Coupling and Bernoullicity in random-cluster and Potts models

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Abstract

An explicit coupling construction of random-cluster measures is presented. As one of the applications of the construction, the Potts model on amenable Cayley graphs is shown to exhibit at every temperature the mixing property known as Bernoullicity.

1 Introduction

In the (ferromagnetic) **Potts model**, spins (or colors) from the set $\{1, \ldots, q\}$ are assigned to the vertices of a graph $G = (V, E)$ randomly, in a way that favors configurations where many pairs of neighboring vertices take the same spin value. More precisely, a spin configuration $\xi \in \{1, \ldots, q\}^V$ is assigned probability proportional to

$$\exp\Big(- 2\beta \sum_{[x,y] \in E} \mathbf{1}_{\{\xi(x) \neq \xi(y)\}} \Big)$$

where $\beta \geq 0$ is referred to as the inverse temperature parameter. The case $q = 2$ is known as the Ising model.

The Potts model has received a considerable amount of attention in the statistical mechanics and probability literature for several decades. In the last decade, perhaps the most important tool for analyzing the Potts model has been the **random-cluster model**, which is a kind of edge representation of the Potts model. It was introduced by Fortuin and Kasteleyn [14], and has been heavily exploited in the study of Potts models since the seminal papers by Swendsen and Wang [35], Edwards and Sokal [12], and Aizenman, Chayes, Chayes and Newman [2]. One of the main points of working with the random-cluster representation, rather than directly with the Potts model, is that

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questions about spin correlations in the latter turn into questions about connectivity probabilities in the former, thereby allowing powerful percolation techniques to come into play. Another interesting aspect of the random-cluster representation is that it makes sense also for noninteger q.

This paper is a contribution to the study of random-cluster and Potts models on infinite lattices. After recalling some necessary prerequisites in Section 2, we come in Sections 3 and 4 to the two main purposes of this paper, which are the following:

- In Section 3, we present a useful device for the analysis of random-cluster and Potts models, namely an explicit pointwise **dynamical construction** of random-cluster measures. The construction provides natural couplings between random-cluster measures with different parameter values or different boundary conditions. To some extent, this construction can be viewed as known and our presentation of it can to the same extent be viewed as expository; it consists of putting together a few well-known ingredients from Grimmett [16], Propp and Wilson [30], and Häggström, Schonmann and Steif [21].
- In Section 4, we apply the dynamical construction from the preceding section to show that the Potts model with fixed-spin boundary condition on $\mathbb{Z}^d$ (and more generally on amenable Cayley graphs) exhibits a rather strong mixing condition known as **Bernoullicity**. Our proof appears to be the simplest to date even in cases where the result was known previously.

Finally, some additional consequences of, and questions on, the dynamical construction are discussed in Section 5.

2 Preliminaries

The following subsections are devoted to recalling known material that will be used in later sections. The random-cluster and Potts models are introduced in Sections 2.3 and 2.4, respectively. Before that, however, we recall some graph terminology in Section 2.1 and some basics on stochastic domination in Section 2.2. A general reference for this background material is Georgii, Häggström and Maes [15].

2.1 Some graph terminology

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We shall always assume either that the graph is finite, or that it is countably infinite and locally finite. An edge $e \in E$ will often be denoted $[x, y]$. The number of edges incident to a vertex x is called the **degree** of x. For $W \subset V$, we define the (inner) **boundary** ∂W of W as

$$\partial W := \{x \in W : \exists y \in V \setminus W \text{ such that } [x, y] \in E\} . \tag{1}$$

A **graph automorphism** of G is a bijective mapping $\gamma : V \to V$ with the property that for all $x, y \in V$, we have $[\gamma x, \gamma y] \in E$ if and only if $[x, y] \in E$. Write $\mathrm{Aut}(G)$ for the group of all graph automorphisms of G. To each $\gamma \in \mathrm{Aut}(G)$, there is a corresponding mapping $\tilde{\gamma} : E \to E$ defined by $\tilde{\gamma}[x, y] := [\gamma x, \gamma y]$. The graph G is said to be **transitive** if and only if for some (any) $x \in V$, one has that for any $y \in V$ there exists $\gamma \in \mathrm{Aut}(G)$ such that $\gamma x = y$. One says that G is **quasi-transitive** if and only if for some finite subset $\{x_1, \ldots, x_n\}$ of V, one has that for any $y \in V$ there exists $\gamma \in \mathrm{Aut}(G)$ such that $\gamma x_i = y$ for some x_i.

A probability measure μ on $\{0,1\}^E$ is said to be **automorphism invariant** if for any n, any $e_1, \ldots, e_n \in E$, any $i_1, \ldots, i_n \in \{0,1\}$, and any $\gamma \in \mathrm{Aut}(G)$ we have

$$\mu\big(\{X \in \{0,1\}^E : \ X(e_1) = i_1, \ldots, X(e_n) = i_n\}\big)$$
$$= \mu\big(\{X \in \{0,1\}^E : \ X(\tilde{\gamma}(e_1)) = i_1, \ldots, X(\tilde{\gamma}(e_n)) = i_n\}\big)\,.$$

In the sequel, we shall simplify the notation and omit the "$\{X \in \{0,1\}^E : \ \}$" as used in the preceding equation.

A graph property that turns out to be important in many situations is amenability: An infinite graph G is said to be **amenable** if

$$\inf \frac{|\partial W|}{|W|} = 0\,,$$

where the infimum ranges over all finite $W \subset V$, and $|\cdot|$ denotes cardinality. There are various alternative definitions of amenability of a graph that coincide for transitive graphs (and more generally for graphs of bounded degree), but not in general.

For any graph G and $x \in V$, define the **stabilizer** $S(x)$ as the set of graph automorphisms that fix x, i.e.,

$$S(x) := \{\gamma \in \mathrm{Aut}(G) : \ \gamma x = x\}\,.$$

For $x, y \in V$, define

$$S(x)y := \{z \in V : \ \exists \gamma \in S(x) \text{ such that } \gamma y = z\}\,.$$

When $\mathrm{Aut}(G)$ is given the weak topology generated by its action on V, all stabilizers are compact subgroups of $\mathrm{Aut}(G)$ because G is locally finite and connected. A transitive graph G is said to be **unimodular** if for all $x, y \in V$ we have the symmetry

$$|S(x)y| = |S(y)x|\,.$$

Another important class of graphs is the class of Cayley graphs. If Γ is a finitely generated group with generating set $\{g_1, \ldots, g_n\}$, then the **Cayley graph** associated with Γ and that particular set of generators is the (unoriented) graph $G = (V, E)$ with vertex set $V := \Gamma$, and edge set

$$E := \{[x, y] : \ x, y \in \Gamma, \exists i \in \{1, \ldots, n\} \text{ such that } x g_i = y\}\,.$$

Obviously, a Cayley graph is transitive, and furthermore it is not hard to show that it is unimodular. Most graphs that have been studied in percolation theory are Cayley graphs. Examples include $\mathbb{Z}^d$ (which, with a slight abuse of notation, is short for the graph with vertex set $\mathbb{Z}^d$ and edges connecting pairs of vertices at Euclidean distance 1 from each other), and the regular tree $\mathbf{T}_n$ in which every vertex has exactly $n+1$ neighbors. The graph $\mathbb{Z}^d$ is amenable, while $\mathbf{T}_n$ is nonamenable for $n \geq 2$. Also studied are certain nonamenable tilings of the hyperbolic plane (see, e.g., [6] and [18]), and further examples can be obtained, e.g., by taking Cartesian products of two or more Cayley graphs.

