*Random Discrete Structures*, ed. D. Aldous and R. Pemantle, Springer, New York, 1996, pp. 185–198.

# How Fast and Where Does a Random Walker Move on a Random Tree?

By Russell Lyons

**Abstract.** The oldest model of random trees is that of Galton and Watson. In joint work with Robin Pemantle and Yuval Peres, described informally here, we consider simple random walk on the family tree $T$ of a supercritical Galton-Watson branching process. First we calculate the speed (i.e., rate of escape) of the walk; the answer is simple, yet surprising. Then we show that the resulting harmonic measure has a.s. strictly smaller dimension than that of the whole boundary of $T$. Concretely, this means that after $T$ is picked but before the random walk takes place, a subtree of $T$ of a smaller exponential growth rate can be specified such that with overwhelming probability, the random walk particle will never exit this subtree (!).

We present an exposition of some recent joint work with Robin Pemantle and Yuval Peres (see Lyons, Pemantle and Peres (1995)). Consider the tree of Figure 1. This was generated pseudorandomly by beginning with one vertex at the top, called the **root**, and adding 1 or 2 children with equal probability (in this case, it was 2 children). Then each of these children also has 1 or 2 children with equal probability and independently, and so on forever (only 9 generations are represented in the figure). We shall refer to this tree as the **1-2 tree**.

A random walker starts at the root of the tree and chooses among the edges incident to the root at random, walking to the adjacent vertex. Then the walker again chooses among the incident edges *including the one leading back to the root* at random and moves to the adjacent vertex. This goes on forever. Such a random walk is called **simple random walk**. Since there are, on average, 1.5 edges leading away from the root and only 1 leading toward the root, the walker eventually stops returning to the root and, in fact, escapes to infinity a.s. The questions we address are:

- How fast does the walker escape? In other words, after a long time, about how far is the walker from the root?

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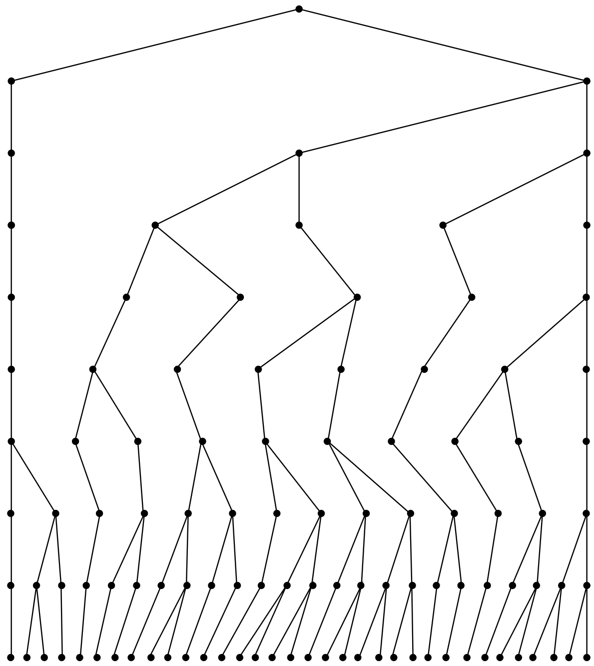

**Figure 1.** A typical random tree.

- Where does the walker go? In other words, what are the chances of escaping along the various paths to infinity?

The 1-2 tree is actually only an example of the general set-up we treat: In general, each particle (or vertex) has $k$ children with probability $p_k$, where $\sum_{k=1}^{\infty} p_k = 1$. (We exclude the trivial case of $p_1 = 1$.) This gives the family tree of a **Galton-Watson** branching process. The sequence $\{p_k\}$ is called the **progeny distribution**. The 1-2 tree is the special case where $p_1 = p_2 = 1/2$. Again, simple random walk is transient by the intuition that on average, there are $m := \sum_{k=1}^{\infty} p_k k > 1$ edges out of each vertex leading away from the root. Note that when we say that simple random walk is transient, we actually mean that for *almost every* Galton-Watson tree for a fixed progeny distribution, the random walk is transient. This intuition was first proved correct in an unpublished supplement to Grimmett and Kesten (1984).

It is well known that an equivalent assertion comes from electrical networks: Assume that each edge of the tree is a unit conductor. Then transience is equivalent to the effective conductance from the root to infinity being positive (see, e.g., Doyle and Snell (1984)). We illustrate this with a couple of simple examples before turning back to Galton-Watson trees.

