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Random Walks and the Growth of Groups

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Abstract. The critical value separating transience from recurrence for the amount of radial drift of a random walk on a Cayley graph of any finitely generated group is shown to equal the exponential growth rate of the group.

Marches Aléatoires et la Croissance des Groupes

Résumé. On considère des marches aléatoires avec drift radial sur le graphe de Cayley d'un groupe quelconque finiment engendré. La valeur critique du drift qui sépare le transience du récurrence est égale à la croissance exponentielle du groupe.

Version française abrégée. Plusieures propriétés des marches aléatoires sont, comme il est bien connu, étroitement liées aux propriétés structurelles des groupes infinis. Ici, on donne un rapport précis entre le comportement de la trajectoire de la marche aléatoire et une parmi les plus simples des propriétés structurelles, à savoir, celle de la croissance exponentielle du groupe.

Les marches aléatoires que l'on considère sont naturelles du point de vue du graphe de Cayley, mais elles ne sont pas invariantes sous l'action du groupe. D'autre part, aucune hypothèse n'est faite sur le groupe, à part qu'il soit finiment engendré.

Soit G un groupe finiment engendré et soit X son graphe de Cayley. Notons l'identité de G par $\emptyset$, puisque c'est le produit nul des générateurs. On note $|g|$ la distance dans X entre g et

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$\emptyset$. Soit M_n le nombre d'éléments de G à distance n de l'identité et soit $\mathrm{gr}\,(G) := \lim_{n\to\infty} M_n^{1/n}$ la **croissance** de G. Etant donné $\lambda \geq 1$, on définit la marche aléatoire RW_λ sur X comme suit. Soit $\deg v$ le degré du sommet $v \in X$ et soit $\deg^- v$ le nombre d'arêtes qui relient v à un sommet w avec $|w| = |v| - 1$. Alors la probabilité de transition de v à un voisin w est

$$p(v,w) := \begin{cases} \lambda/(\deg v + (\lambda - 1)\deg^- v) & \text{si } |w| = |v| - 1 \\ 1/(\deg v + (\lambda - 1)\deg^- v) & \text{autrement.} \end{cases}$$

C'est-à-dire, à partir d'un sommet quelconque v, il est λ fois plus probable de passer sur une arête qui relie v à un sommet plus proche à $\emptyset$ que sur chaque autre arête. (Pour $\lambda = 1$, c'est la marche aléatoire simple.) Notre résultat principal est le suivant.

THÉORÈME 1.1. *Si $\lambda < \mathrm{gr}\,(G)$, alors RW_λ est transiente, tandis que si $\lambda > \mathrm{gr}\,(G)$, alors RW_λ est récurrente positive.*

Démonstration. La marche RW_λ correspond au réseau électrique sur X où l'arête entre u et v a la conductivité $\lambda^{-(|u|\wedge|v|)}$. Si $\lambda > \mathrm{gr}\,(G)$, on a $\sum M_n \lambda^{-n} < \infty$, d'où la somme de toutes les conductivités est finie. D'après la Proposition 9-131 de Kemeny, Snell et Knapp (1976), il suit que RW_λ est récurrente positive.

Or, soit $\{g_1, \ldots, g_k\}$ le système générateur de G qui donne son graphe X; on peut supposer que ce système contient l'inverse de chacun de ses éléments. Pour tout $v \in G$, il existe un mot unique $[v] := (g_{i_1}, g_{i_2}, \ldots, g_{i_n})$ tel que $v = g_{i_1} g_{i_2} \cdots g_{i_n}$, $n = |v|$, et tel que $(g_{i_1}, \ldots, g_{i_n})$ soit minimal pour l'ordre léxicographique, c'est-à-dire, si $(g_{i'_1}, \ldots, g_{i'_n})$ est un autre mot dont le produit donne v et m est le premier j tel que $i_j \neq i'_j$, alors $i_m < i'_m$. Soit T le sous-graphe de X qui contient tous les sommets avec les arêtes entre les paires (u,v) où $|u| + 1 = |v|$ et $[u]$ est un segment initial de $[v]$.