2.2 Stochastic domination

Let E be any finite or countably infinite set. (In our applications, E will be an edge set; hence the notation.) For two configurations $\xi, \xi' \in \{0,1\}^E$, we write $\xi \preccurlyeq \xi'$ if $\xi(e) \leq \xi'(e)$ for all $e \in E$. A function $f : \{0,1\}^E \to \mathbf{R}$ is said to be increasing if $f(\xi) \leq f(\eta)$ whenever $\xi \preccurlyeq \eta$. For two probability measures μ and μ' on $\{0,1\}$, we say that μ is **stochastically dominated** by μ', writing $\mu \overset{\mathcal{D}}{\preccurlyeq} \mu'$, if

$$\int_{\{0,1\}^E} f d\mu \leq \int_{\{0,1\}^E} f d\mu' \tag{2}$$

for all bounded increasing f.

By a **coupling** of μ and μ', or of two random objects X and X' with distributions μ and μ', we simply mean a joint construction of two random objects with the prescribed distributions on a common probability space.

By Strassen's Theorem (see, e.g., [25]), $\mu \overset{\mathcal{D}}{\preccurlyeq} \mu'$ is equivalent to the existence of a coupling $\mathbf{P}$ of two random objects X and X' with distributions μ and μ', such that $\mathbf{P}(X \preccurlyeq X') = 1$. We call such a coupling a **witness** to the stochastic domination (2).

A useful tool for establishing stochastic domination is the well-known Holley's Inequality. For $E' \subset E$ and $\xi \in \{0,1\}^E$, we let $\xi(E')$ denote the restriction of ξ to E'.

Lemma 2.1 (Holley's Inequality). *Let E be finite, and let μ and μ' be probability measures on $\{0,1\}^E$ that assign positive probability to all elements of $\{0,1\}^E$. Suppose that μ and μ' satisfy*

$$\mu(X(e) = 1 \,|\, X(E \setminus \{e\}) = \xi) \leq \mu'(X(e) = 1 \,|\, X(E \setminus \{e\}) = \xi')$$

for all $e \in E$, and all $\xi, \xi' \in \{0,1\}^{E \setminus \{e\}}$ such that $\xi \preccurlyeq \xi'$. Then $\mu \overset{\mathcal{D}}{\preccurlyeq} \mu'$.

This is not the most general form of Holley's Inequality, but one that is sufficient for our purposes. For a proof, see, e.g., [15] (Theorem 4.8).

We shall also need the notion of weak convergence of probability measures on $\{0,1\}^E$, when E is countably infinite. For such probability measures $\mu_1, \mu_2, \ldots$ and μ, we say that μ is the (weak) limit of μ_i as $i \to \infty$ if $\lim_{i\to\infty} \mu_i(A) = \mu(A)$ for all cylinder events A.

2.3 The random-cluster model

Let $G = (V, E)$ be a finite graph. An element ξ of $\{0,1\}^E$ will be identified with the subgraph of G that has vertex set V and edge set $\{e \in E : \ \xi(e) = 1\}$. An edge e with $\xi(e) = 1$ (resp. $\xi(e) = 0$) is said to be open (resp. closed). A central quantity to the random-cluster model is the number of connected components of ξ, which will be denoted $\|\xi\|$. We emphasize that in the definition of $\|\xi\|$, isolated vertices in ξ also count as connected components.

The **random-cluster measure** $\mathsf{RC} := \mathsf{RC}^G_{p,q}$ (sub- and superscripts will be dropped whenever possible) with parameters $p \in [0,1]$ and $q > 0$, is defined as the probability measure on $\{0,1\}^E$ that to each $\xi \in \{0,1\}^E$ assigns probability

$$\mathsf{RC}(\xi) := \frac{q^{\|\xi\|}}{Z} \prod_{e \in E} p^{\xi(e)}(1-p)^{1-\xi(e)}\,, \tag{3}$$

where $Z := Z^G_{p,q} := \sum_{\xi \in \{0,1\}^E} q^{\|\xi\|} \prod_{e \in E} p^{\xi(e)}(1-p)^{1-\xi(e)}$ is a normalizing constant making RC a probability measure.

When $q = 1$, we see that all edges are independently open and closed with respective probabilities p and $1-p$, so that we get the usual i.i.d. bond percolation model on G. All other choices of q yield dependence between the edges. Throughout the paper, we shall assume (as in most studies of the random-cluster model) that $q \geq 1$. The main reason for doing so is that when $q \geq 1$, the conditional probability in eq. (4) below becomes increasing not only in p but also in ξ, and this allows some very powerful stochastic domination arguments, based on Holley's Inequality (Lemma 2.1), to come into play; these are not available for $q < 1$. Furthermore, it is only random-cluster measures with $q \in \{2, 3, \ldots\}$ that have proved to be useful in the analysis of Potts models.

It is immediate from the definition that if X is a $\{0,1\}^E$-valued random object with distribution RC, then we have, for each $e = [x, y] \in E$ and each $\xi \in \{0,1\}^{E \setminus \{e\}}$, that

$$\mathsf{RC}\big(X(e) = 1 \bigm| X(E \setminus \{e\}) = \xi\big) = \begin{cases} p & \text{if } x \leftrightarrow y, \\ \frac{p}{p+(1-p)q} & \text{otherwise,} \end{cases} \tag{4}$$

where $x \leftrightarrow y$ is the event that there is an open path (i.e., a path of open edges) from x to y in $X(E \setminus \{e\})$. As a first application of Holley's Inequality, we get from (4) that

$$\mathsf{RC}^G_{p,q}\big(X \in \cdot \bigm| X(E') = \xi\big) \overset{\mathcal{D}}{\preccurlyeq} \mathsf{RC}^G_{p,q}\big(X \in \cdot \bigm| X(E') = \xi'\big) \tag{5}$$

whenever $E' \subseteq E$ and $\xi \preccurlyeq \xi'$.

Our next task is to define the random-cluster model on infinite graphs. Let $G = (V, E)$ be infinite and locally finite. The definition (3) of random-cluster measures does not work in this case, because there are uncountably many different configurations $\xi \in \{0,1\}^E$. Instead, there are two other approaches to defining random-cluster measures on infinite graphs: one via limiting procedures, and the other via local specifications, also known as the Dobrushin-Lanford-Ruelle (DLR) equations. We shall sketch the first approach.

Let $V_1, V_2, \ldots$ be a sequence of finite vertex sets increasing to V in the sense that $V_1 \subset V_2 \subset \ldots$ and $\bigcup_{i=1}^\infty V_i = V$. For any finite $K \subseteq V$, define

$$E(K) := \big\{[x, y] \in E : \;\; x, y \in K\big\},$$

set $E_i := E(V_i)$ and note that $E_1, E_2, \ldots$ increases to E in the same sense that $V_1, V_2, \ldots$ increases to V. Let ∂V_i be the (inner) boundary of V_i (defined as in (1)). Also set $G_i := (V_i, E_i)$, and let $\mathsf{FRC}^{G,i}_{p,q}$ be the probability measure on $\{0,1\}^E$ corresponding to picking $X \in \{0,1\}^E$ by letting $X(E_i)$ have distribution $\mathsf{RC}^{G_i}_{p,q}$ and setting $X(e) := 0$ for all $e \in E \setminus E_i$. Since the projection of $\mathsf{FRC}^{G,i}_{p,q}$ on $\{0,1\}^{E \setminus E_i}$ is nonrandom, we can also view $\mathsf{FRC}^{G,i}_{p,q}$ as a measure on $\{0,1\}^{E_i}$, in which case it coincides with $\mathsf{RC}^{G_i}_{p,q}$. Applying (5) to the graph G_i with $E' := E_i \setminus E_{i-1}$ and $\xi \equiv 0$ gives

$$\mathsf{FRC}^{G,i-1}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G,i}_{p,q},$$

so that

$$\mathsf{FRC}^{G,1}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G,2}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \cdots . \tag{6}$$

This implies the existence of a limiting (as $i \to \infty$) probability measure $\mathsf{FRC}^G_{p,q}$ on $\{0,1\}^E$. This limit is independent of the choice of $\{V_i\}_{i=1}^\infty$, and we call it the random-cluster measure on G with **free boundary condition** (hence the F in FRC) and parameters p and q.