First, simple random walk on the nonnegative integers is the same as on the 1-ary tree of Figure 2. If each edge has unit conductance, then each edge also has unit resistance. As these edges are in series and resistances in series add, we see that there is infinite resistance to infinity, whence 0 effective conductance. This proves the well-known fact that such a random walk is recurrent.

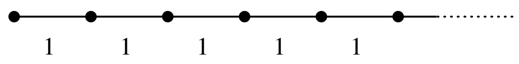


**Figure 2.** The nonnegative integers.

Second, consider simple random walk on the binary tree (Figure 3). By symmetry, the vertices of a given generation are at the same voltage when a battery is connected between the root and infinity. Therefore, they may be shorted together with no change in the effective conductance. This produces the network of Figure 4. Since conductances in parallel can be added, we have an equivalent network of Figure 5. Finally, adding the resultant resistances in series gives an effective conductance of 1.

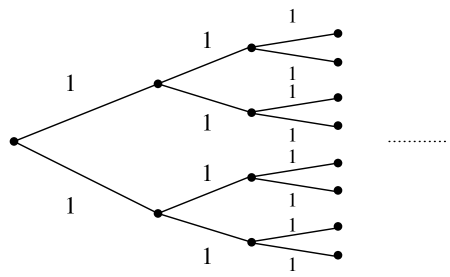

**Figure 3.** The binary tree.

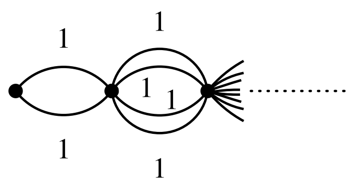

**Figure 4.** Shorting the generations.

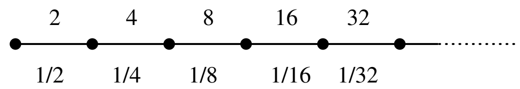


**Figure 5.** Adding the conductances (on top) from Figure 4 and then the resistances (on bottom).

For the 1-2 tree, it follows by comparison with the binary tree that the conductance is a random variable lying in (0, 1). Its density is shown in Figure 6. (In fact, it is not known that the conductance has an absolutely continuous distribution, but it certainly looks that way from the graph: the derivative of the c.d.f. is never more than 2.5.) From this information, we can calculate that the probability that the random walker never returns to the root is about 29.7%.

Figure 6 is calculated by an iterative solution to a functional equation. If $\mathcal{C}$ is the conductance of a Galton-Watson tree, then the c.d.f. $F$ of $\mathcal{C}/(\mathcal{C}+1)$ satisfies the equation

$$F(s) = \begin{cases} \sum_k p_k F^{*k}\left(\frac{s}{1-s}\right), & \text{if } s \in (0,1); \\ 0, & \text{if } s \le 0; \\ 1, & \text{if } s \ge 1 \ . \end{cases}$$

The only other solution to this equation is the Heaviside function.

