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ZERO SETS FOR SPACES OF ANALYTIC FUNCTIONS

by Russell LYONS & Alex ZHAI (*)

Abstract. — We show that under mild conditions, a Gaussian analytic function $\boldsymbol{F}$ that a.s. does not belong to a given weighted Bergman space or Bargmann–Fock space has the property that a.s. no non-zero function in that space vanishes where $\boldsymbol{F}$ does. This establishes a conjecture of Shapiro [21] on Bergman spaces and allows us to resolve a question of Zhu [24] on Bargmann–Fock spaces. We also give a similar result on the union of two (or more) such zero sets, thereby establishing another conjecture of Shapiro [21] on Bergman spaces and allowing us to strengthen a result of Zhu [24] on Bargmann–Fock spaces.

Résumé. — On montre que sous des conditions faibles, une fonction analytique gaussienne $\boldsymbol{F}$ qui n'appartient pas p.s. à un espace pondéré de Bergman ou de Bargmann–Fock donné a p.s. la propriété qu'il n'existe pas de fonction non-nulle dans cette espace qui s'annule où $\boldsymbol{F}$ s'annule. Ceci démontre une conjecture de Shapiro [21] sur les espaces de Bergman et nous permet de résoudre une question de Zhu [24] sur les espaces de Bargmann–Fock. On donne aussi un résultat similaire sur la réunion de deux (ou plus) tels ensembles de zéros, montrant ainsi une autre conjecture de Shapiro [21] sur les espaces de Bergman et nous permettant de renforcer un résultat de Zhu [24] sur les espaces de Bargmann–Fock.

1. Introduction

Zeros of Gaussian analytic functions were originally studied by Paley and Wiener [13], Kac [7, 8], and Rice [15, 16]. Since then, many more mathematicians and physicists have been interested in such zero sets. For some of the history, see [22] and [6]. Those sources also give surveys of certain aspects of zero sets of Gaussian analytic functions as random objects. The topic of the present paper, however, is not mainly zero sets of Gaussian analytic functions as random objects, but as tools to understand zero

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sets in standard spaces of analytic functions. In particular, we consider the (weighted) Bergman spaces in the unit disk and the (weighted) Bargmann–Fock spaces in the entire plane, for which we give a unified treatment. In Subsection 1.1, we give a brief history of what is known for zero sets of functions in these spaces, focused on results relevant to ours. More can be found in Chapter 4 of [4] and Chapter 4 of [3], which are devoted to zero sets of Bergman spaces, and Chapter 5 of [25], which is devoted to zero sets of Bargmann–Fock spaces.

Let μ be a finite measure on $(0,\infty)$, not identically 0. Write $r_\mu := \inf\{r\,;\ \mu(r,\infty)=0\} \in (0,\infty]$, and assume that $\mu(\{r_\mu\})=0$. For $p \in (0,\infty)$, write $A^p(\mu)$ for the set of analytic functions f defined for $|z|<r_\mu$ that satisfy

$$\int_0^{r_\mu}\int_0^1 \left|f(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r) < \infty.$$

When $r_\mu = 1$, these spaces are referred to as *weighted Bergman spaces*, whereas when $r_\mu=\infty$, they are called *weighted Bargmann–Fock spaces*. Clearly all spaces $A^p(\mu)$ when r_μ is finite are isomorphic to weighted Bergman spaces. Denote the unit disk by $\mathbb{D} := \{z\,;\ |z|<1\}$. The unweighted Bergman spaces $A^p(\mathbb{D})$ correspond to $\mathrm{d}\mu(r) = 2r\mathbb{1}_{[0,1]}(r)\,\mathrm{d}r$. The most-studied weights are $\mathrm{d}\mu(r) = 2(1-r^2)^\alpha\mathbb{1}_{[0,1]}(r)\,\mathrm{d}r$ $(\alpha>-1)$, in which case the corresponding Bergman spaces are denoted $A^p_\alpha(\mathbb{D})$. By contrast, the most-studied Bargmann–Fock spaces are defined differently, with μ depending on p, namely, $\mathrm{d}\mu(r) = p\alpha r\, e^{-p\alpha r^2/2}\,\mathrm{d}r$ $(\alpha>0)$, in which case the corresponding Bargmann–Fock spaces are denoted $B^p_\alpha(\mathbb{C})$. An older definition of $B^p_\alpha(\mathbb{C})$ was used by [24], where μ did not depend on p; in our notation, this was the space $B^p_{2\alpha/p}(\mathbb{C})$. By [25, Theorem 2.10], $B^p_\alpha(\mathbb{C}) \subsetneq B^q_\alpha(\mathbb{C})$ for $0<p<q<\infty$.

A *standard complex Gaussian* random variable is one whose density with respect to Lebesgue measure on $\mathbb{C}$ is $z\mapsto e^{-|z|^2}/\pi$. We always consider the zero set $Z(f)$ of an analytic function f as a multiset or a sequence, where each zero w is listed with its multiplicity, which is m if $z\mapsto f(z)/(z-w)^m$ is analytic and does not vanish at w.

Our main result is the following.

Theorem 1.1. — *Let μ be a finite measure on $(0,\infty)$ with $\mu(\{r_\mu\})=0$. Let $p\in(0,\infty)$. Suppose that $a_n \geqslant 0$ satisfy $\limsup_{n\to\infty} a_n^{1/n}<\infty$ and $r\mapsto\sum_{n=0}^\infty a_n^2 r^{2n} \notin L^{p/2}(\mu)$. Let $\boldsymbol{F}(z) := \sum_{n=0}^\infty a_n\zeta_n z^n$ for $|z|<r_\mu$, where ζ_n are independent complex Gaussian random variables. Then a.s. the only analytic function $f\in A^p(\mu)$ with $Z(f)\supseteq Z(\boldsymbol{F})$ is $f\equiv 0$.*

Note that if $r \mapsto \sum_{n=0}^{\infty} a_n^2 r^{2n} \notin L^{p/2}(\mu)$ for all $p > p_0$, then by considering a countable set of $p > p_0$, we may conclude that a.s. for all $p > p_0$, the only analytic function $f \in A^p(\mu)$ with $Z(f) \supseteq Z(\boldsymbol{F})$ is $f \equiv 0$.

The following corollary, in the special case where $\mu_1 = \mu_2$, was known for Bergman spaces [5]. The corollary follows from Theorem 1.1 and (1.7).