Next, define $\mathsf{WRC}^{G,i}_{p,q}$ as the probability measure on $\{0,1\}^E$ corresponding to first setting $X(E \setminus E_i) \equiv 1$, and then picking $X(E)$ in such a way that

$$\mathsf{WRC}^{G,i}_{p,q}\big(X(E_i) = \xi\big) = \frac{q^{\|\xi\|^*}}{Z} \prod_{e \in E_i} p^{\xi(e)}(1-p)^{1-\xi(e)}$$

where $\|\xi\|^*$ is the number of connected components of ξ **that do not intersect** ∂V_i, and Z is again a normalizing constant. Similarly as in (6), we get

$$\mathsf{WRC}^{G,1}_{p,q} \overset{\mathcal{D}}{\succcurlyeq} \mathsf{WRC}^{G,2}_{p,q} \overset{\mathcal{D}}{\succcurlyeq} \cdots$$

(with the inequalities reversed compared to (6)), and thus also a limiting measure $\mathsf{WRC}^G_{p,q}$ that we call the random-cluster measure on G with **wired boundary condition** and parameters p and q.

Note that the free and wired random-cluster measures FRC and WRC are both automorphism invariant. This follows from their construction, in particular from the independence of the choice of $\{G_i = (V_i, E_i)\}_{i=1}^\infty$.

2.4 The Potts model

Fix a finite graph $G = (V, E)$ and the inverse temperature parameter $\beta \geq 0$. We define the **Gibbs measure for the q-state Potts model on G at inverse temperature** β, denoted $\mathsf{Pt} := \mathsf{Pt}^G_{q,\beta}$, as the probability measure that to each $\omega \in \{1, \ldots, q\}^V$ assigns probability

$$\mathsf{Pt}(\omega) := \frac{1}{Z} \exp\left(-2\beta \sum_{[x,y] \in E} \mathbf{1}_{\{\omega(x) \neq \omega(y)\}}\right),$$

where Z is yet another normalizing constant. The main link between random-cluster and Potts models is the following well-known result. (See, e.g., [35].)

Proposition 2.2. *Fix a finite graph G, an integer $q \geq 2$ and $p \in [0,1]$. Pick a random edge configuration $X \in \{0,1\}^E$ according to the random-cluster measure $\mathsf{RC}^G_{p,q}$. Then, for each connected component $\mathcal{C}$ of X, pick a spin uniformly from $\{1, \ldots, q\}$, and assign this spin to all vertices of $\mathcal{C}$. Do this independently for different connected components. The $\{1, \ldots, q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{Pt}^G_{q,\beta}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$.*

This provides the way (mentioned in the introduction) to reformulate problems about pairwise dependencies in the Potts model into problems about connectivity probabilities in the random-cluster model. Aizenman et al. [2] were the first to exploit such ideas to obtain results about the phase transition behavior of the Potts model, and the technique has been of much use since then.

The case of infinite graphs is slightly more intricate. Let $G = (V, E)$ be infinite and locally finite, and let $\{G_i := (V_i, E_i)\}_{i=1}^\infty$ be as in Section 2.3. For $q \in \{2, 3, \ldots\}$

and $\beta \geq 0$, define probability measures $\left\{\mathsf{FPt}^{G,i}_{q,\beta}\right\}_{i=1}^{\infty}$ on $\{1,\ldots,q\}^V$ in such a way that the projection of $\mathsf{FPt}^{G,i}_{q,\beta}$ on $\{1,\ldots,q\}^{V_i}$ equals $\mathsf{Pt}^{G_i}_{q,\beta}$, and the spins on $V \setminus V_i$ are i.i.d. uniformly distributed on $\{1,\ldots,q\}$ and independent of the spins on V_i. Using Proposition 2.2, one can show that $\mathsf{FPt}^{G,i}_{q,\beta}$ has a limiting distribution $\mathsf{FPt}^{G}_{q,\beta}$ as $i \to \infty$.

Furthermore, for a fixed spin $r \in \{1,\ldots,q\}$, define $\mathsf{WPt}^{G,i}_{q,\beta,r}$ to be the distribution corresponding to picking $X \in \{1,\ldots,q\}^V$ by letting $X(V \setminus V_i) \equiv r$, and letting $X(V_i)$ be distributed according to $\mathsf{Pt}^{G_i}_{q,\beta}$ **conditioned on the event that** $X(\partial V_i) \equiv r$. Again, it turns out that $\mathsf{WPt}^{G,i}_{q,\beta,r}$ has a limiting distribution as $i \to \infty$, and we denote it by $\mathsf{WPt}^{G}_{q,\beta,r}$.

The existence of the limiting distributions $\mathsf{FPt}^{G}_{q,\beta}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$ are nontrivial results, and in fact the shortest route to proving them goes via random-cluster arguments: First carry out the stochastic monotonicity arguments for the random-cluster model outlined in Section 2.3, and then use Propositions 2.3 and 2.4 below.

A probability measure μ on $\{1,\ldots,q\}^V$ is said to be a Gibbs measure (in the DLR sense) for the q-state Potts model on G at inverse temperature β, if it admits conditional distributions such that for all $v \in V$, all $r \in \{1,\ldots,q\}$, and all $\omega \in \{1,\ldots,q\}^{V\setminus\{v\}}$, we have

$$\mu\big(X(v) = r \mid X(V \setminus \{v\}) = \omega\big) = \frac{1}{Z}\exp\Big(-2\beta \sum_{[v,y]\in E} \mathbf{1}_{\{\omega(y)\neq r\}}\Big), \tag{7}$$

where the normalizing constant Z may depend on v and ω but not on r. The limiting measures $\mathsf{FPt}^{G}_{q,\beta}$ and $\mathsf{WPt}^{G}_{q,\beta,r}$ are both Gibbs measures in this sense.

The following extensions of Proposition 2.2 provide the relations between FRC and WRC on one hand, and FPt and WPt on the other.

Proposition 2.3. *Let G be an infinite locally finite graph, and fix $q \in \{2,3,\ldots\}$ and $p \in [0,1]$. Pick a random edge configuration $X \in \{0,1\}^E$ according to $\mathsf{FRC}^{G}_{p,q}$. Then, for each connected component $\mathcal{C}$ of X independently, pick a spin uniformly from $\{1,\ldots,q\}$, and assign this spin to all vertices of $\mathcal{C}$. The $\{1,\ldots,q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{FPt}^{G}_{q,\beta}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$.*

Proposition 2.4. *Let G, p and q be as in Proposition 2.3. Pick a random edge configuration $X \in \{0,1\}^E$ according to the random-cluster measure $\mathsf{WRC}^{G}_{p,q}$. Then, for each* **finite** *connected component $\mathcal{C}$ of X independently, pick a spin uniformly from $\{1,\ldots,q\}$, and assign this spin to all vertices of $\mathcal{C}$. Finally assign value r to all vertices of infinite connected components. The $\{1,\ldots,q\}^V$-valued random spin configuration arising from this procedure is then distributed according to the Gibbs measure $\mathsf{WPt}^{G}_{q,\beta,r}$ for the q-state Potts model on G at inverse temperature $\beta := -\frac{1}{2}\log(1-p)$.*

3 A dynamical construction

Let $G = (V,E)$ be infinite and locally finite, and let $\{G_i := (V_i,E_i)\}_{i=1}^{\infty}$ be as in Section 2. We know from Section 2.3 that

$$\mathsf{FRC}^{G,1}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G,2}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \cdots \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \cdots \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G,2}_{p,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G,1}_{p,q}\,. \tag{8}$$

Other well-known stochastic inequalities are that for $p_1 \leq p_2$ and $i \in \{1, 2, \ldots\}$, we have

$$\mathsf{FRC}^{G,i}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G,i}_{p_2,q}\,, \tag{9}$$

$$\mathsf{FRC}^{G}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^{G}_{p_2,q}\,, \tag{10}$$

$$\mathsf{WRC}^{G,i}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G,i}_{p_2,q}\,, \tag{11}$$

and

$$\mathsf{WRC}^{G}_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{WRC}^{G}_{p_2,q}\,. \tag{12}$$

For all of the above stochastic inequalities, it is desirable to find some natural construction of couplings that witness them. What we shall construct in this section is a coupling of all of the above probability measures (for all $p \in [0, 1]$, $q \geq 1$ and $i \in \{1, 2, \ldots\}$) *simultaneously* that provides witnesses to the stochastic inequalities (8)–(12) above. Some additional useful aspects of the construction are the following.