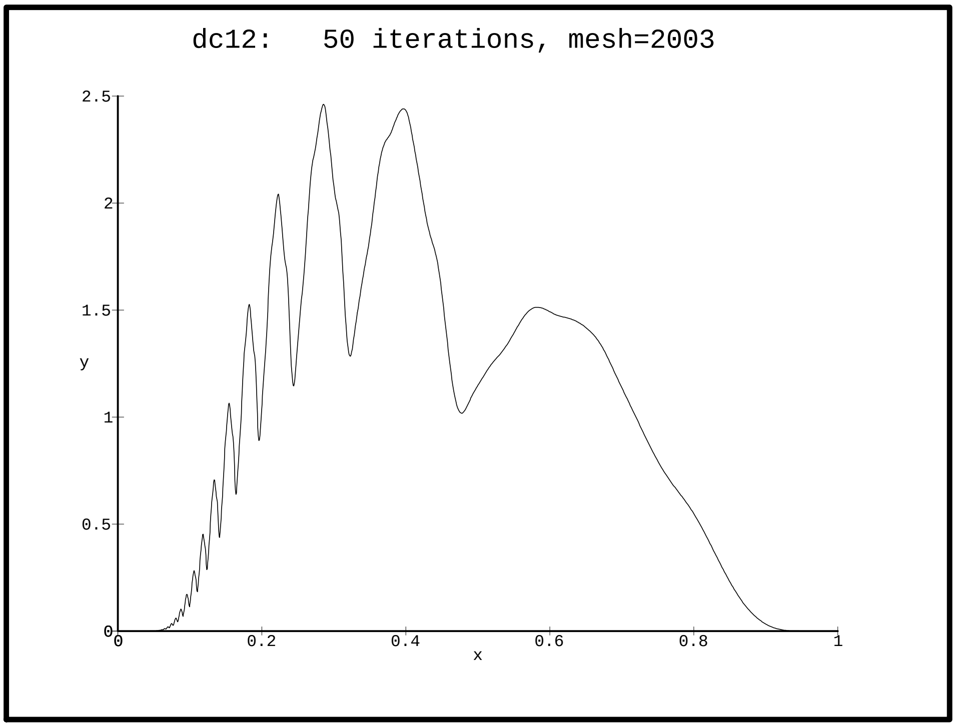


**Figure 6.** The apparent density of the conductance of the 1-2 tree.

Figure 6 reflects the stochastic self-similarity of the 1-2 tree. For example, suppose that the tree begins without branching, i.e., the root has only one child. Then what follows this child is no bigger than a binary tree and so this descendant subtree has conductance at most 1. Therefore, the whole tree has conductance at most 1/2 (see Figure 7). The wide peak at the right of Figure 6 is thus due entirely to those trees which begin with two children of the root. The peak immediately to its left is due, at first approximation, to an unspecified number of generations without branching, while the $n$th peak to the left of it is due to $n$ generations without branching (and an unspecified continuation). Of course, the next level of approximation deals with further resolution of the peaks.

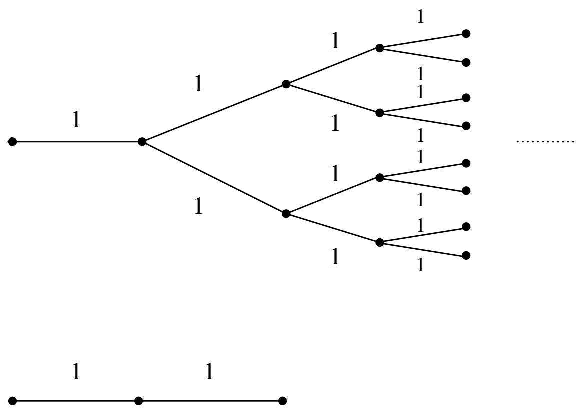

**Figure 7.** The upper bound of a 1-2 tree that starts with one child of the root.

Let $X_n$ be the position of the random walker on a Galton-Watson tree. Denote by $|X_n|$ its (graph) distance to the root. The question of speed, or rate of escape, is due to David Aldous and asks for the limit

$$\lim_{n\to\infty} \frac{|X_n|}{n} =?$$

Consider first the $m$-ary tree ($p_m = 1$). The stochastic process $|X_n|$ is the same as a random walk on $\mathbf{N}$ but with transition probabilities $m/(m+1)$ to the right and $1/(m+1)$ to the left at each positive integer. Since the walk always steps to the right when it is at 0, this is not a stationary process. Nevertheless, it is intuitive that the speed should be given by the strong law of large numbers as

$$\frac{m}{m+1}(+1) + \frac{1}{m+1}(-1) = \frac{m-1}{m+1}.$$

Now consider the 1-2 tree. Half of its vertices have 1 child and half have 2, so perhaps the speed is

$$\frac{1}{2}\cdot\frac{1-1}{1+1} + \frac{1}{2}\cdot\frac{2-1}{2+1} = \frac{1}{6}.$$

Actually, this seems naive since it doesn't take into account how much time the walker spends at different vertices. Indeed, recall that on a finite graph, the stationary distribution for simple random walk is proportional to the degree of a vertex. From this, we might guess that the speed on the 1-2 tree is

$$\frac{2}{5} \cdot \frac{1-1}{1+1} + \frac{3}{5} \cdot \frac{2-1}{2+1} = \frac{1}{5}\,.$$

Or perhaps neither of these guesses is correct? We shall see in formula (1) below.

