Errata to

Probability on Trees and Networks

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p. 4: In the last sentence of Example 1.1, switch "even" and "odd".

p. 22: End the last sentence of Exercise 2.1(c) with "for irreducible chains".

p. 41: Replace "the (almost surely unique) closest path to L in the Hausdorff metric" by "recursively growing a path by successively adding a vertex closest to L". Change "Figure 2.2" to "Figure 2.3".

p. 61: In Exercise 2.35, change "$x, y, z \in \mathsf{V}$" to "x, y, z be vertices in a finite network."

pp. 64–65: In each of Exercises 2.62 and 2.63, insert "simple" before "network".

p. 68: In Exercise 2.100(b), the lower bound on $p_n(\mathbf{0}, \mathbf{0})$ is only for even n.

p. 78: On line 12, change $x \neq a$ to $x \neq a, z$. On line 13, change "all vertices x" to "all vertices $x \neq z$".

p. 79: Change "number of edges with negative flow" to "number of edges with nonpositive flow" both times.

p. 106: In the first paragraph, replace "Can we interpret it" by "The distribution of the resulting spanning tree again does not depend on the choice of root nor on the ordering of vertices: The corresponding version of Lemma 4.2 is that given any stacks under the states, we pop cycles in any order, provided that the stack under each vertex is considered for popping infinitely often. In that case, the order in which cycles are popped is irrelevant in the sense that every order pops the same colored cycles. The proof is similar to that for finite networks because cycles are finite. Can we interpret such random spanning trees". Delete the last sentence of the paragraph and its footnote.

p. 114: Just before (4.17), e should be e_1 both times.

p. 118: In Corollary 4.9, change the first occurrence of "graph" to "network". In Exercise 4.15, change "a path" to "a simple path" and $\tau_{x_{k+1},\ldots,x_n}$ to $\tau_{\{x_{k+1},\ldots,x_n\}}$.

p. 119, line −2: Insert "simple" before "network".

p. 128: The three probabilities should be $(1 + 2/n)^2/4$, $(1 - 4/n^2)/2$, and $(1 - 2/n)^2/4$.

p. 138: Above (5.6), change $|e| + 1$ to $|e|$.

p. 142: At the end of the first paragraph, change $\mathbf{P}[e \in \omega] > 0$ to $\mathbf{P}[o \leftrightarrow e] > 0$.

p. 145: In Example 5.20, change "the corresponding eigenvectors with $\|v_k\| = 1$" to "a corresponding system of orthonormal eigenvectors". Change "eigenvector has positive entries” to “eigenvector, say, v_1, has positive entries”. Change $P^n(i,j) \sim 2v_k(i)v_k(j)\rho^n$ to $P^n(i,j) \sim 2v_1(i)v_1(j)\rho^n$.

p. 153: Change the last paragraph of the proof of Proposition 5.27 to the following:

Now note that the set of surviving rays forms a closed subset of ∂T. For the culmination of the proof, assume first that T is locally finite. Then ∂T is compact by Exercise 1.1, so the surviving rays a.s. form a perfect set, which, when it is nonempty, must have the cardinality of the continuum. This completes the proof in this first case. Now let T be general. Enumerate the children of each vertex x as $x(1), x(2), \ldots$ in any order. Let R^x be the event that x is on at least one surviving ray. Fix $\epsilon > 0$. Let $N(x)$ be minimal so that $\mathbf{P}\big[\bigcup_{n=1}^{N(x)} R^{x(k)}\big] \ge (1 - \epsilon/2^{|x|+1})\,\mathbf{P}[R^x]$. Define V' to be the set of vertices $x(n)$ such that $n \le N(x)$ for some x, as well as the root, o, of T, and let T' be the largest tree with vertices from V' that contains o. Then T' is locally finite, and the probability that T' contains a surviving ray is at least $(1-\epsilon)\,\mathbf{P}[R^o]$, whence the probability that T' contains a continuum of surviving rays is at least $(1-\epsilon)\,\mathbf{P}[R^o]$. Since this holds for all $\epsilon > 0$, the proof is done. ◀

p. 158: In line 6, change Y to X. In the third paragraph from the bottom, p_1 should be p and $Q_d(p)$ should be $Q_b(p)$ both times.