(A1) Not only are FRC and WRC automorphism invariant separately, but also their joint behavior in our coupling is automorphism invariant. This remains true also if we consider the realizations simultaneously for different parameter values. See Section 5.1, where we describe an application where this property is crucial.

(A2) If G is obtained as an automorphism-invariant percolation process on another graph H, then the construction is easily set up in such a way that the joint distribution of G and the random-cluster measures on G becomes an automorphism-invariant process on H. (See [21] for an example where an analogous property turns out to be important in the context of Ising models with external field on percolation clusters.)

Nevertheless, there are still some desirable aspects of couplings of random-cluster processes for which we do not know whether or not they hold for our construction; see Conjecture 5.1 and Question 5.2 in the final section.

The construction is based on time dynamics for the random-cluster model. Such time dynamics have previously been considered, e.g., by Bezuidenhout, Grimmett and Kesten [8] and by Grimmett [16] for the random-cluster model on $\mathbb{Z}^d$. To some extent our construction will resemble Grimmett's analysis. However, one feature of our construction that differs from Grimmett's is that the dynamics are run "from the past" rather than "into the future", along the lines of the very fashionable CFTP (coupling from the past) algorithm of Propp and Wilson [30]; see also [36] for an early treatment of dynamics from the past, and [11] for a survey putting the ideas in a more general mathematical context. For the case of finite graphs, CFTP was applied to simulate the random-cluster model in [30]. Simulation on infinite graphs would require additional arguments, but our purpose is not simulation; rather, it is to gain some theoretical information. For models other than the random-cluster model, CFTP ideas have been extended to the setting of infinite graphs in [7], [22] and [21], but in all those cases the interaction of the dynamics had a strictly local character, which is not the case in our context. Another feature of our construction is the simultaneity in the parameter space. Such simultaneity, which is related to the level-set representations of Higuchi [23], appears in both [16] and [30]; Propp and Wilson use the term "omnithermal" to denote this particular feature of the construction.

Let us start with a simple finite case: how do we construct a $\{0,1\}^{E_i}$-valued random element with distribution $\mathsf{RC}_{p,q}^{G_i}$ (equivalently, with distribution $\mathsf{FRC}_{p,q}^{G,i}$)? If we are content with getting something that has only approximately the right distribution, then the following dynamical approach works fine: Define some ergodic Markov chain whose unique equilibrium distribution is $\mathsf{RC}_{p,q}^{G_i}$, and run it for time T starting from an arbitrary initial state ξ. If T is large enough, then the distribution of the final state is close to $\mathsf{RC}_{p,q}^{G_i}$, regardless of the choice of ξ.

In particular, we may proceed as follows. To each edge $e \in E_i$, we independently assign an i.i.d. sequence $(\phi_1^e, \phi_2^e, \ldots)$ of exponential random variables with mean 1, and an independent i.i.d. sequence $(U_1^e, U_2^e, \ldots)$ of uniform $[0,1]$ random variables. For $e \in E_i$ and $k = 1, 2, \ldots$, let $\tau_k^e := \phi_1^e + \ldots + \phi_k^e$, so that $(\tau_1^e, \tau_2^e, \ldots)$ are the jump times of a unit rate Poisson process. Now define a $\{0,1\}^{E_i}$-valued continuous-time Markov chain $\{{}^{\xi}\bar{X}_{p,q}^{G_i}(t)\}_{t \geq 0}$ with starting state ${}^{\xi}\bar{X}_{p,q}^{G_i}(0) := \xi$ and evolution as follows. For $e := [x,y] \in E_i$, the value of ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)(e)$ does not change other than (possibly) at the times $\tau_1^e, \tau_2^e, \ldots$, at which times it takes the value

$$ {}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(e) := \begin{cases} 1 & \text{if } U_k^e < p \text{ and } x \leftrightarrow y \text{ in } {}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(E_i \setminus \{e\}) \\ 1 & \text{if } U_k^e < \frac{p}{p+(1-p)q} \text{ and } \neg\left(x \leftrightarrow y \text{ in } {}^{\xi}\bar{X}_{p,q}^{G_i}(\tau_k^e)(E_i \setminus \{e\})\right) \\ 0 & \text{otherwise,} \end{cases} \tag{13} $$

where $\neg$ denotes negation. (Note that a.s., $\tau_k^e \neq \tau_j^{e'}$ for all j, k when $e \neq e'$.) It is easy to see that this Markov chain is irreducible and reversible with $\mathsf{RC}_{p,q}^{G_i}$ as stationary distribution, so that indeed ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)$ converges in distribution to $\mathsf{RC}_{p,q}^{G_i}$ as $t \to \infty$. Note also that since $p \geq \frac{p}{p+(1-p)q}$, the chain preserves the partial order $\preccurlyeq$ on $\{0,1\}^{E_i}$; in other words, for all $t \geq 0$ we have

$$ {}^{\xi}\bar{X}_{p,q}^{G_i}(t) \preccurlyeq {}^{\eta}\bar{X}_{p,q}^{G_i}(t) \quad \text{whenever} \quad \xi \preccurlyeq \eta \,. \tag{14} $$

To get a $\{0,1\}^{E_i}$-valued random object whose distribution is precisely $\mathsf{RC}_{p,q}^{G_i}$, we need to consider some limit as $t \to \infty$. On the other hand, ${}^{\xi}\bar{X}_{p,q}^{G_i}(t)$ does not converge in any a.s. sense, so this may appear not to be feasible.

The solution, which turns the convergence in distribution into a.s. convergence, is to run the dynamics from the past up to time 0, rather than from time 0 into the future. For $T \geq 0$, define the $\{0,1\}^{E_i}$-valued continuous-time Markov chain

$$ \left\{ {}^{\text{free}}_{-T}X_{p,q}^{G_i}(t) \right\}_{t \in [-T, 0]} $$

with starting state ${}^{\text{free}}_{-T}X_{p,q}^{G_i}(-T) \equiv 0$ and the following evolution, similar to the one of ${}^{\xi}\bar{X}_{p,q}^{G_i}$. The value at an edge $e := [x,y] \in E_i$ changes only at times $(\ldots, -\tau_2^e, -\tau_1^e)$, when it takes the value

$$ {}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(e) := \begin{cases} 1 & \text{if } U_k^e < p \text{ and } x \leftrightarrow y \text{ in } {}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(E_i \setminus \{e\}) \\ 1 & \text{if } U_k^e < \frac{p}{p+(1-p)q} \text{ and } \neg\left(x \leftrightarrow y \text{ in } {}^{\text{free}}_{-T}X_{p,q}^{G_i}(-\tau_k^e)(E_i \setminus \{e\})\right) \\ 0 & \text{otherwise,} \end{cases} \tag{15} $$

as in (13). We have, for $0 \leq T_1 \leq T_2$, that

$$ {}^{\text{free}}_{-T_1}X_{p,q}^{G_i}(0) \preccurlyeq {}^{\text{free}}_{-T_2}X_{p,q}^{G_i}(0) $$

(essentially because of (14)), so by monotonicity ${}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)$ has an a.s. limit ${}^{\text{free}}X^{G_i}_{p,q} \in \{0,1\}^{E_i}$, defined by setting ${}^{\text{free}}X^{G_i}_{p,q}(e) := \lim_{T\to\infty} {}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)(e)$ for each $e \in E_i$. Clearly, ${}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)$ has the same distribution as ${}^{\xi}\bar{X}^{G_i}_{p,q}(T)$ with $\xi \equiv 0$, so ${}^{\text{free}}_{-T}X^{G_i}_{p,q}(0)$ converges in distribution to $\mathsf{RC}^{G_i}_{p,q}$ as $T \to \infty$. Hence ${}^{\text{free}}X^{G_i}_{p,q}$ has distribution $\mathsf{RC}^{G_i}_{p,q}$, and if we furthermore define ${}^{\text{free}}X^{G,i}_{p,q} \in \{0,1\}^E$ by setting

$$
{}^{\text{free}}X^{G,i}_{p,q}(e) := \begin{cases} {}^{\text{free}}X^{G_i}_{p,q}(e) & \text{for } e \in E_i \\ 0 & \text{otherwise} \end{cases}
$$

for each $e \in E$, then ${}^{\text{free}}X^{G,i}_{p,q}$ has distribution $\mathsf{FRC}^{G,i}_{p,q}$.