To find correctly the speed on Galton-Watson trees, it is useful first to see how to rigorize the calculation of speed for $m$-ary trees. One way of doing this is to extend the random walk on $\mathbf{N}$ given by the distance process $|X_n|$ to a stationary random walk on $\mathbf{Z}$. We then have true i.i.d. pushes to the right and their averages converge, by the strong law of large numbers, to their expectation.

The analogue we need for Galton-Watson trees is a Markov chain whose state space consists of rooted unlabelled trees. The walk is regarded as being at the root, so that when it takes a step to a neighboring vertex, the new state is the same tree except for the fact that the root has moved. Thus, though the state space is uncountable, from any state, only a finite number of states can be visited next.

Suppose that, somehow, we have a stationary ergodic probability measure $\mu_{\mathsf{SRW}}$ for this Markov chain. We'd like the speed to be an ergodic average of changes in distance to the initial position $X_0$. But this suffers from the same problem as did our intuitive calculation of the speed on an $m$-ary tree: the summands won't then be stationary. One way to fix this, analogous to the rigorization for the $m$-ary tree, is by a standard trick: we use "horodistance" from a boundary point.

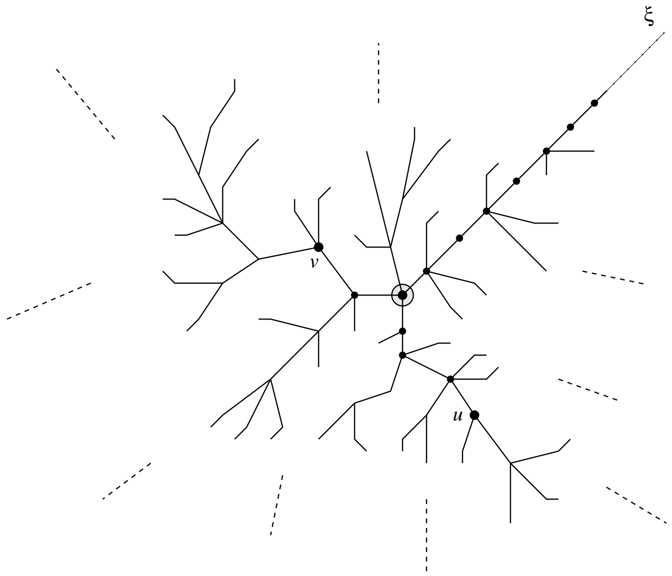


**Figure 8.** The horodistance from $v$ to $u$ as seen from the boundary point $\xi$ is 2.

A **boundary point** $\xi$ is an equivalence class of rays, where two rays are **equivalent** if they have an infinite number of vertices in common. In Figure 8, the two rays from $v$ and $u$ going out to the upper right are equivalent. They meet at the shaded vertex, $u \wedge_\xi v$. The **signed distance** from $v$ to $u$ as seen from $\xi$ is defined to be

$$[u - v]_\xi := |u - u \wedge_\xi v| - |v - u \wedge_\xi v|,$$

where $|x - y|$ denotes the distance between $x$ and $y$ in the tree. The intuitive notion of vertices at a given horodistance from a boundary point is given by the concept of a **horosphere**. This is a set of vertices such that any pair of them have signed distance 0 from one to the other as seen from the boundary point. The horospheres are drawn in Figure 9 as horizontal levels. The picture makes clear the additivity of signed distance:

$$[u - w]_\xi = [u - v]_\xi + [v - w]_\xi .$$

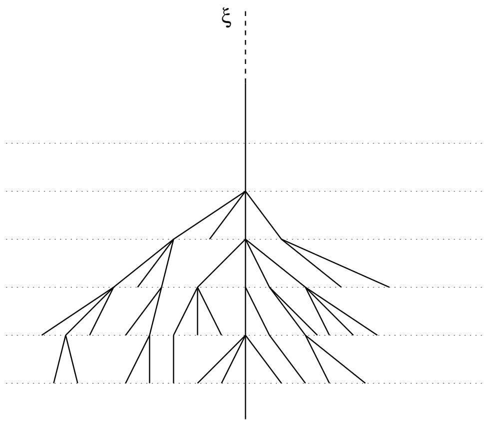


**Figure 9.** Horospheres centered at the boundary point $\xi$.

Now the random walk tends to a boundary point in the following sense: The shortest paths $\overline{X_0 X_n}$, by virtue of transience of the random walk, have more and more edges eventually constant, defining (a.s.) a limiting ray in an equivalence class which we denote $X_\infty$. Provided the boundary point $\xi \neq X_\infty$, there is a (random) constant $c$ such that for all but finitely many $n$,

$$|X_n - X_0| = [X_n - X_0]_\xi + c .$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} |X_n - X_0| = \lim_{n\to\infty} \frac{1}{n} [X_n - X_0]_\xi = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} [X_{k+1} - X_k]_\xi .$$