Corollary 1.2. — *Let $R \in (0, \infty]$. Let μ_i $(i = 1, 2)$ be finite measures with $r_{\mu_i} = R$. Let $p_i \in (0, \infty)$ $(i = 1, 2)$. Suppose that there exist $a_n \geqslant 0$ that satisfy $\limsup_{n\to\infty} a_n^{1/n} < \infty$ and $r \mapsto \sum_{n=0}^{\infty} a_n^2 r^{2n} \in L^{p_1/2}(\mu_1) \setminus L^{p_2/2}(\mu_2)$. Then there is a function $f \in A^{p_1}(\mu_1)$ such that the only $g \in A^{p_2}(\mu_2)$ with $Z(g) \supseteq Z(f)$ is $g \equiv 0$.*

We will actually prove a quantitative version of Theorem 1.1. Write $A(\mu)$ for the set of functions that are analytic in $\{z\,;\ |z| < r_\mu\}$. For $s \leqslant r_\mu$ and $f \in A(\mu)$, write $Z_s(f)$ for the multiset of z with $f(z) = 0$ and $0 < |z| < s$. Denote

$$\|f\|_{A^p(\mu,s)} := \left(\int_0^s \int_0^1 \left| f(re^{2\pi i\theta}) \right|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r) \right)^{1/p}.$$

We also abbreviate

$$\|f\|_{A^p(\mu)} := \|f\|_{A^p(\mu, r_\mu)}\,.$$

Given a sequence $\langle a_n\,;\ n \geqslant 0 \rangle$, write $a^{(r)}$ for the sequence $\langle a_n r^n\,;\ n \geqslant 0 \rangle$ and $\|a^{(r)}\|_2$ for its ℓ^2-norm.

Theorem 1.3. — *Let $a_n \geqslant 0$ satisfy $R^{-1} := \limsup_{n\to\infty} a_n^{1/n} < \infty$ and $a_0 \neq 0$. Let $\boldsymbol{F}(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n$ for $|z| < R$, where ζ_n are independent complex Gaussian random variables. Then for all finite measures μ with $r_\mu = R$ and $\mu(\{R\}) = 0$, all $p \in (0, \infty)$, and all $s \in (0, R]$,*

$$\mathbf{E}\left[\max\left\{ \frac{|f(0)|}{\|f\|_{A^p(\mu,s)}}\,;\ 0 \not\equiv f \in A(\mu),\ Z(f) \supset Z_s(\boldsymbol{F}) \right\} \right] \leqslant \frac{\sqrt{\pi}\, a_0}{\left(\int_0^s \|a^{(r)}\|_2^p \,\mathrm{d}\mu(r) \right)^{1/p}}. \tag{1.1}$$

Proof of Theorem 1.1 from Theorem 1.3. — Consider $0 \not\equiv f \in A(\mu)$ with $Z(\boldsymbol{F}) \subset Z(f)$; we will show that $\|f\|_{A^p(\mu)} = \infty$.

Without loss of generality, we may shift the indices of the a_n so that $a_0 \neq 0$, since this does not affect the condition $\|a^{(r)}\|_2 \notin L^p(\mu)$, and it does not change $Z_R(\boldsymbol{F})$. Thus, Theorem 1.3 applies.

If $f(0) \neq 0$, then the result follows directly from (1.1) by taking $s = R$. Otherwise, we may reduce to this case: Let m denote the order of vanishing of f at 0, and let $g(z) := f(z)/z^m$, so that $g \in A(\mu)$ and $g(0) \neq 0$. We then have $Z(g) \supset Z_R(\boldsymbol{F})$, from which we conclude that $\|g\|_{A^p(\mu)} = \infty$. This clearly implies that also $\|f\|_{A^p(\mu)} = \infty$, as desired. □

We also establish the following theorem, which relates to unions of zero sets in the special case $b \equiv -1$ upon observing that $Z(\boldsymbol{F}^N - 1) = \bigcup_{k=0}^{N-1} Z(\boldsymbol{F} - e^{2\pi ik/N})$.

THEOREM 1.4. — *Let μ be a finite measure on $(0,\infty)$ with $\mu(\{r_\mu\}) = 0$. Let $p \in (0,\infty)$. Suppose that $a_n \geqslant 0$ satisfy $\limsup_{n\to\infty} a_n^{1/n} < \infty$ and $r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \notin L^{p/2}(\mu)$. Let $\boldsymbol{F}(z) := \sum_{n=0}^\infty a_n \zeta_n z^n$ for $|z| < r_\mu$, where ζ_n are independent complex Gaussian random variables. Let $b \in A(\mu)$ and N be a positive integer. Then a.s. the only analytic function $f \in A^{p/N}(\mu)$ with $Z(f) \supseteq Z(\boldsymbol{F}^N + b)$ is $f \equiv 0$.*

This establishes the full conjecture of Shapiro [21] and enlarges the set of b to which it applies. Since $Z(\boldsymbol{F} \pm 1)$ are a.s. simple (see [14, Lemma 28]) and $Z(\boldsymbol{F} \pm 1)$ are disjoint, we obtain the following corollary.

COROLLARY 1.5. — *Let $R \in (0,\infty]$. Let μ_i $(i = 1,2)$ be finite measures with $r_{\mu_i} = R$ and $\mu_i(\{R\}) = 0$. Let $p_i \in (0,\infty)$ $(i = 1,2)$. Suppose that there exist $a_n \geqslant 0$ that satisfy $\limsup_{n\to\infty} a_n^{1/n} < \infty$ and $r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n} \in L^{p_1/2}(\mu_1) \setminus L^{p_2/2}(\mu_2)$. Then there are functions $f_1, f_2 \in A^{p_1}(\mu_1)$ such that $Z(f_1) \cap Z(f_2) = \varnothing$ and the only $g \in A^{p_2/2}(\mu_2)$ with $Z(g) \supseteq Z(f_1) \cup Z(f_2)$ is $g \equiv 0$.*

Again, we prove a quantitative version of Theorem 1.4:

THEOREM 1.6. — *Let $a_n \geqslant 0$ satisfy $R^{-1} := \limsup_{n\to\infty} a_n^{1/n} < \infty$. Let $\boldsymbol{F}(z) := \sum_{n=0}^\infty a_n \zeta_n z^n$ for $|z| < R$, where ζ_n are independent complex Gaussian random variables. Then for all finite measures μ with $r_\mu = R$ and $\mu(\{R\}) = 0$, all $p \in (0,\infty)$, all $b \in A(\mu)$, all positive integers N, and all $s \in (0,R]$,*

$$\mathbf{E}\left[\max\left\{\frac{|f(0)|^{1/N}}{\|f\|_{A^p(\mu,s)}^{1/N}}\,;\ 0 \not\equiv f \in A(\mu),\ Z(f) \supset Z_s(\boldsymbol{F}^N + b)\right\}\right] \leqslant \frac{c}{\left(\int_0^s \|a^{(r)}\|_2^p \,\mathrm{d}\mu(r)\right)^{1/p}},$$

where

$$c := \left(a_0^{4N}(2N)! + 4|b(0)|^2 a_0^{2N} N! + |b(0)|^4\right)^{1/4N} \Gamma\left(\frac{2N-1}{4N-1}\right)^{\frac{4N-1}{4N}}.$$

Therefore, if $\|a^{(r)}\|_2 \notin L^p(\mu)$, then a.s. every $f \in A(\mu)$ with $Z(f) \supset Z(\boldsymbol{F}^N + b)$ satisfies $\|f\|_{A^p(\mu)} = \infty$.