Now suppose that we have defined the random variables $(\phi^e_1, \phi^e_2, \ldots)$ and $(U^e_1, U^e_2, \ldots)$ for all $e \in E$ (and not just all $e \in E_i$) in the obvious way. By another application of the order-preserving property (14), we get that

$$
{}^{\text{free}}X^{G,1}_{p,q} \preccurlyeq {}^{\text{free}}X^{G,2}_{p,q} \preccurlyeq \ldots
$$

so that the limiting object ${}^{\text{free}}X^{G}_{p,q}$, defined by taking ${}^{\text{free}}X^{G}_{p,q}(e) := \lim_{i\to\infty} {}^{\text{free}}X^{G,i}_{p,q}(e)$, exists. For any cylinder set $A \in \{0,1\}^E$, we have

$$
\mathbf{P}\left({}^{\text{free}}X^{G}_{p,q} \in A\right) = \lim_{i\to\infty} \mathbf{P}\left({}^{\text{free}}X^{G,i}_{p,q} \in A\right) = \lim_{i\to\infty} \mathsf{FRC}^{G,i}_{p,q}(A) = \mathsf{FRC}^{G}_{p,q}(A) \qquad (16)
$$

so that ${}^{\text{free}}X^{G}_{p,q}$ has distribution $\mathsf{FRC}^{G}_{p,q}$. Thus, to summarize the construction so far, what we have is a coupling of $\{0,1\}^E$-valued random objects ${}^{\text{free}}X^{G,1}_{p,q}, {}^{\text{free}}X^{G,2}_{p,q}, \ldots$ and ${}^{\text{free}}X^{G}_{p,q}$ that witnesses the stochastic inequalities in the first half of (8).

Next, we go on to construct, in analogous fashion, the corresponding objects for wired random-cluster measures. For $T \geq 0$, define the $\{0,1\}^E$-valued continuous-time Markov chain

$$
\left\{{}^{\text{wired}}_{-T}X^{G,i}_{p,q}(t)\right\}_{t\in[-T,0]}
$$

with starting configuration ${}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-T) \equiv 1$. Edges $e \in E \setminus E_i$ remain in state 1 forever, while the value of an edge $e := [x,y] \in E_i$ is updated at times $(\ldots, -\tau^e_2, -\tau^e_1)$, when it takes the value

$$
{}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-\tau^e_k)(e) := \begin{cases} 1 & \text{if } U^e_k < p \text{ and } A(x,y,i,p,q,e,k) \\ 1 & \text{if } U^e_k < \frac{p}{p+(1-p)q} \text{ and } \neg A(x,y,i,p,q,e,k) \\ 0 & \text{otherwise;} \end{cases} \qquad (17)
$$

here, $A(x,y,i,p,q,e,k)$ is the event $\left\{x \overset{\partial V_i}{\longleftrightarrow} y \text{ in } {}^{\text{wired}}_{-T}X^{G,i}_{p,q}(-\tau^e_k)(E_i \setminus \{e\})\right\}$, where, in turn, $x \overset{\partial V_i}{\longleftrightarrow} y$ denotes the event that either

(a) there is an open path from x to y (not using e), or

(b) both x and y have open paths (not using e) to ∂V_i.

It is immediate from the definition of $\mathsf{WRC}^{G,i}_{p,q}$ that the conditional $\mathsf{WRC}^{G,i}_{p,q}$-probability that an edge $e := [x,y] \in E_i$ is open, given the status of all other edges, is p or

$p/[p+(1-p)q]$, depending on whether or not the event $x \overset{\partial V_i}{\longleftrightarrow} y$ happens. It follows that the distribution of ${}^{\text{wired}}_{-T}X^{G,i}_{p,q}(0)$ tends to $\mathsf{WRC}^{G,i}_{p,q}$ as $T \to \infty$. Moreover, the dynamics in (17) preserves $\preccurlyeq$ similarly as in (14), implying that

$$ {}^{\text{wired}}_{-T_1}X^{G,i}_{p,q} \succcurlyeq {}^{\text{wired}}_{-T_2}X^{G,i}_{p,q} $$

whenever $0 \le T_1 \le T_2$. This establishes the existence of a limiting $\{0,1\}^E$-valued random object ${}^{\text{wired}}X^{G,i}_{p,q}$ defined by ${}^{\text{wired}}X^{G,i}_{p,q}(e) := \lim_{T\to\infty} {}^{\text{wired}}X^{G,i}_{p,q}(0)(e)$ for each $e \in E$. Clearly, ${}^{\text{wired}}X^{G,i}_{p,q}$ has distribution $\mathsf{WRC}^{G,i}_{p,q}$. Another use of the $\preccurlyeq$-preserving property of the dynamics (17) shows that

$$ {}^{\text{wired}}X^{G,1}_{p,q} \succcurlyeq {}^{\text{wired}}X^{G,2}_{p,q} \succcurlyeq \cdots , $$

so that we have a limiting object ${}^{\text{wired}}X^{G}_{p,q} \in \{0,1\}^E$ defined by setting ${}^{\text{wired}}X^{G}_{p,q}(e) := \lim_{i\to\infty} {}^{\text{wired}}X^{G,i}_{p,q}(e)$ for each $e \in E$. By arguing as in (16), we get that ${}^{\text{wired}}X^{G}_{p,q}$ has distribution $\mathsf{WRC}^{G}_{p,q}$. The random objects ${}^{\text{wired}}X^{G,1}_{p,q}, {}^{\text{wired}}X^{G,2}_{p,q}, \ldots$ and ${}^{\text{wired}}X^{G}_{p,q}$ witness the stochastic inequalities in the second half of (8).

In order to fully establish that we have a witness to (8), it remains to show that ${}^{\text{free}}X^{G}_{p,q}$ and ${}^{\text{wired}}X^{G}_{p,q}$ witness the middle inequality in (8), i.e., we need to show that ${}^{\text{free}}X^{G}_{p,q} \preccurlyeq {}^{\text{wired}}X^{G}_{p,q}$. From the observations that the right-hand sides of (15) and (17) are increasing in the configurations on $E_i \setminus \{e\}$, and that for each such configuration the right-hand side of (17) is greater than that of (15), we get that

$$ {}^{\text{free}}_{-T}X^{G,i}_{p,q}(t) \preccurlyeq {}^{\text{wired}}_{-T}X^{G,i}_{p,q}(t) $$

for any $i \in \{1,2,\ldots\}$, $T \ge 0$ and $t \in [-T,0]$. By taking $t := 0$, letting $T \to \infty$ and then $i \to \infty$, we get

$$ {}^{\text{free}}X^{G}_{p,q} \preccurlyeq {}^{\text{wired}}X^{G}_{p,q} \tag{18} $$

as desired. Hence our coupling is a witness to all the inequalities in (8).

