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Homogeneous Partial Derivatives of Radial Functions

by Russell Lyons and Kevin Zumbrun

The following surprising identity for differentiation of radial functions by homogeneous partial differential operators appears to be new. For a polynomial $P(x_1, \ldots, x_n)$, write, as usual, $P(D) := P(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Write $r := (x_1^2 + \cdots + x_n^2)^{1/2}$.

Theorem. *Let P be a polynomial of n variables homogeneous of degree h. Let f be a function of one variable. Then*

$$P(D)f(r) = \sum_{k=0}^{\lfloor h/2 \rfloor} \frac{1}{2^k k!} \Delta^k P(x) \cdot \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r).$$

Proof. It suffices to prove this when P is a monomial. The assertion for monomials is established by induction on the degree h as follows. Note that for all i and all f,

$$\frac{1}{x_i}\frac{\partial}{\partial x_i} f(r) = \frac{1}{r}\frac{\partial}{\partial r} f(r)\,.$$

Also, by induction on $k \geq 0$, we have that

$$\Delta^k (x_i P(x)) = x_i \Delta^k P(x) + 2k \frac{\partial}{\partial x_i} \Delta^{k-1} P(x).$$

Therefore, if

$$P(D)f(r) = \sum_{k=0}^{\infty} \frac{1}{2^k k!} \Delta^k P(x) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r),$$

then for any i,

$$\begin{aligned}
(x_i P)(D) f(r) &= \frac{\partial}{\partial x_i} P(D) f(r) \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2^k k!} \frac{\partial}{\partial x_i} \left[\Delta^k P(x)\right] \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k} f(r) + \frac{1}{2^k k!} \Delta^k P(x) \cdot x_i \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{h-k+1} f(r) \right] \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2^k k!} \frac{1}{2(k+1)} \left\{ \Delta^{k+1}(x_i P(x)) - x_i \Delta^{k+1} P(x) \right\} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-(k+1)} f(r) + \right. \\
&\qquad \left. \frac{1}{2^k k!} \Delta^k P(x) \cdot x_i \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-k} f(r) \right] \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k k!} \Delta^k (x_i P(x)) \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{(h+1)-k} f(r),
\end{aligned}$$

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which is the desired induction step. ∎

Let $K[g](x_1, \ldots, x_n) := r^{2-n} g(x_1/r^2, \ldots, x_n/r^2)$ be the Kelvin transform of g. We were motivated to discover the above identity in order to understand the following result (Axler, Bourdon and Ramey 1992, p. 90) which is crucial to the fast algorithm of Axler and Ramey (199?) for symbolic solution of the Dirichlet problem on balls with polynomial data. Another derivation of the following corollary with more conceptual understanding comes from the representation theory of $SO(n)$, the special orthogonal group acting on $\mathbf{R}^n$. It was known to Axler, Bourdon and Ramey as well as to Lenard (1992).

COROLLARY. *Let P be a harmonic polynomial of n variables homogeneous of degree h. Then*

$$P = c_h K[P(D) f_n(r)],$$

where $f_n(r) := r^{2-n}$ for $n > 2$, $f_n(r) := \log r$ if $n = 2$, $c_h := \prod_{i=1}^{h} (4 - n - 2i)^{-1}$ for $n > 2$, and $c_h := (-1)^{h-1}/[2^{h-1}(h-1)!]$ if $n = 2$.

Proof. Since K is an involution, it suffices to show that

$$c_h^{-1} K[P] = P(D) f_n(r).$$

This is now an immediate consequence of the above theorem and the definition of K. ∎

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