p. 162: After line 7, add “For $x \in \mathsf{V}(T)$, write Rays_x for the set of surviving rays (starting at o) that pass through x.”

p. 165: At the top, add “For $x \in \mathsf{V}(T)$, write Rays_x for the set of surviving rays (starting at o) that pass through x.”

p. 172: In Exercise 5.70(c), replace “$[0, 1/2] \times [0, 2]$ rectangles” by “rectangles of shape $1/2 \times 1$” in the hint.

p. 189: Before (6.28), change $n \ge 1$ to $n \ge 0$.

p. 196, line −2: Change “contains” to “intersects” and add “and edges” after “vertices”.

p. 201: In (6.38), change $\mathcal{P}_i A$ to $\mathcal{P}_i(A)$.

p. 271: In Exercise 7.42, add “with one end” after “transitive graph”.

p. 298, line −17: Change Γ to γ.

p. 300: In the statement and proof of Theorem 8.47, replace “open” by “open, nonempty”.

p. 338: Replace Exercise 9.50 by the following: Let G and G' be networks with bounded conductances, resistances, and degrees. Suppose that there is a rough embedding from G to G' such that each vertex of G' is within some constant distance of the image of $\mathsf{V}(G)$.

(a) Find an example where G has unique currents but G' does not.

(b) Find an example where G' has unique currents but G does not.

p. 350: Change “In general, suppose that G is a simple plane network whose plane dual $G^\dagger$ is locally finite.” to “In general, suppose that G is a proper, simple plane network whose plane dual $G^\dagger$ is locally finite.”

p. 365: In Theorem 10.36, insert “proper,” before ”simple”.

p. 366: In Proposition 10.37, insert "proper," before "simple".

p. 386: In Exercise 10.33, both subtractions of "2" should be replaced by subtractions of expected degrees in the wired spanning forest.

p. 414, line −3: Change $X\mu$ to $X\nu$.

p. 420: Replace "We may couple all these random variables so that . . . (in virtue of the first paragraph)." with "For each n and i, $S_{i,j}^{(n)}$ and $V_{i,j}^{(n)}$ are identically distributed with mean 1 (since they pertain to $n-i-1$ generations of independent critical Galton–Watson trees). In virtue of the first paragraph, we may couple them so that $S_{i,j}^{(n)} = V_{i,j}^{(n)}$ for $j \le X_i^{(n)}$ and $0 \le X_i'' - X_i^{(n)} \le \widehat{L}_{i+1}$. Then $X_i^{(n)} - X_i'' \to 0$ in measure as $n \to \infty$."

p. 424, line 3: Change $|Y_n|$ to Y_n^2.

p. 430: In the conclusion of Lemma 13.5, change $\left|T_k\big(P/\|P\|_\pi\big)\right|_\pi$ to $\left\|T_k\big(P/\|P\|_\pi\big)\right\|_\pi$.

p. 435, line −1: Change "last" to "penultimate".

pp. 438–439: Replace the proof of Lemma 13.15 with the following:

Proof. Define $\alpha \colon \{1, \ldots, n\} \to \mathbb{R}$ by $\alpha(i) := \mathbf{E}[f(Z_1) \mid Z_0 = i] - f(i) = \sum_{j=1}^n p_{ij} f(j) - f(i)$. Notice that

$$M_s := f(Z_s) - f(Z_0) - \sum_{u=0}^{s-1} \alpha(Z_u)$$

is a martingale (with respect to the natural filtration of $Z_0, \ldots, Z_t$). Write $\widetilde{Z}_s := Z_{t-s}$ for $0 \le s \le t$. Because the distribution of $(Z_0, \ldots, Z_t)$ is the same as that of $(\widetilde{Z}_0, \ldots, \widetilde{Z}_t)$,