It remains to be demonstrated that the coupling is also a witness to the inequalities (9)–(12). Note first that the right-hand sides of (15) and (17) are increasing not only in the configurations on $E_i \setminus \{e\}$, but also in p. It follows that for $p_1 \le p_2$ we have

$$ {}^{\text{free}}_{-T}X^{G,i}_{p_1,q}(t) \preccurlyeq {}^{\text{free}}_{-T}X^{G,i}_{p_2,q}(t) $$

and

$$ {}^{\text{wired}}_{-T}X^{G,i}_{p_1,q}(t) \preccurlyeq {}^{\text{wired}}_{-T}X^{G,i}_{p_2,q}(t) $$

for all $i \in \{1,2,\ldots\}$, $T \ge 0$ and $t \in [-T,0]$. Taking $t := 0$ and letting $T \to \infty$ yields

$$ {}^{\text{free}}X^{G,i}_{p_1,q} \preccurlyeq {}^{\text{free}}X^{G,i}_{p_2,q} $$

and

$$ {}^{\text{wired}}X^{G,i}_{p_1,q} \preccurlyeq {}^{\text{wired}}X^{G,i}_{p_2,q} , $$

witnessing (9) and (11). Letting $i \to \infty$, we get

$$ {}^{\text{free}}X^{G}_{p_1,q} \preccurlyeq {}^{\text{free}}X^{G}_{p_2,q} \tag{19} $$

and

$$ {}^{\text{wired}}X^G_{p_1,q} \preccurlyeq {}^{\text{wired}}X^G_{p_2,q}\,, \tag{20} $$

finally witnessing (10) and (12). In fact, examination also shows that as long as $p_1 \le p_2$ and $p_1/[(1-p_1)q_1] \le p_2/[(1-p_2)q_2]$, we have

$$ {}^{\text{free}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{free}}X^G_{p_2,q_2}\,, \tag{21} $$

$$ {}^{\text{wired}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{wired}}X^G_{p_2,q_2}\,, \tag{22} $$

and

$$ {}^{\text{free}}X^G_{p_1,q_1} \preccurlyeq {}^{\text{wired}}X^G_{p_2,q_2}\,, \tag{23} $$

witnessing more general well-known stochastic inequalities [13].

Property (A1) of the coupling is obvious from the construction. In order for (A2) to be true, we need only define random variables $\{\phi_k^e, U_k^e\}_{e\in E(H),i=1,2,\ldots}$ for all edges in H and to take them to be independent of the percolation process that yields G from H.

4 Bernoullicity

Let Γ be a closed subgroup of $\mathrm{Aut}(G)$ with $G = (V,E)$ being any connected graph. We shall be most interested in two cases: (1) that G is the Cayley graph of Γ with respect to some finite generating set of Γ; and (2) that $\Gamma = \mathrm{Aut}(G)$ and G is quasi-transitive. Let S and T be arbitrary state spaces. For $\gamma \in \Gamma$, define the map $\theta_\gamma : S^V \to S^V$ (or $\theta_\gamma : T^V \to T^V$) by setting $\theta_\gamma\omega(x) := \omega(\gamma^{-1}x)$ for each $x \in V$. A measurable mapping $f : (S^V,\mu) \to (T^V,\nu)$ is said to be **Γ-equivariant** if it commutes with these actions of Γ, i.e., if $f(\theta_\gamma\omega) = \theta_\gamma\big(f(\omega)\big)$ for all $\gamma \in \Gamma$ and μ-a.e. $\omega \in S^V$; it is called **measure-preserving** if $\nu = \mu\circ f^{-1}$. The action of Γ on (T^V,ν) is called **free** if for ν-a.e. $x \in T^V$, the only element in Γ that leaves x fixed is the identity.

We say that a probability measure ν on T^V is a **Γ-factor of an i.i.d. process** if there exists a T^V-valued random element X with distribution ν, a state space S, an S^V-valued random element Y with distribution μ, and a Γ-equivariant measure-preserving mapping $f : (S^V,\mu) \to (T^V,\nu)$ such that

(i) Y is an i.i.d. process, and

(ii) $X = f(Y)$.

In case G is the Cayley graph of Γ, if S can be taken to be finite and f can be taken to be an invertible mapping, then (Γ,ν) is said to be **Bernoulli**, a mixing property of fundamental importance in ergodic theory. In [29, p. 127], it is shown that the following definition is a proper extension of the preceding definition: An action (Γ,ν) is said to be **Bernoulli** if it is a free Γ-factor of a Poisson process on Γ. We shall prove, using the dynamical construction in Section 3, that Bernoullicity holds for the wired Potts model on $\mathbb{Z}^d$, and more generally on many amenable quasi-transitive graphs. We shall need the following condition. Let $S_n(x)$ denote the set of points at distance n from a vertex x. Consider the condition on Γ that

$$ \forall x \in V \quad \forall y \in \Gamma x \setminus \{x\} \quad \exists \text{ infinitely many } n \qquad S_n(x) \ne S_n(y)\,. \tag{24} $$

Theorem 4.1. *Let G be a Cayley graph of any amenable group Γ or be any amenable graph with a closed automorphism group Γ acting quasi-transitively on G and satisfying (24). Let $q \in \{2, 3, \ldots\}$, $r \in \{1, \ldots, q\}$, and $\beta \geq 0$. Then the Gibbs measure $\mathsf{WPt}^G_{q,\beta,r}$ is Bernoulli with respect to the action of Γ.*

For the $\mathbb{Z}^d$ case, this was previously known only for the cases where either $q = 2$ (the Ising model) or β is sufficiently small; see, e.g., [28], [24] and [34]. For the Ising model result on amenable graphs, see [1], while for a proof of a stronger property than Bernoullicity in the case of β small, using CFTP ideas, see [22]. The paper [21] uses ideas similar to ours to prove that the Ising model is Bernoulli.

Remark 4.2. Actually, we shall prove a slightly stronger result, which is the best possible. That is, we shall show that as long as i.i.d. variables on the vertices of G yield a free action of Γ, then $\mathsf{WPt}^G_{q,\beta,r}$ is Bernoulli. It is not clear when the full automorphism group $\mathrm{Aut}(G)$ satisfies this freeness condition, so we have supplied the condition (24).

We call an i.i.d. process (S^V, μ) **standard** if S is a standard Borel space and the marginal of μ on S is Borel. Ornstein and Weiss [29] show that when Γ is amenable and discrete, then (Γ, ν) is Bernoulli iff it is a free Γ-factor of a standard i.i.d. process. More generally, we have the following result:

Lemma 4.3. *Let V be a countable set and Γ be a closed subgroup of the symmetric group on V. Suppose that all orbits of the Γ-action on V are infinite and that Γ is amenable, unimodular, and not the union of an increasing sequence of compact proper subgroups of Γ. Further, suppose that for each $x \in V$, the Γ-stabilizer of x is compact. Then every free Γ-factor of a standard i.i.d. process (S^V, μ) is Bernoulli.*

Proof. Assume that there is some free Γ-factor ν of a standard i.i.d. process (S^V, μ), since otherwise there is nothing to prove. Let Z_n be i.i.d. Poisson point processes on Γ with Haar measure as the underlying intensity measure. By [29, Theorem III.6.5], the product process $\langle Z_n : \ n \geq 1\rangle$ is Bernoulli. We shall show that ν is a Γ-factor of $\langle Z_n : \ n \geq 1\rangle$, whence is a factor of a Poisson process, whence is Bernoulli.

