathbb{N}$ with multiple edges and loops. We may ignore the loops; and there are $\tilde{M}_n$ edges between $n-1$ and n. Since they are in parallel, they are equivalent to a single edge whose conductance is the sum of the conductances, $\tilde{M}_n\lambda^{-n}$. These new edges are in series, so they are equivalent to a single edge (from 0 to infinity) with resistance the sum of the resistances, $\sum_n \lambda^n/\tilde{M}_n$. This gives the proposition. ∎

Next, consider the **Cayley graphs** of finitely generated groups: if G is a group generated by $\{g_1, \ldots, g_k\}$ as a semigroup, then we form a graph with vertices G and edges $\{(v, vg_i);\ v \in G,\ 1 \le i \le k\}$. Because $\{g_i\}$ generates G, the graph is connected. It is also **homogeneous**, that is, its automorphism group is transitive on vertices. Some examples of Cayley graphs with natural generating sets appear in Figure 7.

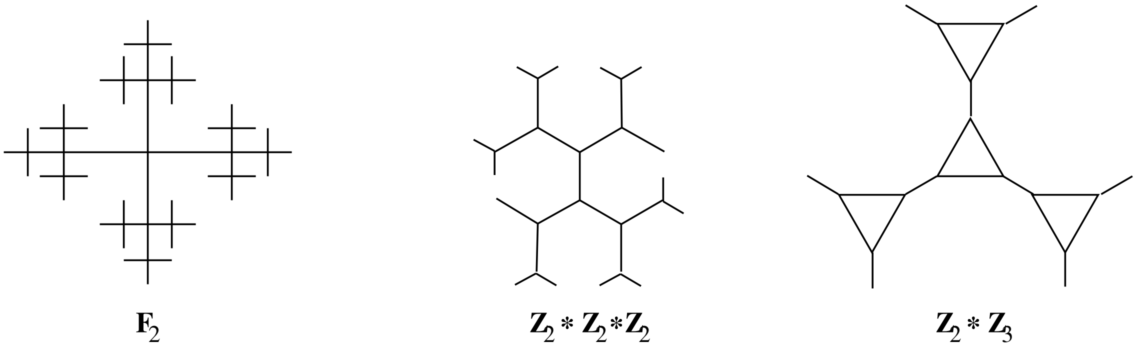

Figure 7. The Cayley graphs of the free group on 2 letters; the free product of $\mathbf{Z}_2$ with itself 3 times $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$; and the free product $\mathbf{Z}_2 * \mathbf{Z}_3$.

Although Cayley graphs are homogeneous, they are generally not spherically symmetric. Still, if M_n denotes the number of vertices at distance n from the identity, then $\lim M_n^{1/n}$ exists since the triangle inequality implies that $M_{m+n} \le M_m M_n$. Is $\lambda_c(G) = \lim M_n^{1/n}$?

If we identify the vertices at a given distance from the identity, as in the proof of Proposition 14, we may not preserve the effective conductance from the identity to infinity, but we cannot decrease it. Hence, the calculation in the proof of Proposition 14 shows that if $\sum_n \lambda^n/M_n = \infty$, then RW_λ is recurrent. In particular, this is the case when $\lambda > \lim M_n^{1/n}$. What about when $\lambda < \lim M_n^{1/n}$? It turns out that RW_λ is then transient.

To prove transience for a given λ, it suffices to prove that a subgraph is transient: this is the same as saying that removing edges cannot increase effective conductance. The easiest subgraph to analyze would be a subtree of G, while in order to have the greatest likelihood of being transient, it should be as big as possible, i.e., a **spanning tree** (one that includes every vertex). Here is one: for each $v \in G$, there is a unique word $(g_{i_1}, g_{i_2}, \ldots, g_{i_n})$ such that $v = g_{i_1} g_{i_2} \cdots g_{i_n}$, $n = |v|$, and $(g_{i_1}, \ldots, g_{i_n})$ is **lexicographically minimal** with these properties, i.e., if $(g_{i'_1}, \ldots, g_{i'_n})$ is another and m is the first j such that $i_j \neq i'_j$, then $i_m < i'_m$. Call this word w_v. Let T be the subgraph of G containing all vertices and with u adjacent to v when $|u| + 1 = |v|$ and w_u is an initial segment of w_v (or vice versa). We root T at the identity.

Suppose that w_v is broken arbitrarily as the concatenation of two words w_1 and w_2. Let v_i be the product of the generators in w_i $(i = 1, 2)$. Then $v = v_1 v_2$. If for some i, we had $w_{v_i} \neq w_i$, then we could substitute the word w_{v_i} for w_i and find another word w whose product was v yet would be either shorter than w_v or come earlier lexicographically than w_v. Either of these circumstances would contradict the definition of w_v as minimal, whence $w_i = w_{v_i}$ for both i.

This fact for $i = 1$ shows that T is a tree; for $i = 2$, it shows that T is subperiodic. Since T is spanning and since distances to the identity in T are the same as in G, we have $\operatorname{gr} T = \lim M_n^{1/n}$. Since T is subperiodic, Furstenberg's Theorem 2 shows that $\operatorname{br} T = \lim M_n^{1/n}$. Hence RW_λ is transient on T for $\lambda < \lim M_n^{1/n}$ by Theorem 5, whence on G as well. This gives the following theorem of Lyons (1995):

THEOREM 15. *RW_λ on a Cayley graph has critical value λ_c equal to the growth rate of the graph.*

Of course, this theorem makes Cayley graphs look spherically symmetric from a probabilistic point of view. Such a conclusion, however, would be too hasty, for there are Cayley graphs with the following very surprising property (Lyons, Pemantle and Peres 1996): define the **speed** (or rate of escape) of RW_λ as the limit of the distance from the identity at time n divided by n as $n \to \infty$, if the limit exists. On any spherically symmetric graph, the speed of RW_λ (if it exists) is monotone decreasing in λ. However, there is a Cayley graph such that the speed is 0 when $\lambda = 1$ but is strictly positive when $1 < \lambda < (1 + \sqrt{5})/2$.

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