Of course, what allows Gaussian series to have these properties is that such series have many zeros. A quantitative form of this property is what lies behind our results. Recall that by the arithmetic mean-geometric mean inequality (or Jensen's inequality) and Jensen's formula, every $f \in A(\mu)$ with $f(0) \neq 0$ satisfies

$$\begin{aligned} \|f\|_{A^p(\mu)}^p &= \int_0^{r_\mu} \int_0^1 \left|f(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r) \\ &\geqslant \int_0^{r_\mu} \exp \int_0^1 \log\left|f(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r) \\ &= \int_0^{r_\mu} |f(0)|^p \prod_{z \in Z(f)} \max\left\{\frac{r^p}{|z|^p}, 1\right\} \mathrm{d}\mu(r). \end{aligned} \tag{1.2}$$

In general, this inequality can be very far from an equality; for two simple examples, consider $f(z) := (1-z)^{-1}$ and $p \geqslant 2$ or $f(z) := e^{(1-z)^{-1}}$ and all p. What we will show, in contrast, is that for $f = \boldsymbol{F}$, a.s. finiteness of the right-hand side of (1.2) implies a.s. finiteness of the left-hand side and even finiteness of the expectation of the left-hand side. This is reminiscent of Fernique's theorem (Appendix A), but the functional on the right-hand side does not satisfy the hypotheses of Fernique's theorem. Moreover, Fernique's theorem gives finiteness of a moment defined in terms of the original functional, whereas here, the $A^p(\mu)$-norm is, as we just illustrated, not in any way a function of the right-hand side.

Theorem 1.7. — *Let $a_n \geqslant 0$ satisfy $R^{-1} := \limsup_{n\to\infty} a_n^{1/n} < \infty$ and $a_0 \neq 0$. Let $\boldsymbol{F}(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n$ for $|z| < R$, where ζ_n are independent complex Gaussian random variables. Then for all finite measures μ with $r_\mu = R$ and $\mu(\{R\}) = 0$ and all $p \in (0,\infty)$, the following are equivalent:*

(1) $\int_0^R \exp \int_0^1 \log|\boldsymbol{F}(re^{2\pi i\theta})|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r) < \infty$ *a.s.;*

(2) $\mathbf{E}\left[\|\boldsymbol{F}\|_{A^p(\mu)}^p\right] < \infty$*;*

(3) $\mathbf{E}\left[\int_0^R \exp \int_0^1 \log|\boldsymbol{F}(re^{2\pi i\theta})|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r)\right] < \infty$*;*

(4) $\int_0^R \exp \int_0^1 \log|\boldsymbol{F}(re^{2\pi i\theta})|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r) < \infty$ *with positive probability.*

Moreover, for all $s \in (0,R]$,

$$\mathbf{E}\left[\frac{|\boldsymbol{F}(0)|}{\left(\int_0^s \exp \int_0^1 \log|\boldsymbol{F}(re^{2\pi i\theta})|^p \,\mathrm{d}\theta \,\mathrm{d}\mu(r)\right)^{1/p}}\right] \leqslant \frac{\sqrt{\pi}\,\Gamma(1+p/2)^{1/p}\, a_0}{\mathbf{E}\left[\|\boldsymbol{F}\|_{A^p(\mu,s)}^p\right]^{1/p}}. \tag{1.3}$$

The equivalence shown here may be surprising; indeed, in discussing his conjecture, Shapiro [21] wrote that the arithmetic mean-geometric mean inequality "seems to give away too much."

1.1. History of Zero Sets

Given a collection A of analytic functions, say that Z is an *A-zero set* if there is some function in A whose zero set equals Z. There is no geometric characterization known for a set of points in $\mathbb{D}$ to be an $A^p(\mathbb{D})$-zero set, but there are necessary conditions known that are not far from known sufficient conditions. It is also known that no condition depending solely on the moduli of the points can be both necessary and sufficient. For further discussion, let $\mathbf{z}$ be a countable multiset in $\mathbb{D}$ and write

$$\varphi_{\mathbf{z}}(r) := \sum_{\substack{z \in \mathbf{z}, \\ |z| \leqslant r}} (1 - |z|) \,.$$

The situation for zeros of Bergman functions contrasts strongly with that for the Hardy spaces,

$$H^p(\mathbb{D}) := \left\{ f \in A^0(\mathbb{D})\,;\ \sup_{r<1} \int_0^1 \left| f(re^{2\pi i\theta}) \right|^p \,\mathrm{d}\theta < \infty \right\},$$

where for all $p \in (0, \infty]$, the *Blaschke condition*

$$\varphi_{\mathbf{z}}(1) < \infty$$

is necessary and sufficient to be an $H^p(\mathbb{D})$-zero set. For every $p \in (0, \infty)$, the Blaschke condition is sufficient to be an $A^p(\mathbb{D})$-zero set (since $H^p(\mathbb{D}) \subset A^p(\mathbb{D})$), while the condition

$$\sum_{z \in \mathbf{z} \setminus \{0\}} \frac{(1 - |z|)}{\log^{1+\epsilon} (1 - |z|)^{-1}} < \infty$$

is known to be necessary for every $\epsilon > 0$ but not for $\epsilon = 0$ [5]. On the other hand, if a subset of $\mathbf{z}$ lies on a line (or in a Stolz angle), then the Blaschke condition for that subset is also necessary for $\mathbf{z}$ to be an $A^p(\mathbb{D})$-zero set [19]. Combining the preceding results, we deduce that the moduli alone do not determine whether a point set is an $A^p(\mathbb{D})$-zero set.