Let W be a selection of one point from each orbit of the action of Γ on V. Given $v \in V$, let $X_n(v)$ be the number of points in Z_n that take o to v for $v \in V$, where $\{o\} = W \cap \Gamma v$. Since Γ is a countable union of translates of stabilizers, each stabilizer has positive finite Haar measure, so that $X_n(v)$ is a nontrivial Poisson random variable. Also, the random variables $\langle X_n(v) : \ n \geq 1, v \in V\rangle$ are mutually independent. Since X_n is a Γ-factor of Z_n, it follows that $\langle X_n : \ n \geq 1\rangle$ is a Γ-factor of $\langle Z_n\rangle$. Since every standard i.i.d. process (S^V, μ) is a Γ-factor of $\langle X_n : \ n \geq 1\rangle$ and ν is a factor of (S^V, μ), we obtain the result we want. $\square$

We also need the following fact:

Lemma 4.4. *If G is a quasi-transitive amenable graph, then* $\mathrm{Aut}(G)$ *is amenable, unimodular, and not the union of an increasing sequence of compact proper subgroups.*

Proof. $\mathrm{Aut}(G)$ is amenable and unimodular by results of Soardi and Woess [33] and Salvatori [31]; see also [4] for another proof. Furthermore, in this case $\mathrm{Aut}(G)$ is generated by, say, the compact set $\Delta := \{\gamma \in \mathrm{Aut}(G) : \ d(o, \gamma o) \leq 2r + 1\}$, where r is such that every vertex of G is within distance r of some vertex in $\mathrm{Aut}(G)o$ and $d(\cdot, \cdot)$ denotes distance in G. Thus, if Γ_n are compact increasing subgroups of $\mathrm{Aut}(G)$ whose union is $\mathrm{Aut}(G)$, we have $\bigcap_{n\geq 1}(\Delta \setminus \Gamma_n) = \emptyset$, whence for some n, we have $\Delta \subseteq \Gamma_n$. Since Δ generates $\mathrm{Aut}(G)$, it follows that $\Gamma_n = \mathrm{Aut}(G)$. $\square$

Because of the above, Theorem 4.1 is established once the following lemma is proved:

Lemma 4.5. *For any graph G, any subgroup Γ of $\mathrm{Aut}(G)$, any $q \in \{2, 3, \ldots\}$ and $r \in \{1, \ldots, q\}$, and any $\beta \geq 0$, the Gibbs measure $\mathsf{WPt}^G_{q,\beta,r}$ is a Γ-factor of a standard i.i.d. process. If either (i) Γ is countable and every element of Γ other than the identity moves an infinite number of vertices or (ii) Γ satisfies condition (24), then the action of Γ on $\mathsf{WPt}^G_{q,\beta,r}$ is free.*

Proof. Let the degree of G be d. For each $x \in V$, let $N_x = \{Z^x_1, \ldots, Z^x_d\}$ be the set of neighbors of x in any fixed order.

Take

$$S := \big\{[0,\infty) \times [0,1]\big\}^{\{1,2,\ldots\}\times\{1,\ldots,d\}} \times [0,1]^d \times [0,1] \times \{1, \ldots, q\}\,.$$

Let

$$\Big\{\phi^j_k(x), U^j_k(x), U^j_*(x), U^*(x), \sigma(x) :\ k = 1, 2, \ldots,\ j = 1, \ldots, d,\ x \in V\Big\}$$

be independent random variables with $\phi^j_k(x)$ exponential of mean 1, $U^j_k(x)$, $U^j_*(x)$, and $U^*(x)$ uniform $[0,1]$, and $\sigma(x)$ uniform on $\{1, \ldots, q\}$. For each $x \in V$, put

$$Y(x) := \Big(\big(\phi^j_k(x), U^j_k(x)\big)_{k=1,2,\ldots,j=1,\ldots,d}, \big(U^j_*(x)\big)_{j=1,\ldots,d}, U^*(x), \sigma(x)\Big)\,.$$

Set $p := 1 - e^{-2\beta}$, and construct a $\{0,1\}^E$-valued edge configuration $X^G_{p,q}$ with distribution $\mathsf{WRC}^G_{p,q}$ by the dynamical construction in Section 3, where for each $e \in E$ we take

$$(\phi^e_k, U^e_k)_{k=1,2,\ldots} := \big(\phi^j_k(x), U^j_k(x)\big)_{k=1,2,\ldots}\,, \tag{25}$$

where $x \in V$ and $j \in \{1, \ldots, d\}$ are chosen in such a way that $e = [x, Z^x_j]$, and, if we denote $y := Z^x_j$ and j' is such that $x = Z^y_{j'}$, then $U^j_*(x) < U^{j'}_*(y)$. This choice of x and j is a.s. unique.

From $X^G_{p,q}$, we obtain the desired spin configuration $X \in \{1, \ldots, q\}^V$ with distribution $\mathsf{WPt}^G_{q,\beta,r}$ by assigning spins to the connected components of $X^G_{p,q}$ as in Proposition 2.4: All vertices in infinite connected components in $X^G_{p,q}$ are assigned value r, whereas the vertices of each finite connected component $\mathcal{C}$ are assigned value $\sigma(x)$, where x is the vertex in $\mathcal{C}$ that minimizes $U^*(x)$. It is obvious that this mapping $Y \mapsto X$ from S^V to $\{1, \ldots, q\}^V$ is $\mathrm{Aut}(G)$-equivariant, and that the resulting spin configuration has distribution $\mathsf{WPt}^G_{q,\beta,r}$. Hence $\mathsf{WPt}^G_{q,\beta,r}$ is a factor of a standard i.i.d. process.

To see that the action of Γ on $\mathsf{WPt}^G_{q,\beta,r}$ is free under the additional hypotheses (i) stated in the lemma, it suffices to show that for any $\gamma \in \Gamma$ other than the identity, $\mathbf{P}[\theta_\gamma X = X] = 0$. From the hypotheses, we may find an infinite set W of vertices such that $\gamma x \notin W$ for all $x \in W$ and $\gamma x \neq \gamma y$ for distinct $x, y \in W$. Because of (7), by repeated conditioning we see that there is some $c < 1$ such that for any $x_1, \ldots, x_n \in W$, we have $\mathbf{P}[X(x_i) = X(\gamma^{-1}x_i)$ for all $i = 1, \ldots, n] \leq c^n$. Therefore $\mathbf{P}[\theta_\gamma X = X] = 0$.

Consider now the hypothesis (ii). Again because of (7), there is some $c < 1$ such that if A and A' are two finite sets of vertices that are not identical, then the chance is at most c that the number of spins in A equal to 1 is the same as the number of spins in A' equal to 1, even given all spins outside $A \cup A'$. Suppose that $x \neq y$ and x and y are in the same orbit. Let $W(x,y)$ be the set of spin configurations such that for some n, the number of spins in $S_n(x)$ equal to 1 differs from the number in $S_n(y)$. By our assumption and the fact just noted, it follows that $W(x,y)$ has probability 1. Hence so does $W := \bigcap_{x,y} W(x,y)$. It is clear that Γ acts freely on W. $\square$

5 Further remarks on the coupling construction

5.1 Critical behavior of the random-cluster model

Let us mention another application of the pointwise construction in Section 3. Consider the random-cluster model on an infinite quasi-transitive graph G at some fixed value of q. We shall let p vary. Clearly, by stochastic monotonicity, the $\mathsf{FRC}^G_{p,q}$- and $\mathsf{WRC}^G_{p,q}$-probabilities of having some infinite open cluster are increasing in p. Furthermore, by ergodicity, these probabilities must be 0 or 1 for any given p (although the $\mathsf{FRC}^G_{p,q}$-probability does not necessarily equal the $\mathsf{WRC}^G_{p,q}$-probability). Hence, there exist critical values $p_c^{\mathrm{free}} := p_c^{\mathrm{free}}(G,q)$ and $p_c^{\mathrm{wired}} := p_c^{\mathrm{wired}}(G,q)$ such that

$$\mathsf{FRC}^G_{p,q}(\exists \text{ at least one infinite cluster}) = \begin{cases} 0 & \text{for } p < p_c^{\mathrm{free}}, \\ 1 & \text{for } p > p_c^{\mathrm{free}} \end{cases} \tag{26}$$

and

$$\mathsf{WRC}^G_{p,q}(\exists \text{ at least one infinite cluster}) = \begin{cases} 0 & \text{for } p < p_c^{\mathrm{wired}}, \\ 1 & \text{for } p > p_c^{\mathrm{wired}}. \end{cases} \tag{27}$$

A very natural question is whether or not there is an infinite cluster at criticality. In [18], we proved that when G is a unimodular nonamenable quasi-transitive graph, then the answer is no for FRC. In other words,

$$\mathsf{FRC}^G_{p_c^{\mathrm{free}},q}(\exists \text{ at least one infinite cluster}) = 0\,. \tag{28}$$

The proof in [18] of (28) uses, as a key ingredient, the existence of an automorphism-invariant coupling of the measures $\mathsf{FRC}^G_{p,q}$ for different p that witnesses the stochastic domination (10). Such a coupling was provided in Section 3 of the present paper.