We can now make the summands stationary as follows. Extend the walk backwards in time and let $X_{-\infty} := \lim_{n\to\infty} \overline{X_0 X_{-n}}$. Since $X_{-\infty} \neq X_\infty$ a.s., we may take $\xi := X_{-\infty}$. The ergodic theorem then gives that the speed is the expectation of, say, the 0th term in the sum above:

$$\iint [X_1 - X_0]_{X_{-\infty}} \, d\vec{X} \, d\mu_{\mathsf{SRW}}(T) \, .$$

To evaluate this, note that $X_{-n}$ is gotten by running an independent simple random walk from $X_0$. Thus $X_1$ is independent of $X_{-\infty}$ and is uniform on the neighbors of $X_0$. That is,

$$[X_1 - X_0]_{X_{-\infty}} = \begin{cases} 1 & \text{with probability } (d-1)/d, \\ -1 & \text{with probability } 1/d, \end{cases}$$

where $d := \deg(X_0) = \deg(\mathrm{root}(T))$. Therefore, the speed is

$$\int \frac{\deg(\mathrm{root}(T)) - 2}{\deg(\mathrm{root}(T))} \, d\mu_{\mathsf{SRW}}(T) \, .$$

It remains now to find $\mu_{\mathsf{SRW}}$. Of course, there are many stationary ergodic measures for the Markov chain under consideration. As a trivial example, use the measure concentrated on the regular tree, all of whose vertices have degree 3. What we want is a stationary measure that will give information about Galton-Watson trees. But we cannot use the measure given by the standard Galton-Watson branching process: this would not be stationary since the root has stochastically one fewer neighbor than all the other vertices. Instead, we claim that the following "augmented" Galton-Watson measure is stationary. This measure is defined just like ordinary Galton-Watson measure except that the number of children of the root (only) is one more; i.e., the root has $k+1$ children with probability $p_k$ and these children all have independent standard Galton-Watson descendant trees. This is the same as generating a Galton-Watson tree and then adding one more edge to the root, to which we append another independent Galton-Watson tree: see Figure 10. To see that augmented Galton-Watson measure is stationary, consider a random step from the root. The particle is then at a vertex to which is attached

- a Galton-Watson tree labelled **GW** in the figure
- and another edge, at the end of which is another Galton-Watson tree.

But this is precisely the same stochastic figure as the particle sees from the original root.

Now augmented Galton-Watson measure has the property that all vertices, including the root, have degree $k+1$ with probability $p_k$. We simply plug this measure into the last equation above to get that the speed is

$$\sum_k p_k \frac{k-1}{k+1} \, . \tag{1}$$

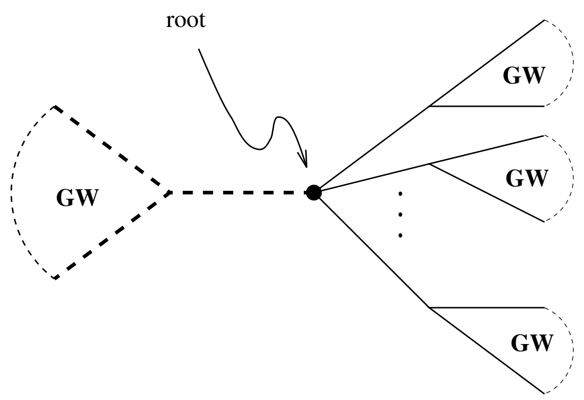


**Figure 10.** An augmented Galton-Watson tree.

Although this is the speed for simple random walk on augmented Galton-Watson trees, it is intuitively clear that it is also the speed on standard Galton-Watson trees. In particular, the speed on the 1-2 tree is $1/6$, as the naive calculation had it. Of course, this is because the stationary measure is *not* proportional to the degree of the root; the intuition from finite graphs turns out to be misleading. By way of comparison, note that the argument from finite graphs would give a speed of $(m-1)/(m+1)$. Concavity of the function $x \mapsto (x-1)/(x+1)$ shows that this is always strictly larger than the true speed (1) except in the trivial case where $p_m = 1$. If $m$ is an integer, then this shows that the speed on a Galton-Watson tree with mean $m$ is smaller than on the corresponding deterministic tree, even though both have the same growth rate.