T est un arbre jouissant de la propriété suivante: Soit T^v le sous-arbre de T formé de tous les sommets w telle que $[v]$ est un segment initial de $[w]$. Alors, pour tout $v \in T$, il existe une injection de T^v en T qui préserve les incidences et qui envoie v à $\emptyset$. Cette propriété a la conséquence que $\mathrm{br}\, T = \mathrm{gr}\, T$ à cause d'un théorème de Furstenberg (1967) (voir aussi Lyons (1990), Proposition 3.1). Mais évidemment, $\mathrm{gr}\, T = \mathrm{gr}\, G$, donc pour $\lambda < \mathrm{gr}\, G$, la marche RW_λ restreinte à T est transiente (Lyons (1990), Theorem 4.3). D'après la loi de monotonicité de Rayleigh, la marche RW_λ est aussi transiente sur X. ∎

§**1. Introduction.** Several properties of random walk are well known to be closely related to significant structural properties of infinite groups. Here, we give a precise relationship of one of the simplest of structural properties, namely, that of the exponential growth rate of a group, to sample-path behavior of random walk. Although a precise probabilistic interpretation is known for the polynomial growth rate of a group (Hebisch and Saloff-Coste (1993), Theorem 5.1), no precise interpretation has been known for the exponential growth rate.

The random walks that we consider are natural from the viewpoint of the Cayley graph, but they are not group invariant. On the other hand, absolutely no assumptions on the group are made, other than that it be finitely generated.

To understand our principal result, let G be a finitely generated group and let X be its Cayley graph. We write the identity element of G as 0, without intending to imply commutativity. The distance between points in X is the usual graph distance; in particular, the distance from g to 0 is written $|g|$. Denote the sphere of radius n by $S_n := \{g\,;\ |g| = n\}$. Its cardinality is written $M_n := |S_n|$. It is clear that $\{M_n\}$ is a submultiplicative sequence, whence the limit $\mathrm{gr}\,(G) := \lim_{n\to\infty} M_n^{1/n}$ exists; we call this the **growth** of G. Given $\lambda \geq 1$, we define a nearest-neighbor random walk on X denoted RW_λ as follows. Let $\deg v$ be the degree of a vertex $v \in X$ and let $\deg^- v$ stand for the number of edges connecting v to $S_{|v|-1}$. Then the transition probability from v to an adjacent vertex w is

$$p(v,w) := \begin{cases} \lambda/(\deg v + (\lambda - 1)\deg^- v) & \text{if } w \in S_{|v|-1} \\ 1/(\deg v + (\lambda - 1)\deg^- v) & \text{otherwise.} \end{cases}$$

That is, from any vertex v, each edge connecting v to a vertex closer to 0 is λ times more likely to be taken than any other edge incident to v. (For $\lambda = 1$, this is simple random walk.) Such random walks are sometimes referred to as "homesick", with λ being the measure of homesickness. They have been studied before primarily in the context of trees (e.g., Berretti and Sokal (1985), Krug (1988), Lawler and Sokal (1988), Lyons (1990)). Our main result is the following.

THEOREM 1.1. *If $\lambda < \mathrm{gr}\,(G)$, then RW_λ is transient, while if $\lambda > \mathrm{gr}\,(G)$, then RW_λ is positive recurrent.*

This result may be surprising for several reasons. Consider, for example, the group $G_1 := \mathbf{Z} \ltimes_\alpha \sum_{x\in\mathbf{Z}} \mathbf{Z}_2$, a semidirect product of $\mathbf{Z}$ with the direct sum of copies of $\mathbf{Z}_2$ indexed by $\mathbf{Z}$, where the action α of $\mathbf{Z}$ on the direct sum is

$$\big(\alpha(x)(f)\big)(y) := f(y - x) \qquad (x, y \in \mathbf{Z},\ f \in \sum_{x\in\mathbf{Z}} \mathbf{Z}_2)\,,$$

where we write $\sum_{x\in\mathbf{Z}} \mathbf{Z}_2$ as maps from $\mathbf{Z}$ to $\mathbf{Z}_2$ (see Kaimanovich and Vershik (1983), Section 6.1). Generators of G_1 are $(1, \mathbf{0})$, $(-1, \mathbf{0})$, and $(0, \mathbf{1}_0)$. This solvable group has exponential growth, but the speed of simple random walk is zero. Yet by the above theorem, homesick random walk is transient for values of the homesickness parameter λ greater than 1.

Now, natural Følner sets for the group G_1 are the "boxes"

$$\{(n, f)\,;\ |n| \le N \text{ and for } |x| > N\ f(x) = 0\}$$

for $N \ge 1$. Thus, it is reasonable to take averages over boxes. Doing so, we see that half the vertices are incident to one edge leading farther away from 0 and two edges leading closer to 0, while the other half are incident to two edges leading farther away from 0 and one leading closer. There are no edges between points at the same distance from 0. From this point of view, it looks like simple random walk is not, on average, pushed away from 0. Indeed, the speed of simple random walk is zero, but this makes the transience of homesick random walk with $\lambda > 1$ still more surprising.