A set W is called a *set of uniqueness for* $A(\mu)$ if the only $f \in A(\mu)$ with $Z(f) \supseteq W$ is $f \equiv 0$. Horowitz [5] showed that for $0 < p < q < \infty$, there exists an $A^p(\mathbb{D})$-zero set that is an $A^q(\mathbb{D})$-uniqueness set. In fact, he showed that if $f \in A^q(\mathbb{D})$ with zero set $\langle z_k\,;\ k \geqslant 1\rangle$ ordered so that $|z_k|$ is increasing and $f(0) \neq 0$, then

$$\sup_n n^{-1/q} \prod_{k=1}^{n} \frac{1}{|z_k|} < \infty, \tag{1.4}$$

whereas for every $p < q$, there is some $f \in A^p(\mathbb{D})$ with zero set $\langle z_k\,;\ k \geqslant 1\rangle$ ordered so that $|z_k|$ is increasing and $f(0) \neq 0$ satisfying

$$\sup_n n^{-1/q} \prod_{k=1}^{n} \frac{1}{|z_k|} = \infty.$$

(Since (1.4) depends only on the moduli, it is not sufficient to be a zero set.) This distinction among the zero sets for different p was refined by Shapiro [21]: for $0 < p < \infty$, there exists $f \in A^p(\mathbb{D})$ whose zeros are not the zeros of any function in $A^{p+}(\mathbb{D})$, where $A^{p+}(\mathbb{D}) := \bigcup_{q>p} A^q(\mathbb{D})$.(1) Shapiro [21] did this by using random (Gaussian) series, as we detail soon.(2)

Later works [1, 10, 12] considered random angles for fixed moduli, culminating in the following result.

Theorem 1.8. — *Let $0 < p < \infty$ and $\mathbf{z} = \langle z_n\,;\ n \geqslant 1\rangle \subset \mathbb{D}$. Let θ_n be independent uniform $[0,1]$ random variables. If there exists $\epsilon > 0$ such that*

$$\int_0^1 e^{p\varphi_{\mathbf{z}}(r)} \log^{(1+\epsilon)}(1-r)^{-1}\,\mathrm{d}r < \infty, \tag{1.5}$$

then a.s. $\langle z_n e^{2\pi i\theta_n}\,;\ n \geqslant 1\rangle$ is an $A^p(\mathbb{D})$-zero set. If $q > p$, then the condition (1.5) is not sufficient for $\langle z_n e^{2\pi i\theta_n}\,;\ n \geqslant 1\rangle$ to be a.s. an $A^q(\mathbb{D})$-zero set.

The Blaschke condition shows that the union of two $H^p(\mathbb{D})$-zero sets is again an $H^p(\mathbb{D})$-zero set. Horowitz [5] also showed that although the union of two $A^p(\mathbb{D})$-zero sets is an $A^{p/2}(\mathbb{D})$-zero set (trivially: just multiply the functions), it need not be an $A^q(\mathbb{D})$-zero set if $q > p/2$. This was again strengthened by Shapiro [21] to show that it need not be an $A^{(p/2)+}(\mathbb{D})$-zero set.(3)

(1) In that paper, [20] is cited for the first proof of this existence. However, he seems to have misinterpreted the order of quantifiers. Instead, the novelty of [20] was to extend the allowed set of weights from those in [5].

(2) Actually, there was a gap in his proof: in the middle of page 168 where the quantity $I(r)$ is being bounded below, going from the integral over $\mathbb{T}$ to $E^\omega(n)$ throws away a part that may be negative, so the inequality does not follow. Thus, it seems that our proof of Corollary 1.2 is the first valid proof of [21, Theorem 1 (i) implies (iii)].

(3) The same gap as noted in the previous footnote applies to this result, but is filled by the proof of our Corollary 1.5.

Many of the above results were extended to weighted Bergman spaces. For example, for $(p,\alpha) \in (0,\infty)\times(-1,\infty)$, Horowitz [5] studied the zero sets of the spaces $A^p_\alpha(\mathbb{D})$, showing that they were distinct classes of sets for pairs with distinct values of $(\alpha+1)/p$, provided that $\alpha \geqslant 0$. He asked whether it sufficed that the pairs (p,α) be distinct. The proviso that $\alpha \geqslant 0$ was removed by Sedletskiĭ [17]. The full question was answered affirmatively by Sevast′yanov and Dolgoborodov [18]. Our Corollary 1.2 easily establishes the full result of [18] by using Theorem 1 of [11], which implies that for $\alpha > -1,\ q > 0$, and $c_k \geqslant 0$,

$$\int_0^1 \left(\sum_{k=0}^{\infty} c_k r^k\right)^q (1-r)^\alpha \,\mathrm{d}r < \infty \iff \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\sum_{k=2^n}^{2^{n+1}-1} c_k\right)^q < \infty.$$

In the above expressions, we take $c_{2k} = a_k^2$ and $c_{2k+1} = 0$, and we make a judicious choice of a_k so that the convergence behavior is different for distinct pairs (p,α) and (p',α'). The most interesting case is when $(\alpha+1)/p = (\alpha'+1)/p'$ and $p < p'$, which can be handled by taking $q = p/2$, $a_{2^n}^2 = 2^{2n(\alpha+1)/p}\cdot n^{-2/p}$, and $a_k = 0$ when k is not a power of 2.

Very little is known about the zero sets of functions in the Bargmann–Fock spaces, even for $p = 2$. Zhu [24] showed that if $f \in B^p_\alpha(\mathbb{C})$ with $f(0) \neq 0$ and we write $Z(f) = \mathbf{z}$ as a sequence in increasing order of modulus, then $\inf_n |z_n|/\sqrt{n} > 0$. On the other hand, classical results show that if $\mathbf{z}$ satisfies $\sum_n |z_n|^{-2} < \infty$, then there is some $f \in B^p_\alpha(\mathbb{C})$ with $Z(f) = \mathbf{z}$ (see [25, Theorem 5.3]).

The paper [2] considered particular stationary random point processes and showed that for $p = 2$, the critical density for being a $B^2_1(\mathbb{C})$-zero set is 1. Zhu [25, p. 203] gives examples showing that a $B^2_\alpha(\mathbb{C})$-zero set and a $B^2_\alpha(\mathbb{C})$-uniqueness set can differ by just one point, and that for all $p,q \in (0,\infty)$ and $n \in (2/q,\infty)\cap\mathbb{Z}$, for every nontrivial $B^p_\alpha(\mathbb{C})$-zero set W, removing any n points from W yields a $B^q_\alpha(\mathbb{C})$-zero set.