$$\widetilde{M}_s := f(\widetilde{Z}_s) - f(\widetilde{Z}_0) - \sum_{u=0}^{s-1} \alpha(\widetilde{Z}_u)$$

is a martingale (with respect to the natural filtration of $\widetilde{Z}_0, \ldots, \widetilde{Z}_t$). Note that

$$\widetilde{M}_{t-s} - \widetilde{M}_t = f(\widetilde{Z}_{t-s}) - f(\widetilde{Z}_t) + \sum_{u=t-s}^{t-1} \alpha(\widetilde{Z}_u) = f(Z_s) - f(Z_0) + \sum_{j=1}^{s} \alpha(Z_j)\,.$$

Averaging this with the definition of M_s and rearranging yields

$$f(Z_s) - f(Z_0) = \frac{1}{2}\Big(M_s + \widetilde{M}_{t-s} - \widetilde{M}_t + \alpha(Z_0) - \alpha(Z_s)\Big)\,.$$

Therefore,

$$2 \max_{0 \le s \le t} |f(Z_s) - f(Z_0)| \le \max_{0 \le s \le t} |M_s| + \max_{0 \le s \le t} |\widetilde{M}_s| + |\widetilde{M}_t| + |\alpha(Z_0)| + \max_{1 \le s \le t} |\alpha(Z_s)|\,.$$

It follows that

$$2\Big\|\max_{0 \le s \le t} |f(Z_s) - f(Z_0)|\Big\| \le \Big\|\max_{0 \le s \le t} |M_s|\Big\| + \Big\|\max_{0 \le s \le t} |\widetilde{M}_s|\Big\| + \Big\|\widetilde{M}_t\Big\| + \Big\|\alpha(Z_0)\Big\| + \Big\|\max_{1 \le s \le t} |\alpha(Z_s)|\Big\|,$$

where all norms are in $L^2(\mathbf{P})$.

We will use Doob's L^2-maximal inequality for martingales $\langle N_s \rangle$ (see, for example, Theorem 5.4.3 in Durrett (2010)),

$$\mathbf{E}\Big[\max_{0 \le s \le t} N_s^2\Big] \le 4\,\mathbf{E}\big[N_t^2\big] = 4\bigg(\mathbf{E}\big[N_0^2\big] + \sum_{s=1}^{t} \mathbf{E}\big[(N_s - N_{s-1})^2\big]\bigg) .$$

In our case, $M_0 = 0$ and for $s > 0$,

$$M_s - M_{s-1} = f(Z_s) - f(Z_{s-1}) - \alpha(Z_{s-1}) = f(Z_s) - f(Z_{s-1}) - \mathbf{E}\big[f(Z_s) - f(Z_{s-1}) \bigm| Z_{s-1}\big]$$

is orthogonal to $\mathbf{E}\big[f(Z_s) - f(Z_{s-1}) \bigm| Z_{s-1}\big]$ in $L^2(\mathbf{P})$, whence

$$\mathbf{E}\big[(M_s - M_{s-1})^2\big] + \mathbf{E}\big[\alpha(Z_{s-1})^2\big] = \mathbf{E}\big[|f(Z_s) - f(Z_{s-1})|^2\big] =: V ,$$

where V does not depend on $s > 0$ by stationarity. Write V_1 and V_2 for the expectations on the left-hand side. A similar equation holds for $\widetilde{M}$. This gives us

$$\Big\|\max_{0 \le s \le t} |M_s|\Big\| + \Big\|\max_{0 \le s \le t} |\widetilde{M}_s|\Big\| + \Big\|\widetilde{M}_t\Big\| \le 2\|M_t\| + 3\|\widetilde{M}_t\| = 5\|M_t\| = 5\sqrt{tV_1} .$$