Finally, in the context of homesick random walks on trees, the growth rate is not the decisive quantity even for such crude behavior as transience/recurrence (Lyons (1990)).

On the other hand, there is an analogue of Theorem 1.1 for arbitrary trees when growth is changed to another quantity called "branching number" (Lyons (1990)). If the tree is sufficiently regular (in any of various senses), then Theorem 1.1 for this tree holds as written above with growth. Clearly, the Cayley graph of a group is "regular" in being vertex-transitive, and one would expect this regularity to carry over to some other properties. Indeed, the method of proof will be to find a spanning tree of the Cayley graph which is sufficiently regular that Theorem 1.1 (with growth) holds for it.

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§**2. Trees.** Homesick random walk is reversible and thus corresponds to an electrical network on X (see, e.g., Doyle and Snell (1984) or Kemeny, Snell and Knapp (1976), Chapter IX, Section 10). Indeed, on any connected graph X with a distinguished vertex 0 and with uniformly bounded degree, consider the electrical network whose conductances are given by

$$C(u, v) := \lambda^{-(|u| \wedge |v|)}\,,$$

where u and v are adjacent vertices and $|\cdot|$ indicates graph distance to 0. The corresponding random walk is $\mathsf{RW}_\lambda(X)$. Let $M_n(X)$ be the number of vertices of X at distance n from 0 and define $\operatorname{gr}(X) := \liminf M_n(X)^{1/n}$. Set

$$\lambda_c(X) := \inf\,\{\lambda\,;\ \mathsf{RW}_\lambda(X) \text{ is recurrent}\}\ .$$

(Note that neither $\operatorname{gr}(X)$ nor $\lambda_c(X)$ depend on the choice of distinguished vertex. This first fact is clear, while the second is immediate from considerations of the energy of flows.)

In case X is a tree T, we also define a **cutset** to be a collection Π of vertices such that every infinite path from 0 that includes infinitely many vertices must intersect Π. The **branching number** of T is then

$$\operatorname{br} T := \inf\left\{\lambda > 1\,;\ \inf_{\Pi} \sum_{v \in \Pi} \lambda^{-|v|} = 0\right\}$$

(Lyons (1990)).

We call a subtree T of X **geodesic** if for every vertex $v \in T$, the distance from v to 0 is the same in T as in X.

Our basic tool is the following proposition:

PROPOSITION 2.1. *If a graph X has a geodesic subtree T with $\operatorname{br} T = \limsup M_n(X)^{1/n}$, then $\lim M_n(X)^{1/n}$ exists and equals $\lambda_c(X)$. Moreover, RW_λ is positive recurrent for $\lambda > \lambda_c(X)$.*

Proof. If $\lambda > \limsup M_n(X)^{1/n}$, then $\sum M_n \lambda^{-n} < \infty$, whence the sum of all the conductances is finite. Therefore (Kemeny, Snell and Knapp (1976), Proposition 9–131), $\mathsf{RW}_\lambda(X)$ is positive recurrent. On the other hand, $\operatorname{br} T \le \operatorname{gr} T \le \operatorname{gr} X \le \limsup M_n(X)^{1/n}$, whence all these quantities are equal. In particular, $\liminf M_n(X)^{1/n} = \limsup M_n(X)^{1/n}$ and for $\lambda < \operatorname{br} T = \operatorname{gr} X$, the walk $\mathsf{RW}_\lambda(T)$ is transient (Lyons (1990), Theorem 4.3). Hence the walk $\mathsf{RW}_\lambda(X)$ is transient as well by the Rayleigh monotonicity law. ∎

REMARK. If RW_λ were defined with different conductances for those edges joining vertices at the same distance from 0, the result would not change (though there could be different behavior at the critical parameter). This is seen by shorting or cutting such edges.

We shall also need the following result about trees due to Furstenberg (1967); see also Lyons (1990), Proposition 3.1. Given a tree T and a vertex $v \in T$, let $[0, v]$ denote the shortest path from 0 to v. The subtree T^v starting at v is defined to be the subgraph of T induced by the vertices $\{w \in T\,;\ [0, v] \subseteq [0, w]\}$. A tree T is called **subperiodic** if for every $v \in T$, there is an incidence-preserving map from the subtree T^v into T that sends v to 0. (Actually, the definition in Lyons (1990) is a bit more general than this.)

PROPOSITION 2.2. *If T is a subperiodic tree, then $\operatorname{br} T = \operatorname{gr} T$.*

§3. Cayley Graphs.