Our results give new proofs of results of Zhu [24, 25] and answer his question [25, p. 202 and 209], showing that the zero sets of $B^p_\alpha(\mathbb{C})$ depend on p for fixed α; he had shown that they differ for differing α, whether or not p is fixed [25, Theorem 5.8]. To apply Corollary 1.2 to Zhu's question, we use the following result of Stokes [23]: for $b \geqslant 0$,

$$\lim_{t\to\infty} e^{-t} t^b \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+b)} = 1. \tag{1.6}$$

Given α and p, set $a_n := \sqrt{\alpha^n/\Gamma(n+2/p)}$ and $\boldsymbol{F}(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n$, where ζ_n are independent complex Gaussian random variables. We apply (1.6) with $b = 2/p$ and $t = \alpha r^2$, so that for $q > 0$ and as $r \to \infty$,

$$re^{-\frac{q\alpha r^2}{2}} \left(\sum_{n=0}^{\infty} a_n^2 r^{2n} \right)^{\frac{q}{2}} \asymp re^{-\frac{q\alpha r^2}{2}} \left(e^{\alpha r^2} r^{-\frac{4}{p}} \right)^{\frac{q}{2}} = r^{1-\frac{2q}{p}}.$$

Then by (1.7) and Theorem 1.1, it follows that a.s. $\boldsymbol{F} \in \bigcap_{q>p} B^q_\alpha(\mathbb{C})$ and $Z(\boldsymbol{F})$ is a $B^p_\alpha(\mathbb{C})$-uniqueness set.

Similarly, for $b \geqslant 0$ and $c \in \mathbb{R}$, we have the asymptotic

$$\lim_{t\to\infty} e^{-t} t^b (\log t)^c \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+b)(\log n)^c} = 1.$$

With $a_n := \sqrt{\alpha^n / \left(\Gamma(n+2/p)(\log n)^{4/p}\right)}$, $b = 2/p$, $c = 4/p$, and $t = \alpha r^2$, we obtain by a similar calculation that a.s. $\boldsymbol{F} \in B^p_\alpha(\mathbb{C})$ and the only $f \in \bigcup_{q<p} B^q_\alpha(\mathbb{C})$ with $Z(f) \supseteq Z(\boldsymbol{F})$ is $f \equiv 0$.

We also strengthen Zhu's result ([24] or [25, Theorem 5.4]) that there is a union of two disjoint $B^p_\alpha(\mathbb{C})$-zero sets that is a $B^p_\alpha(\mathbb{C})$-uniqueness set. Indeed, by Theorem 1.4 and Proposition 1.9, we can find disjoint $B^p_\alpha(\mathbb{C})$-zero sets Z_1, Z_2 such that the only $f \in \bigcup_{q>p} B^{q/2}_{2p\alpha/q}(\mathbb{C})$ with $Z(f) \supseteq Z_1 \cup Z_2$ is $f \equiv 0$; taking $q := 2p$ gives Zhu's result. (Note that $B^{q/2}_{2p\alpha/q}(\mathbb{C})$ decreases in q by [25, Corollary 2.8].)

1.2. Shapiro's Approach

Consider $\boldsymbol{F}(z) := \sum_{n=0}^{\infty} a_n \zeta_n z^n$, where ζ_n are IID standard complex Gaussian random variables and $a_n > 0$ satisfy $\limsup_{n\to\infty} a_n^{1/n} \leqslant 1$. Because $\mathbf{E}\left[\log^+ |\zeta_0|\right] < \infty$, we also have $\limsup_{n\to\infty} |\zeta_n|^{1/n} = 1$ a.s. by the Borel–Cantelli lemma, whence a.s. $\boldsymbol{F}(z)$ converges for all $z \in \mathbb{D}$ to an analytic function.

Let μ be a finite measure with $r_\mu = 1$. Write $L^{p+}(\mu) := \bigcup_{q>p} L^q(\mu)$. Shapiro [21] showed that the following are equivalent:

(1) $\left(r \mapsto \|a^{(r)}\|_2\right) \in L^p(\mu) \setminus L^{p+}(\mu)$;
(2) a.s. $\boldsymbol{F} \in A^p(\mu) \setminus A^{p+}(\mu)$;
(3) a.s. $\boldsymbol{F} \in A^p(\mu)$ and the only function in $A^{p+}(\mu)$ that vanishes everywhere that $\boldsymbol{F}$ does is the 0 function.

In addition, he showed that when (1) holds, *a.s. $\boldsymbol{F} \pm 1 \in A^p(\mu)$ and the only function in $A^{(p/2)+}(\mu)$ that vanishes on $Z(\boldsymbol{F}^2 - 1)$ is the 0 function.*

He conjectured that the following strengthening holds:

$$\left(r \mapsto \|a^{(r)}\|_2\right) \notin L^p(\mu) \Longrightarrow$$

a.s. the only function in $A^p(\mu)$ that vanishes on $Z(\boldsymbol{F})$ is the 0 function and the only function in $A^{p/2}(\mu)$ that vanishes on $Z(\boldsymbol{F}^2 - 1)$ is the 0 function.

More generally, he conjectured Theorem 1.4 when $r_\mu = 1$ and b satisfies a certain restriction.

The equivalence of (1) and (2) follows from the following equivalence:

$$\left(r \mapsto \|a^{(r)}\|_2\right) \in L^p(\mu) \quad \Longleftrightarrow \quad \text{a.s. } \boldsymbol{F} \in A^p(\mu). \tag{1.7}$$

(This equivalence is valid for $r_\mu = \infty$ as well.) To see this, note that for each $z \in \mathbb{D}$, the random variable $\boldsymbol{F}(z)$ has the same distribution as $\|a^{(|z|)}\|_2 \zeta_0$. Thus, Tonelli's theorem yields

$$\mathbf{E}\left[\|\boldsymbol{F}\|^p_{A^p(\mu)}\right] = \mathbf{E}\left[|\zeta_0|^p\right] \cdot \int_0^1 \|a^{(r)}\|_2^p \, \mathrm{d}\mu(r). \tag{1.8}$$

The forward implication of (1.7) is now immediate. The reverse implication is a consequence of (1.8) and Fernique's theorem, which tells us that if $\boldsymbol{F}$ a.s. belongs to $A^p(\mu)$, then there exist some $c_0, c_1 > 0$ such that $\mathbf{E}\left[\exp\{c_0 \|\boldsymbol{F}\|^{c_1}_{A^p(\mu)}\}\right] < \infty$. (See Appendix A for a statement and proof of a general form of Fernique's theorem.)

The usefulness of Shapiro's approach comes partly from his implicit observation that given μ and p, there exists $\left(r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n}\right) \in L^{p/2}(\mu) \setminus \bigcup_{q>p} L^{q/2}(\mu)$. This follows from the lemma in Section 3 of [20], where he considers analytic functions, not just real power series. For completeness, we give a short proof and extension here.