An "explanation" for the smaller speed on Galton-Watson trees arises from the fact that simple random walk essentially remains on a subtree of strictly smaller exponential growth rate. We shall discuss the proof of this in a moment. This fact raises yet another question: The growth rate of a Galton-Watson tree is a.s. $\log m$; if $d$ denotes the growth rate of this smaller subtree, so that $d < \log m$, then is the speed (1) less than $(e^d - 1)/(e^d + 1)$?

To prove that there are such smaller subtrees on which the random walk essentially remains, consider the following finitistic problem. Grow a Galton-Watson tree for $n$ generations. Start a simple random walk at the root and kill it when it reaches level $n$. This gives a probability distribution on the vertices at level $n$. Given $\alpha \in (0,1)$, let $K_T^n(\alpha)$ be the minimum number of such vertices needed to form a set of probability at least $\alpha$. How does $K_T^n(\alpha)$ grow with $n$? It turns out that

$$\lim_{n\to\infty} \frac{1}{n} \log K_T^n(\alpha) = d \quad \text{a.s.} \tag{2}$$

For the 1-2 tree, for example, we have that $d \approx 0.38 \approx \log 1.47$.

For the infinite Galton-Watson tree, we said that there is a subtree containing, with probability $\alpha$ arbitrarily close to 1, the path of simple random walk. This subtree is formed, intuitively, from the $K_T^n(\alpha)$ vertices needed to achieve a set of probability $\alpha$ and their "neighbors". Since the walk has positive speed, it can't go very far from the points first visited at level $n$.

This hitting measure on level $n$ is a finitistic harmonic measure. Its asymptotics (2) are analyzed by considering, instead, harmonic measure on the boundary of the infinite tree. For this purpose, instead of thinking of boundary points as equivalence classes, we choose a representative ray emanating from the root. The distribution of $X_\infty$ is **harmonic measure**, denoted $\mathsf{HARM}_T$. We shall need a metric on $\partial T$, the boundary of $T$. If $\xi$ and $\eta$ are boundary points with $n$ edges in common, then we define the distance to be $e^{-n}$. (See Figure 11.)

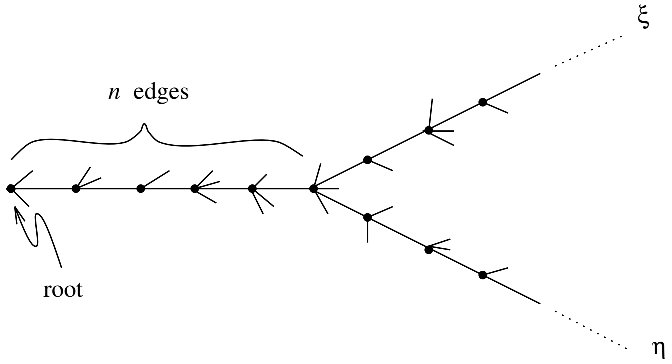


**Figure 11.** The metric on the boundary of a tree.

Although we shall not use it directly, we briefly discuss Hausdorff dimension. In the above metric, the boundary becomes compact and has Hausdorff dimension $\log m$ a.s. (Hawkes 1981). Define the dimension of a probability measure $\theta$ on $\partial T$ as

$$\dim \theta := \min\{\dim E\,;\ \theta(E) = 1\}\,.$$

This measures how much $\theta$ spreads out; in (2), spread is indeed the quantity of interest. The smaller $\dim \theta$ is, the more $\theta$ is concentrated on a small set (even though its support may be the whole boundary). Of course, $\dim \theta \le \dim \partial T = \log m$ a.s. for any $\theta$. In particular, the dimension of harmonic measure is the number $d$ of (2) that we want to show is strictly less than $\log m$ (unless $p_m = 1$).