In addition,

$$\begin{aligned}\Big\|\alpha(Z_0)\Big\| + \Big\|\max_{1 \le s \le t} |\alpha(Z_s)|\Big\| &= \sqrt{V}_2 + \mathbf{E}\Big[\max_{1 \le s \le t} \alpha(Z_s)^2\Big]^{1/2} \\ &\le \sqrt{V}_2 + \mathbf{E}\bigg[\sum_{s=1}^{t} \alpha(Z_s)^2\bigg]^{1/2} = \big(\sqrt{t} + 1\big)\sqrt{V}_2 .\end{aligned}$$

Summing up, dividing by 2, squaring, and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\mathbf{E}\Big[\max_{0 \le s \le t} |f(Z_s) - f(Z_0)|^2\Big] &\le \Big(5\sqrt{tV_1} + \big(\sqrt{t} + 1\big)\sqrt{V}_2\Big)^2/4 \le \big(25t + t + 2\sqrt{t} + 1\big)(V_1 + V_2)/4 \\ &= \big(26t + 2\sqrt{t} + 1\big)V/4 \le 7tV\end{aligned}$$

for $t \ge 2$, as desired. ◀

p. 443: V_i should be assumed finite.

p. 447, line −2: Change λ^{-j} to λ^{j}.

p. 457: Replace the sentence beginning "An extension to the much harder case" by "An extension to the much harder case of (biased) random walks on directed covers was achieved by Takacs (1997), with a further extension to certain random trees by Takacs (1998)."

p. 460: Just before Exercise 3.15, change "smaller" to "greater".

p. 463: In Exercise 13.23, add the hypothesis that G be simple.

p. 464: In Exercise 13.26, change -2 to -3 and $4/3$ to $(5+\sqrt{7})/6 = 1.27^+$. In Exercise 13.31, change $\sqrt{d}$ to $\sqrt{d+2}$.

pp. 465–466: In each of Exercises 13.44–13.47, insert "simple" before "network".

p. 468: Replace "biased simple random walk" with "biased nearest-neighbor random walk".

p. 470, line −3: Change x to o.

p. 472: Remove the period in (14.3).

p. 473: Just before Theorem 14.10, change "lemma" to "theorem".

p. 476: Move the sentence preceding Example 14.14 to between that example and the next one. Change "Theorem 14.16" to "Lemma 14.16".

p. 479: In the first display, change $\mathbf{1}_{\{o\}}$ to $\mathbf{0}$.

p. 482: In Definition 14.26, precede (i) by "there is a metric on $\mathsf{V} \cup \Theta$ such that".

p. 484: In the proof that *(iii)* $\Rightarrow$ *(iv)*, add after $\mathbf{b}(\Omega_1)$, ": Here, we extend ν_o to the σ-field generated by $\mathscr{F}_\Theta$ and those $E \subset \Theta$ for which $\mathbf{b}^{-1}(E)$ lies in a subset of $\mathscr{B}_{\mathsf{V}^\mathbb{N}}$ that has $\mathbf{P}_o$-measure 0. With this extension, $\nu_o(\mathbf{b}(\Omega_1)) = 1$".

p. 485: In Exercise 14.10, change $\mathsf{V}^\mathbb{N}$ to Ω. Change $\mathbf{b} \circ \pi_n$ to $\pi_n \circ \mathbf{b}$.

p. 492: In the proof of Theorem 14.39, replace $\widehat{X}_n$ by (Ψ_n, X_n).

p. 494: Add to Exercise 14.14, "Show also that if $f \ge f \circ T$ a.s., then $f = f \circ T$ a.s."

p. 498: Change $\beta = \beta \circ T$ to $\beta \ge \beta \circ T$.

p. 499: Change "Exercise 13.2(iii)" to "Exercise 13.2(c)".

p. 500: Change "Exercise 13.2(ii)" to "Exercise 13.2(b)".

p. 505: In Exercise 14.20, change "constant $c > 0$" to "constants $c, d > 0$".