Proposition 1.9. — *Let μ be a finite measure with $0 < r_\mu \leqslant \infty$; if $r_\mu = \infty$, then assume that $\int_0^{r_\mu} r^n \, \mathrm{d}\mu(r) < \infty$ for every $n \geqslant 0$. For all $p \in (0, \infty)$, there exists $\left(r \mapsto \sum_{n=0}^\infty a_n^2 r^{2n}\right) \in L^{p/2}(\mu) \setminus \bigcup_{q>p} L^{q/2}(\mu)$.*

Proof. — For each $M > 0$, let $q := p + 1/M$ and let $s = s(M) < r_\mu$ be close enough to r_μ that $\mu(s, r_\mu) \leqslant M^{-pq/(q-p)}$. We can find $N = N(M)$ large enough so that

$$\int_s^{r_\mu} r^{Np} \, \mathrm{d}\mu(r) \geqslant \frac{1}{2} \int_0^{r_\mu} r^{Np} \, \mathrm{d}\mu(r).$$

We also have by the power-mean inequality that

$$\left(\int_s^{r_\mu} r^{Nq}\,\mathrm{d}\mu(r)\right)^{\frac{2}{q}} \geqslant \mu(s,r_\mu)^{\frac{2}{q}-\frac{2}{p}}\left(\int_s^{r_\mu} r^{Np}\,\mathrm{d}\mu(r)\right)^{\frac{2}{p}}$$
$$\geqslant M^2\left(\int_s^{r_\mu} r^{Np}\,\mathrm{d}\mu(r)\right)^{\frac{2}{p}}.$$

Now, let $n_k := N(2^k)$, and choose $b_k > 0$ so that $\left(\int_0^{r_\mu} b_k^p r^{n_k p}\,\mathrm{d}\mu(r)\right)^{2/p} = 1/2^k$. Then $\sum_{k=1}^\infty b_k^2 r^{2n_k}$ has the desired property: Write

$$\|f\|_p := \left(\int_0^{r_\mu} |f(r)|^p\,\mathrm{d}\mu(r)\right)^{1/p}.$$

If $p \geqslant 2$, then

$$\left\|\sum_{k=1}^\infty b_k^2 r^{2n_k}\right\|_{p/2} \leqslant \sum_{k=1}^\infty \left\|b_k^2 r^{2n_k}\right\|_{p/2} = 1,$$

while if $p < 2$, then

$$\left\|\sum_{k=1}^\infty b_n^2 r^{2n_k}\right\|_{p/2}^{p/2} \leqslant \sum_{k=1}^\infty \left\|b_k^2 r^{2n_k}\right\|_{p/2}^{p/2} < \infty.$$

At the same time, for each $q > p$, consider any j with $q > p + 1/2^j$. Then

$$\left\|\sum_{k=1}^\infty b_k^2 r^{2n_k}\right\|_{q/2} \geqslant \left\|b_j^2 r^{2n_j}\right\|_{q/2} \geqslant b_j^2 2^{2j}\left(\int_{s(2^j)}^{r_\mu} r^{n_j p}\,\mathrm{d}\mu(r)\right)^{\frac{2}{p}}$$
$$\geqslant b_j^2 2^{2j}\left(\frac{1}{2}\int_0^{r_\mu} r^{n_j p}\,\mathrm{d}\mu(r)\right)^{\frac{2}{p}} = 2^{j-2/p}.$$

Since this holds for all such j, it follows that $\left\|\sum_{k=1}^\infty b_k^2 r^{2n_k}\right\|_{q/2} = \infty$. □

2. Proofs

In this section, we prove Theorem 1.3 and then indicate the additional steps needed for the more general Theorem 1.6. At the end, we prove Theorem 1.7.

Proof of Theorem 1.3. — Note that the density of $|\zeta_0|$ with respect to Lebesgue measure on $\mathbb{R}^+$ is $r \mapsto 2re^{-r^2}$. It suffices to prove the theorem for $s < r_\mu$, since the case $s = r_\mu$ follows by taking limits.

Suppose that $0 \notin Z(f) \supseteq Z(\boldsymbol{F})$. Note that $0 \notin Z(\boldsymbol{F})$ a.s. Thus, for $0 < s < r_\mu$, we have a.s. by the arithmetic mean-geometric mean inequality and Jensen's formula that

$$\begin{aligned}\|f\|^p_{A^p(\mu,s)} &= \int_0^s \int_0^1 \left|f(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r)\\ &\geqslant \int_0^s \exp\int_0^1 \log\left|f(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r)\\ &= \int_0^s |f(0)|^p \prod_{z\in Z_s(f)} \max\left\{\frac{r^p}{|z|^p}, 1\right\}\mathrm{d}\mu(r)\\ &\geqslant \int_0^s |f(0)|^p \prod_{z\in Z_s(\boldsymbol{F})} \max\left\{\frac{r^p}{|z|^p}, 1\right\}\mathrm{d}\mu(r)\\ &= |f(0)/\boldsymbol{F}(0)|^p \int_0^s \exp\int_0^1 \log\left|\boldsymbol{F}\left(re^{2\pi i\theta}\right)\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r).\end{aligned}$$

Therefore,

$$\frac{|f(0)|}{\|f\|_{A^p(\mu,s)}} \leqslant a_0|\zeta_0|\left(\int_0^s \exp\int_0^1 \log\left|\boldsymbol{F}(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r)\right)^{-1/p}. \tag{2.1}$$

Recall that for each r and θ, $\boldsymbol{F}(re^{2\pi i\theta})$ is a Gaussian random variable with the same distribution as $\|a^{(r)}\|_2\zeta_0$, where $a^{(r)}_n := a_n r^n$. Write

$$\boldsymbol{G}_r(\theta) := \boldsymbol{F}(re^{2\pi i\theta})/\|a^{(r)}\|_2,$$

so that $\boldsymbol{G}_r(\theta)$ is a standard complex Gaussian random variable for each r and θ. Hölder's inequality and the arithmetic mean-geometric mean inequality yield

$$\begin{aligned}&\left(\int_0^s \exp\int_0^1 \log\left|\boldsymbol{F}(re^{2\pi i\theta})\right|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r)\right)^{-1/p} \\ &\qquad= \left(\int_0^s \|a^{(r)}\|_2^p \exp\int_0^1 \log|\boldsymbol{G}_r(\theta)|^p \,\mathrm{d}\theta\,\mathrm{d}\mu(r)\right)^{-1/p}\\ &\qquad\leqslant \frac{\int_0^s \|a^{(r)}\|_2^p \exp\int_0^1 \log|\boldsymbol{G}_r(\theta)|^{-1}\,\mathrm{d}\theta\,\mathrm{d}\mu(r)}{\left(\int_0^s \|a^{(r)}\|_2^p\,\mathrm{d}\mu(r)\right)^{1+1/p}}\\ &\qquad\leqslant \frac{\int_0^s \|a^{(r)}\|_2^p \int_0^1 |\boldsymbol{G}_r(\theta)|^{-1}\,\mathrm{d}\theta\,\mathrm{d}\mu(r)}{\left(\int_0^s \|a^{(r)}\|_2^p\,\mathrm{d}\mu(r)\right)^{1+1/p}}.\end{aligned} \tag{2.2}$$