Now a nonnegative function on the vertices of a tree taking the value 1 at the root and having the property that the value at any vertex equals the sum of the values at its children is called a **unit flow**. Borel probability measures $\theta$ on $\partial T$ are in 1-1 correspondence with unit flows on $T$ via the assignment to a vertex $v$ the value $\theta(\{\xi\,;\ v\in\xi\})$; i.e., the measure of the set of all rays passing through the vertex $v$. For harmonic measure, as an example, this is the probability that the walk will have the vertex $v$ as the last point visited in its generation.

We are interested in the dimension of harmonic measure, but before its consideration, it is useful to examine first another measure, called **limit uniform** and denoted $\mathsf{UNIF}_T$. We define it as follows through its associated unit flow. Given $k$, consider $n > k$ and all the vertices in generation $k$. Assign temporarily to the vertices in generation $k$ the proportion of their descendants in generation $n$ among all vertices in generation $n$. This is a probability measure on the vertices of level $k$. Now let $n$ tend to infinity and take the limit of this probability measure. It turns out that this converges a.s. and so is well defined a.s. Considering now all $k$, this defines the unit flow of limit uniform measure. By construction, this measure spreads out the most possible; at least, this is so if nothing odd happens in taking the limit. Indeed, one can show that if $\sum_k p_k k \log k < \infty$, then the dimension of limit uniform measure is $\log m$ a.s. (This extends Hawkes (1981), who assumed that $\sum_k p_k k(\log k)^2 < \infty$.)

Now an easy argument shows that the dimension of harmonic measure is an a.s. constant, $d$. What is this constant? Consider the finite tree in Figure 12. Let the height of this tree be $n$. Imagine that Galton-Watson descendant trees continue from the leaves at the bottom. Then limit uniform measure will put about $2^n$ as much mass on the right child of the root as on the left. On the other hand, harmonic measure will not be nearly so unbalanced since the random walk will exit the "pipe" on the left with probability of the order of magnitude $1/n$, not $1/2^n$. This is an extreme illustration of the idea that harmonic measure is more concentrated than limit uniform measure. From this intuition, one expects that $d = \dim \mathsf{HARM}_T < \log m$ a.s. This was first conjectured by Robert Kaufman.

To prove that this is indeed the case, we use another notion, that of Hölder exponent. Given a Borel probability measure $\theta$ on a metric space, define the **Hölder exponent** of $\theta$ at a point $x$ as

$$\lim_{r\downarrow 0} \frac{\log 1/\theta(B_r(x))}{\log 1/r}\,,$$

where $B_r(x)$ denotes the ball of radius $r$ centered at $x$ — if this limit exists. For example, if $\theta$ is Lebesgue measure on $n$-dimensional euclidean space, then it is clear that its Hölder exponent is $n$ at every point. Billingsley (1965) showed that if the Hölder exponent exists and is constant $\theta$-a.s., then it equals $\dim\theta$. He showed this for euclidean space, but the same proof works even more easily on the boundary of trees.

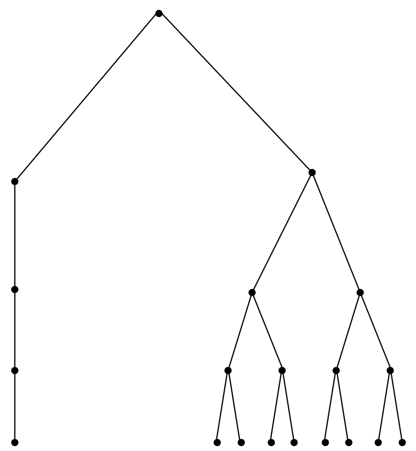

**Figure 12.** The difference between limit uniform and harmonic measures.

If $\xi \in \partial T$ is the ray $\langle \xi_0, \xi_1, \xi_2, \ldots \rangle$, where $\xi_0 = \mathrm{root}(T)$, then the Hölder exponent of $\theta$ at $\xi$ is

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\theta(\xi_n)} \,. \tag{3}$$

Before using this in the case of primary interest, i.e., that of harmonic measure for simple random walk, let us see how to treat an easier case, that of **simple forward random walk**. For this random walk, the walker chooses each time among the children (only) of the present vertex. It cannot go back towards the root. The corresponding unit flow is the **equally-splitting flow** which divides the mass at every vertex equally among its children. We call the associated measure on the boundary of the tree **visibility measure**, denoted $\mathsf{VIS}_T$. For its Hölder exponent, we consider

$$\frac{1}{n} \log \frac{1}{\mathsf{VIS}_T(\xi_n)} = \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\mathsf{VIS}_T(\xi_k)}{\mathsf{VIS}_T(\xi_{k+1})} \,. \tag{4}$$