p. 510: In Exercise 14.24, change γX_j to $X_j \gamma$ and change $C(K) := \sum_{\gamma \in K} \mathbf{P}_\gamma[\forall n \ge 1\ \ X_n \notin K]$ to $C(K) := \sum_{\gamma \in K} \mathbf{P}_{\gamma^{-1}}[\forall n \ge 1\ \ X_n \notin K^{-1}]$. In Exercise 14.32, replace "satisfies Definition 14.26, except that it is not necessarily metrizable" with "allows a Poisson representation in the sense that there is a family $\{\nu_x\,;\ x \in \mathsf{V}\}$ of probability measures on Ξ such that the Gelfand isomorphism $\varphi\colon \mathbf{BH}(\mathsf{V}, P) \to C(\Xi)$ satisfies $u(x) = \int_\Xi \varphi(u)\, d\nu_x$ for all $u \in \mathbf{BH}(\mathsf{V}, P)$ and $x \in \mathsf{V}$".

p. 511: In Exercise 14.36(a), replace the final ν by "ν_0 for some ν_0". In Exercise 14.39, remove the negative sign in the display.

p. 523: In Example 15.14, replace $M^{2\alpha}$ by M^α.

p. 526, line −7: Change $\mathbf{T}$ to T^v.

p. 538: In line −3, change $z^{\dim E}$ to $z^{-\dim E}$.

p. 540: In the next-to-last line of the proof of Proposition 6.5, add a factor of 3 after the equality sign. In Proposition 6.8, change b-ary to b-adic, 2^{-d} to $2^{-d} b^{-d\ell}$, $3^d b^{d(1+\ell)}$ to $3^d b^d$, and $\ell := (1/2)\log_b d$ to $\ell := \lceil (1/2)\log_b d \rceil$.

p. 550: In the line beginning "Now assume (iii)," change $\xi_k \cap T$ to $\xi_k \cap T^*$. Two lines below that, change $u \ngeq x_k$ to $u \nleq x_k$.

p. 554: In Exercise 16.9, change the first occurrence of $\mu^i(E)$ to $\mu^i(E)^2$. In Exercise 16.13, A should be assumed closed. In Exercise 16.14, Λ should be assumed compact.

p. 555: In Exercise 16.18, change "supremum" to "suprema". In Exercise 16.19, change $\text{cap}_1(\Lambda)$ to $\text{cap}_2(\Lambda)$. In Exercise 16.20, replace "Remove" by "Can one remove" and end with a question mark.

p. 556: Change "***ergodic theorem*** theorem" to "***ergodic theorem***".

p. 557: At the end of Exercise 17.1, clarify that (iii) then means that $(\mathbf{p} \times \mu)\big(f(X_0) \neq f(X_1)\big) = 0$.

p. 562: In the proof of Lemma 17.5, μ should be replaced by $\mathsf{DSRW} \times \mu$ and μ' by $\mathsf{SRW} \times \mu'$.

p. 563: In the proof of Theorem 17.7, change $X_1 B$ to $X_1^{-1}B$, $X_1 A$ to $X_1^{-1}A$, sB to $s^{-1}B$, $s^{-1}A$ to sA, $s^{-1}A$ to sA, and in line −5, the first s^{-1} to s.

p. 575: In Exercise 17.9, change "a UNIF_T-path is $\widehat{\mathbf{GW}}_*$" to "(T, ξ) is $\widehat{\mathbf{GW}}_*$ when $T \sim W \cdot \mathbf{GW}$ and ξ is a UNIF_T-path in T."

p. 577, lines 5–6: Change $\int$ to $\iint$.

p. 578: In line 7, $\mathbf{AGW}_{\mathsf{Exit}}$ should be the pushforward of $(\mathsf{SRW} \times \mathbf{AGW})_{\mathsf{Exit}}$ under the map $(\overleftrightarrow{x}, T) \mapsto T$.

p. 579, line 6: Change $y = 1$ to $|y| = 1$.