Let G be a group generated by $\{g_1, \ldots, g_k\}$; we may assume that this generating set contains the inverse of each of its elements since the Cayley graph is unaffected by inclusion of inverses. Vertices u, v are adjacent in the Cayley graph X when there is some i such that $u = vg_i$. Now for each $v \in G$, there is a unique word $[v] := (g_{i_1}, g_{i_2}, \ldots, g_{i_n})$ such that $v = g_{i_1} g_{i_2} \cdots g_{i_n}$, $n = |v|$, and such that $(g_{i_1}, \ldots, g_{i_n})$ is **lexicographically minimal** with these properties, i.e., if $(g_{i'_1}, \ldots, g_{i'_n})$ is another word whose product is v and m is the first j such that $i_j \ne i'_j$, then $i_m < i'_m$. Let T be the subgraph of X containing all vertices and with $u \sim v$ when $|u| + 1 = |v|$ and $[u]$ is an initial segment of $[v]$.

It is easy to see that T is a subperiodic geodesic spanning tree.

Theorem 1.1 now follows from Proposition 2.1 and Proposition 2.2.

REMARK. What type (transient or recurrent) is critical homesick random walk, $\mathsf{RW}_{\operatorname{gr}(G)}$? The Nash-Williams criterion (Nash-Williams (1959), Griffeath and Liggett (1982), T. Lyons (1983)) shows that if $\sum \lambda^n M_n^{-1} = \infty$, then RW_λ is recurrent. Is the converse true? In any case, this is sufficient to show that critical homesick random walk is recurrent in case the growth of the group is not much more than pure exponential. The important case of word-hyperbolic groups is covered by this case since for hyperbolic groups, $M_n \approx c\,\mathrm{gr}\,(G)^n$ (Coornaert (1993), Theorem 7.2). Note too that because of this behavior of M_n, critical homesick random walk is null recurrent.

REMARK. The proof of Theorem 1.1 gives another probabilistic interpretation of the result that $\lambda_c(G) = \mathrm{gr}\,(G)$. Namely, the "average" number of lexicographic extensions of a group element is $\mathrm{gr}\, G$. That is, if each word in the dictionary of G is removed independently with probability $1-p$, then the critical value of p such that with positive probability all initial segments of some infinite "word" remain in the dictionary is $1/\mathrm{gr}\, G$ (see Lyons (1990), Theorem 6.2).

This proof also lends support to the following conjecture of Itai Benjamini: The critical parameter for percolation on the Cayley graph of a finitely generated group is less than one unless the group is a finite extension of $\mathbf{Z}$. Benjamini has established this conjecture for certain classes of groups including the groups of polynomial growth. Now the proof of Theorem 1.1 combined with Lyons (1990), Theorem 6.2, shows that a group of exponential growth contains a spanning tree with critical probability less than one. Therefore, Benjamini's conjecture holds for groups of exponential growth, whence the only case remaining consists of groups of intermediate growth.

It would be interesting to know whether *every* geodesic spanning tree of X has branching number equal to the growth of X.

§**4. Vertex-Transitive Graphs.** We may generalize Theorem 1.1 as follows. An **automorphism** of a graph is an incidence-preserving bijection. A graph is **vertex-transitive** if its automorphism group acts transitively on the vertex set. We shall assume that the degree of the vertices is finite. We choose some vertex 0 of X to be distinguished.

THEOREM 4.1. *If X is a vertex-transitive graph of finite degree, then $\lambda_c(X) = \mathrm{gr}\,(X)$. Moreover, RW_λ is positive recurrent for $\lambda > \lambda_c(X)$.*

Proof. For each neighbor a_i of 0 ($1 \le i \le k$), choose an automorphism g_i of X taking a_i to 0. We write this by action on the right: $a_i g_i = 0$. Then for every path $0 = v_0, v_1, v_2, \ldots, v_n = v$ in X, there is a (unique) word $(g_{i_1}, \ldots, g_{i_n})$ such that for every j, we have $v_j g_{i_1} g_{i_2} \cdots g_{i_j} = 0$. For every vertex v in X, call the shortest lexicographically minimal such word $[v]$, where we minimize over all paths from 0 to v. Let T be the subgraph of X consisting of all vertices and with $u \sim v$ when $|u| + 1 = |v|$ and $[u]$ is a terminal segment of $[v]$. Then T is a subperiodic geodesic spanning tree of X, whence the result follows as for Theorem 1.1. ∎

REMARK. Again, the type of critical homesick random walk can sometimes be determined. If, in addition to the above, X is hyperbolic in the sense of Gromov (under the usual graph metric) with at least three points in its hyperbolic boundary, then critical homesick random walk on X is recurrent: this is because $M_n(X) = O(\mathrm{gr}\,(X)^n)$ (Coornaert (1993), Corollary 6.9).

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