Now the quotient on the right-hand side is, by definition, the number of children of the vertex $\xi_k$. Its distribution is, therefore, the progeny distribution $\{p_k\}$. These quotients are also independent along a ray and of the ray chosen, whence the strong law of large numbers gives that the right-hand side converges to $\sum p_k \log k$ a.s. (Note that by concavity of the log function, this is strictly less than $\log \sum p_k k = \log m$ unless $p_m = 1$. That is, the Hölder exponent, and thus also the Hausdorff dimension, of visibility measure is less than that of the full boundary. The intuition behind this from Figure 12 is immediate. It also suggests that the dimension of $\mathsf{VIS}_T$, i.e., $\sum p_k \log k$, should be less than $d$, but this is unknown.)

In order to treat harmonic measure of simple random walk, we again need to use the ergodic theorem for a Markov chain. This Markov chain has the same state space as before, namely, the space of rooted unlabelled trees, but has the following transition probabilities. Starting from a rooted tree, choose among the children of the root according to their respective harmonic measures (more precisely, the harmonic measures of the sets of rays passing through the children). The new state is then the descendant tree of the child picked, rooted at this child. (See Figure 13.)

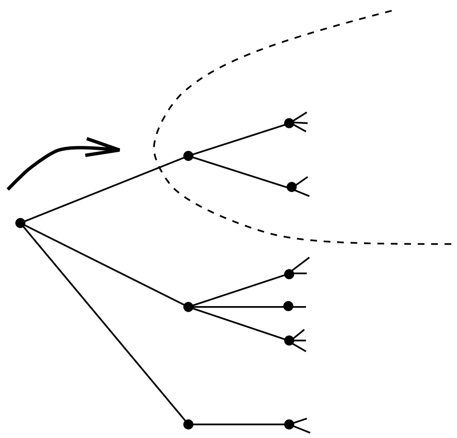

**Figure 13.** A transition of the Markov chain for harmonic measure. Here, the top child is being chosen.

One can show that this Markov chain is ergodic for Galton-Watson measure, but it is not stationary. To get around this same problem that we encountered before, we induce from augmented Galton-Watson measure, the stationary measure for simple random walk, to a certain set of states called "exit points". The resulting measure is denoted $\mu_{\mathsf{HARM}}$. Again, the expression (3) whose limit gives the Hölder exponent is written as a telescoping sum. In (4), this was an average of i.i.d. r.v.'s. Now, it is an ergodic average, so the ergodic theorem gives its limit as

$$\begin{aligned}\iint \log \frac{1}{\mathsf{HARM}_T(\xi_1)}\, d\mathsf{HARM}_T(\xi)\, d\mu_{\mathsf{HARM}}(T) \\ = \int \sum_{|x-\mathrm{root}(T)|=1} \mathsf{HARM}_T(x) \log \frac{1}{\mathsf{HARM}_T(x)}\, d\mu_{\mathsf{HARM}}(T)\,. \end{aligned} \tag{5}$$

Recall Shannon's inequality (the concavity of log): if $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 1$, where $a_i,\, b_i \in (0,\,1)$ and some $a_i \neq b_i$, then

$$\sum_{i=1}^n a_i \log \frac{1}{a_i} < \sum_{i=1}^n a_i \log \frac{1}{b_i}\,.$$

Since harmonic measure and limit uniform measure are a.s. unequal (except on a deterministic Galton-Watson tree), this gives that the expression in (5) is

$$< \int \sum_{|x - \mathrm{root}(T)|=1} \mathsf{HARM}_T(x) \log \frac{1}{\mathsf{UNIF}_T(x)} \, d\mu_{\mathsf{HARM}}(T) \,.$$

(Recall that $\mathsf{UNIF}_T$ denotes limit uniform measure on $\partial T$.) To evaluate this, we use a trick with the ergodic theorem and a certain coboundary. It turns out that this integral is equal to $\log m$. Therefore $d$, the dimension of harmonic measure, is less than $\log m$.

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