Multiplying both sides by $a_0|\zeta_0|$ and using (2.1), we have

$$\frac{|f(0)|}{\|f\|_{A^p(\mu,s)}} \leqslant \frac{a_0|\zeta_0| \cdot \int_0^s \|a^{(r)}\|_2^p \int_0^1 |\boldsymbol{G}_r(\theta)|^{-1}\,\mathrm{d}\theta\,\mathrm{d}\mu(r)}{\left(\int_0^s \|a^{(r)}\|_2^p\,\mathrm{d}\mu(r)\right)^{1+1/p}}. \tag{2.3}$$

Recall that for each r and θ, $\boldsymbol{G}_r(\theta)$ and ζ_0 are both standard complex Gaussians, and $(\boldsymbol{G}_r(\theta), \zeta_0)$ is jointly Gaussian. By a version of Slepian's lemma due to [9], we have

$$\mathbf{E}\left[|\zeta_0| \cdot |\boldsymbol{G}_r(\theta)|^{-1}\right] \leqslant \mathbf{E}\left[|\zeta_0|\right] \cdot \mathbf{E}\left[|\boldsymbol{G}_r(\theta)|^{-1}\right] = 1 \cdot \sqrt{\pi}.$$

Taking expectations in (2.3) and applying the above inequality finishes the proof, except for showing that the maximum on the left-hand side of (1.1) is achieved and is measurable.

To show these properties, note first that the maximum is achieved because of a standard normal-families argument (compare [3, p. 120]). Next, for a finite multiset W, let $p_W(z) := \prod_{w\in W}(z-w)$ be the monic polynomial whose zeros are W (with multiplicity). For any analytic function f whose zeros include W, the function f/p_W is analytic. Therefore,

$$\max\left\{\frac{|f(0)|}{\|f\|_{A^p(\mu,s)}}\,;\ 0 \not\equiv f \in A(\mu),\ Z(f) \supset Z_s(\boldsymbol{F})\right\} \\ = \max\left\{\frac{|f(0)p_{Z_s(\boldsymbol{F})}(0)|}{\left\|fp_{Z_s(\boldsymbol{F})}\right\|_{A^p(\mu,s)}}\,;\ 0 \not\equiv f \in A(\mu)\right\}.$$

Restricting to polynomials f with rational coefficients, we see that this maximum is measurable provided $p_{Z_s(\boldsymbol{F})}$ is measurable. Now there is a measurable set (of probability 0) where $\limsup |\zeta_n|^{1/n} > 1$; off of this set, $Z_s(\boldsymbol{F})$ is finite and can be determined by looking at the values of $\boldsymbol{F}$ on a fixed, countable, dense set of points, thereby proving the desired measurability. □

Remark 2.1. — In fact, Theorem 1.1 may be deduced directly from (2.2) without using Slepian's lemma: Simply take expectations of both sides and use the facts that $\mathbf{E}\left[|\boldsymbol{G}_r(\theta)|^{-1}\right] = \sqrt{\pi}$ and $|\zeta_0| < \infty$ to obtain

$$\mathbf{E}\left[\left(\int_0^s \exp \int_0^1 \log\left|\boldsymbol{F}(re^{2\pi i\theta})\right|^p\,\mathrm{d}\theta\,\mathrm{d}\mu(r)\right)^{-1/p}\right] \leqslant \frac{\sqrt{\pi}}{\left(\int_0^s \|a^{(r)}\|_2^p\,\mathrm{d}\mu(r)\right)^{1/p}}.$$

As $s \uparrow r_\mu$, the right-hand side tends to 0, which already gives Theorem 1.1 via (2.1).

Proof of Theorem 1.6. — We may assume that $a_0 \neq 0$. Suppose that $0 \notin Z(f) \supseteq Z(\boldsymbol{F})$. As before, we have for $0 < s < r_\mu$

$$\left(\frac{|f(0)|}{\|f\|_{A^{p/N}(\mu,s)}}\right)^{1/N} \leqslant |a_0^N \zeta_0^N + b(0)|^{1/N} \cdot \left(\int_0^s \exp \int_0^1 \log \left|\boldsymbol{F}(re^{2\pi i\theta})^N + b(re^{2\pi i\theta})\right|^{p/N} \mathrm{d}\theta \, \mathrm{d}\mu(r)\right)^{-1/p}.$$

Write

$$\boldsymbol{H}_r(\theta) := \left|\boldsymbol{F}(re^{2\pi i\theta})^N + b(re^{2\pi i\theta})\right| / \|a^{(r)}\|_2^N.$$

In the same way as before, we obtain

$$\begin{aligned}(2.4)\quad \left(\int_0^s \exp \int_0^1 \log|\boldsymbol{F}(re^{2\pi i\theta})^N + b(re^{2\pi i\theta})|^{p/N} \mathrm{d}\theta \, \mathrm{d}\mu(r)\right)^{-1/p} \\ = \left(\int_0^s \|a^{(r)}\|_2^p \exp \int_0^1 \log \boldsymbol{H}_r(\theta)^{p/N} \mathrm{d}\theta \, \mathrm{d}\mu(r)\right)^{-1/p} \\ \leqslant \frac{\int_0^s \|a^{(r)}\|_2^p \exp \int_0^1 \log \boldsymbol{H}_r(\theta)^{-1/N} \mathrm{d}\theta \, \mathrm{d}\mu(r)}{\left(\int_0^s \|a^{(r)}\|_2^p \, \mathrm{d}\mu(r)\right)^{1+1/p}} \\ \leqslant \frac{\int_0^s \|a^{(r)}\|_2^p \int_0^1 \boldsymbol{H}_r(\theta)^{-1/N} \mathrm{d}\theta \, \mathrm{d}\mu(r)}{\left(\int_0^s \|a^{(r)}\|_2^p \, \mathrm{d}\mu(r)\right)^{1+1/p}}.\end{aligned}$$

We have for any β that

$$\mathbf{E}\left[\boldsymbol{H}_r(\theta)^{-\beta}\right] = \mathbf{E}\left[|\zeta_0^N + b(re^{2\pi i\theta})/\|a^{(r)}\|_2^N|^{-\beta}\right].$$