It seems reasonable to expect that (28) extends to all quasi-transitive graphs (except those for which the critical value is 1). For $q = 1$, this was conjectured by Benjamini and Schramm [5]. The situation for WRC seems to be more complicated. For instance, as shown in [10] and [17], when G is the regular tree $\mathbf{T}_n$ with $n \geq 2$, we get that the $\mathsf{WRC}^G_{p_c^{\mathrm{wired}},q}$-probability of seeing an infinite cluster is 0 or 1 depending on whether $q \in [1,2]$ or $q > 2$.

5.2 Simultaneity statements

For quasi-transitive graphs, the famous finite-energy argument of Newman and Schulman [27] shows that the number of infinite clusters must (under either FRC or WRC, and for fixed p and q) be an almost sure constant, and either 0, 1 or ∞. For unimodular quasi-transitive graphs, Lyons [26] recently obtained the necessary uniqueness monotonicity statement for deducing that (in addition to the critical values in (26) and (27)), there exist critical values p_u^{free} and p_u^{wired} such that

$$\mathsf{FRC}^G_{p,q}(\exists \text{ a unique infinite cluster}) = \begin{cases} 0 & \text{for } p < p_u^{\mathrm{free}}, \\ 1 & \text{for } p > p_u^{\mathrm{free}} \end{cases} \tag{29}$$

and

$$\mathsf{WRC}^G_{p,q}(\exists \text{ a unique infinite cluster}) = \begin{cases} 0 & \text{for } p < p_u^{\mathrm{wired}}, \\ 1 & \text{for } p > p_u^{\mathrm{wired}}. \end{cases} \tag{30}$$

(For $q = 1$ this goes back to [19] and [32].) See [18] for a detailed discussion of how the four critical values p_c^{free}, p_c^{wired}, p_u^{free} and p_u^{wired} relate to each other.

It is not obvious that, in the coupling of Section 3, (29) and (30) hold simultaneously for all p and q. This is in fact an open problem, and we conjecture the following strengthening, analogous to the simultaneous uniqueness results of [3], [19], [20], and [32]:

Conjecture 5.1. *Let $G = (V, E)$ be connected and quasi-transitive. For a configuration $\xi \in \{0,1\}^E$, write $N(\xi)$ for the number of infinite clusters in ξ. Let $\mathcal{D}$ be the set of quadruples (p_1, p_2, q_1, q_2) such that*

$$p_1 \le p_2 \quad \text{and} \quad \frac{p_1}{(1-p_1)q_1} \le \frac{p_2}{(1-p_2)q_2},$$

with at least one of these inequalities being strict. In the notation of Section 3, we have a.s. for all quadruples $(p_1, p_2, q_1, q_2) \in \mathcal{D}$ simultaneously, each infinite cluster of Y contains $N(X)$ infinite clusters of X, where X and Y may be any of the following three pairs of random variables:

(i) $X = {}^{\mathrm{free}}X^G_{p_1,q_1}$ *and* $Y = {}^{\mathrm{free}}X^G_{p_2,q_2}$,

(ii) $X = {}^{\mathrm{wired}}X^G_{p_1,q_1}$ *and* $Y = {}^{\mathrm{wired}}X^G_{p_2,q_2}$,

(iii) $X = {}^{\mathrm{free}}X^G_{p_1,q_1}$ *and* $Y = {}^{\mathrm{wired}}X^G_{p_2,q_2}$.

5.3 Another open problem

Let us finally discuss another open problem concerning our coupling in Section 3. For $p_1 < p_2$, define

$$\Delta_q(p_1, p_2) := \min\left\{p_2 - p_1,\ \frac{p_2}{p_2 + (1-p_2)q} - \frac{p_1}{p_1 + (1-p_1)q}\right\}$$

and note that $\Delta_q(p_1, p_2) > 0$. For $e \in E$ and $\xi \in \{0,1\}^{E \setminus \{e\}}$, write $A(\xi, e, p, q)$ for the event that ${}^{\mathrm{free}}X^G_{p,q}(E \setminus \{e\}) = \xi$. From the fact that $\mathsf{FRC}^G_{p,q}$ is a DLR random-cluster measure, it follows that for any $e \in E$ and almost any $(\xi, \eta) \in (\{0,1\}^{E \setminus \{e\}})^2$ with respect to the law of $\big({}^{\mathrm{free}}X^G_{p_1,q}(E \setminus \{e\}),\ {}^{\mathrm{free}}X^G_{p_2,q}(E \setminus \{e\})\big)$ under our coupling (which implies that $\xi \preccurlyeq \eta$), we have

$$\mathbf{P}\left({}^{\mathrm{free}}X^G_{p_2,q}(e) = 1 \mid A(\eta, e, p_2, q)\right) - \mathbf{P}\left({}^{\mathrm{free}}X^G_{p_1,q}(e) = 1 \mid A(\xi, e, p_1, q)\right) \ge \Delta_q(p_1, p_2) \tag{31}$$

(and similarly for wired random-cluster measures; everything we say in relation to Question 5.2 applies as well to the wired case as to the free). From this, one is easily seduced into thinking that

$$\mathbf{P}\left({}^{\mathrm{free}}X^G_{p_2,q}(e) = 1\,,\ {}^{\mathrm{free}}X^G_{p_1,q}(e) = 0 \mid A(\eta, e, p_2, q) \cap A(\xi, e, p_1, q)\right) \ge \Delta_q(p_1, p_2)\,, \tag{32}$$

but to conclude this directly from (31) is unwarranted, because conditioning on ξ and η jointly is not the same as conditioning on them separately. It is nevertheless natural to ask whether something like (32) is true. In particular, the following question asks for a weaker property.

Question 5.2. *For $p_1 < p_2$ and $q \geq 1$, does there exist an $\varepsilon > 0$ (depending on p_1, p_2 and q) such that for any $e \in E$ and almost any $(\xi, \eta) \in (\{0,1\}^{E \setminus \{e\}})^2$, we have*

$$\mathbf{P}\left({}^{\text{free}}X^G_{p_2,q}(e) = 1\,,\ {}^{\text{free}}X^G_{p_1,q}(e) = 0 \;\middle|\; A(\eta, e, p_2, q) \cap A(\xi, e, p_1, q)\right) \geq \varepsilon\,?$$

A positive answer to this question (for our coupling or for some other automorphism-invariant witness to the stochastic inequality $\mathsf{FRC}^G_{p_1,q} \overset{\mathcal{D}}{\preccurlyeq} \mathsf{FRC}^G_{p_2,q}$) is precisely the missing ingredient that prevented the authors of [19] from extending their uniqueness monotonicity result for i.i.d. percolation ($q = 1$) for unimodular quasi-transitive graphs to the more general case $q \geq 1$ (i.e., from proving the relations (29) and (30) that were later obtained in [26]). Such a positive answer might perhaps also be an ingredient in applying the reasoning of Schonmann [32] in order to remove the unimodularity assumption in these results.

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