p. 581: In the line preceding (17.17), change "limit" to "liminf": this is easy to see and sufficient, although the limit does exist a.s.

p. 590: Add "***In all the exercises about*** $\mathbf{GW}$ ***or*** $\mathbf{AGW}$***, assume that*** $p_0 = 0$ ***unless stated otherwise.***" In the definition of $D_k(\overleftrightarrow{x}, T)$ in Exercise 17.19, change $j \in \mathbb{N}$ to $j \in \mathbb{Z}$.

p. 592: In Exercise 17.29, remove $\displaystyle\sum_{n=0}^{N-1}$ and change n to N. In Exercise 17.30, change $\theta(x)$ to $\theta_0(x)$.

p. 593: In Exercise 17.34, change $m - 1$ to $m^2 - m$ and insert "when $\mathbf{E}[L^2] < \infty$" after "$\mathbf{E}\big[\mathscr{E}_c(\theta)\big] < \infty$". In Exercise 17.36, insert "augmented" before "Galton-Watson" and "a uniformly chosen neighbor of" after "(not loop-erased) simple random walk from".

p. 603: In the solution to Exercise 3.9, replace $< \frac{a_m}{m} + \frac{a_r}{n} < \beta + \frac{a_r}{n}$ by $= \frac{a_m}{m} \cdot \frac{qm}{qm+r} + \frac{a_r}{n} < \beta \cdot \frac{qm}{qm+r} + \frac{a_r}{n}$.

p. 610: In the solution to Exercise 5.3, $L^{j+1}_{I(i_1,\ldots,i_j)}$ should be $L^{(j+1)}_{I(i_1,\ldots,i_j)}$.

p. 612: In the solution to Exercise 5.29, replace "by the weak law of large numbers and Proposition 5.1, $Z'_{n+k}/Z'_n \xrightarrow{\mathbf{P}} m^k$" with "by the Seneta–Heyde theorem, $Z'_{n+k}/Z'_n \to m^k$ a.s.

given nonextinction".

p. 633: In the solution to Exercise 11.16, it should not be asserted that the bound is at least $(1-\log 2)(1/\rho - 1)$. Such a bound holds, but requires a modified argument.

p. 636: For Exercise 13.23, if G is not simple, then one can make a similar Markov chain using the directed line graph of G.

p. 637: For Exercise 13.31, change $\sqrt{d+\epsilon}$ to $\sqrt{d+2+\epsilon}$.

p. 639: For Exercise 14.6, change $[k, n-1]$ to $(k, n-1]$. For Exercise 14.12, change "folows" to "follows". For Exercise 14.13, replace "biased simple random walk" with "biased nearest-neighbor random walk".

p. 640: Add a comment on Exercise 14.14 to say, "For the second part, consider $\mathbf{E}[g \circ f - g \circ f \circ T]$ for a suitable g." For Exercise 14.23, change "$H(X_1) = 0$" to "$H(X_1) < \infty$". For Exercise 14.24, change "$\sum_{k=0}^{n-1} \sum_{\gamma \in K} \mathbf{P}_o\big[\forall j \in [k, n-1] \;\; \gamma X_j \notin K X_k\big]$" to "$\sum_{i=0}^{n-1} \sum_{\gamma \in K} \mathbf{P}_o\big[\forall j \in (i, n-1] \;\; X_i \gamma \notin X_j K\big]$".

p. 641: Replace the comment for Exercise 14.32 with "As noted by Fursternberg (1971b), the space Ξ can be too large to provide a compactification boundary."

p. 645: For Exercise 17.4, after "3", add "in the reduced tree".

p. 646: For Exercise 17.19(c), change "this is the average number" to "this is a weighted average number"; change Γ_k to $2\Gamma_k - 1$; and change $j \in \mathbb{N}$ to $j \in \mathbb{Z}$ in the definition of $D'_k(\overleftrightarrow{x}, T)$.

p. 655: The journal for Cayley should be *Quart. J. Pure Appl. Math.*