Now ζ_0^N has density $\rho\colon z \mapsto c|z|^{-2(N-1)/N} e^{-|z|^2}$ (with respect to area measure λ_2, for some constant c) that is decreasing in $|z|$. Therefore, given any $\alpha \in \mathbb{C}$, the rearrangement inequality of Hardy and Littlewood yields

$$\begin{aligned}\mathbf{P}\left[|\zeta_0^N| < r\right] = \int_{|z|<r} \rho(z) \, \mathrm{d}\lambda_2(z) \geqslant \int_{|z-\alpha|<r} \rho(z) \, \mathrm{d}\lambda_2(z) \\ = \mathbf{P}\left[|\zeta_0^N - \alpha| < r\right],\end{aligned}$$

which is to say that $|\zeta_0^N|$ is stochastically dominated by $|\zeta_0^N - \alpha|$. Thus, for all $0 < \beta < 2/N$,

$$\mathbf{E}\left[|\zeta_0^N - \alpha|^{-\beta}\right] \leqslant \mathbf{E}\left[|\zeta_0^N|^{-\beta}\right] = \int_0^\infty \frac{e^{-t}}{t^{\beta N/2}} \, \mathrm{d}t = \Gamma(1 - \beta N/2).$$

Therefore,

$$(2.5)\qquad \mathbf{E}\left[\boldsymbol{H}_r(\theta)^{-\beta}\right] \leqslant \Gamma(1 - \beta N/2).$$

Multiply both sides of the inequality (2.4) by $|a_0^N\zeta_0^N + b(0)|^{1/N}$ and use Hölder's inequality to bound the resulting expectation:

$$\begin{aligned}\mathbf{E}\left[|a_0^N\zeta_0^N + b(0)|^{1/N}\cdot \boldsymbol{H}_r(\theta)^{-1/N}\right]&\\ \leqslant \mathbf{E}\left[|a_0^N\zeta_0^N + b(0)|^4\right]^{\frac{1}{4N}}\mathbf{E}\left[\boldsymbol{H}_r(\theta)^{-4/(4N-1)}\right]^{\frac{4N-1}{4N}}&\\ \leqslant \left(a_0^{4N}(2N)! + 4|b(0)|^2a_0^{2N}N! + |b(0)|^4\right)^{1/4N}\Gamma\left(\frac{2N-1}{4N-1}\right)^{\frac{4N-1}{4N}}&,\end{aligned}$$

where in the last inequality, we used (2.5) with $\beta := 4/(4N-1)$. □

Proof of Theorem 1.7. — We established (1.3) during the proof of Theorem 1.3, where we rely on (1.8) and the fact that $\mathbf{E}[|\zeta_0|^p] = \Gamma(1+p/2)$ for an equivalent expression on the right-hand side. That (2) implies (3) follows from the arithmetic mean-geometric mean inequality. That (3) implies (1) and (1) implies (4) are obvious. That (4) implies (2) follows from (1.3) with $s = R$. □

Appendix A. Fernique's Theorem

We present here a general version of Fernique's theorem, not only for use in deriving the background in Subsection 1.2, but also for comparison with our Theorem 1.7.

Theorem A.1. — *Let V be a separable topological vector space. Let $\phi\colon V \to [0,\infty]$ be Borel measurable, $c \in [1,\infty)$, and $c_1, c_2 \in (1,\infty)$ satisfy for all $x, y \in V$ that $\phi(-x) = \phi(x)$, $c_2\phi(x) \leqslant \phi(\sqrt{2}\,x) \leqslant c_1\phi(x)$, and $\phi(x+y) \leqslant c\,(\phi(x)+\phi(y))$. Let X be a random variable with values in V such that if Y has the same distribution as X and is independent of X, then $(\phi(X),\phi(Y))$ has the same joint distribution as $\big(\phi\big((X-Y)/\sqrt{2}\big)$, $\phi\big((X+Y)/\sqrt{2}\big)\big)$. If $\mathbf{P}[\phi(X) < \infty] = 1$, then there are some $\alpha, \beta > 0$ so that $\mathbf{E}\left[e^{\alpha\phi(X)^\beta}\right] < \infty$.*

Proof. — Suppose that $\phi\left((x-y)/\sqrt{2}\right) \leqslant \tau$ and $\phi\left((x+y)/\sqrt{2}\right) > t$. Then $\phi(x-y) \leqslant c_1\tau$ and $\phi(x+y) > c_2t$. Also, $\phi(2y) = \phi(\sqrt{2}^2y) \leqslant c_1^2\phi(y)$, whence $\phi(x+y) \leqslant c\phi(x-y) + cc_1^2\phi(y)$. Therefore $\phi(y) > (c_2t - cc_1\tau)/(cc_1^2)$.

Symmetry gives the same lower bound on $\phi(x)$. It follows that

$$\text{(A.1)}\quad \mathbf{P}[\phi(X) \leqslant \tau]\,\mathbf{P}[\phi(Y) > t] = \mathbf{P}\left[\phi\left((X-Y)/\sqrt{2}\right) \leqslant \tau,\, \phi\left((X+Y)/\sqrt{2}\right) > t\right] \leqslant \mathbf{P}\left[\phi(X) > (c_2 t - c c_1 \tau)/(c c_1^2)\right]^2 .$$

Choose $\tau < \infty$ so that $\mathbf{P}[\phi(X) \leqslant \tau] \geqslant e/(1+e)$. Define recursively $t_0 := (cc_1/c_2)\tau \geqslant \tau$ and $t_{n+1} := (cc_1^2/c_2)t_n + t_0$; thus,

$$t_n = \frac{(cc_1^2/c_2)^{n+1} - 1}{cc_1^2/c_2 - 1} t_0 < c_3 (cc_1^2/c_2)^n t_0$$

for some constant $c_3 < \infty$. The display (A.1) yields

$$\mathbf{P}[\phi(X) > t_{n+1}] = \mathbf{P}[\phi(Y) > t_{n+1}] \leqslant \frac{1+e}{e} \mathbf{P}[\phi(X) > t_n]^2 ,$$

whence if we write $y_n := \frac{1+e}{e}\mathbf{P}[\phi(X) > t_n]$, then $y_{n+1} \leqslant y_n^2$, and so $y_n \leqslant y_0^{2^n} \leqslant e^{-2^n}$. Therefore,

$$\mathbf{P}\left[\phi(X) > c_3 (cc_1^2/c_2)^n t_0\right] \leqslant e^{-2^n} = e^{-(cc_1^2/c_2)^{\beta n}},$$

where $\beta := \log 2/\log(cc_1^2/c_2) > 0$. This means that

$$\mathbf{P}[\phi(X) > t] \leqslant e^{-c_4 t^\beta}$$

for some $c_4 > 0$ and all $t \geqslant t_0$. With $\alpha := c_4/2$, the conclusion may be verified via integration by parts. □

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