

Weak and Weak* Topologies

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For us, the key duality relationship is the Riesz representation theorem, identifying $M(\mathbb{R})$ (i.e., the Banach space of finite, signed measures on \mathbb{R} with norm being total variation) as the dual of $C_0(\mathbb{R})$ (i.e., the space of continuous functions on \mathbb{R} that vanish—tend to 0—at $\pm\infty$, with the sup norm), with $\mu \in M(\mathbb{R})$ defining the linear functional $f \mapsto \int f d\mu$. Furthermore, the positive linear functionals (i.e., those that map nonnegative functions to nonnegative numbers) are precisely the positive measures. This theorem was extended by Kakutani from \mathbb{R} to every locally compact Hausdorff space, such as \mathbb{R}^d . Thus, we recall some basic facts from functional analysis.

If X is a Banach space, then its dual, X^* , denotes the linear space of bounded (= continuous) linear functionals on X . The norm on X^* is

$$\|x^*\| := \sup\{|x^*(x)|; \|x\| = 1\} = \sup\{x^*(x); \|x\| = 1\} = \sup\{x^*(x)/\|x\|; x \neq 0\},$$

which makes X^* into a Banach space. Note that when $X = C_0(\mathbb{R})$, this says that for $\mu \in M(\mathbb{R})$,

$$\|\mu\| = \sup\left\{\int f d\mu; f \in C_0(\mathbb{R}), \|f\| = 1\right\}.$$

The weak* topology on X^* is the smallest topology with respect to which $x^* \mapsto x^*(x)$ is continuous for each $x \in X$. If X is separable (i.e., X has a countable, dense subset), then the weak* topology of X^* is metrizable, so $x_n^* \xrightarrow{w^*} x^*$ iff $\forall x \in X$ $x_n^*(x) \rightarrow x^*(x)$. For example, $\mu_n \xrightarrow{w^*} \mu$ iff $\int f d\mu_n \rightarrow \int f d\mu$ for all $f \in C_0(\mathbb{R})$. (If we required this for all $f \in C_b(\mathbb{R})$, then this would be an extension of weak convergence for probability measures, by the portmanteau theorem. If we required this for all $f \in C_c(\mathbb{R})$ —where “c” means with compact support—, then this would be what is called **vague convergence**, which is important for infinite measures. For finite measures with bounded norm, vague convergence is the same as weak* convergence.)

The Banach–Alaoglu theorem says that the unit ball of X^* is weak* compact (i.e., compact in the weak* topology). Recall that a metric space is compact iff it is sequentially compact. Thus, when X is separable, the unit ball of X^* is weak* sequentially compact. However, this does not guarantee that a sequence of probability measures has a subsequence that converges weak* to a *probability* measure. That involves the concept of tightness, which is somewhat analogous to uniform integrability for functions. A collection \mathcal{M} of probability measures on \mathbb{R} is **tight** if

$$\lim_{a \rightarrow \infty} \inf_{\mu \in \mathcal{M}} \mu(-a, a) = 1,$$

or, equivalently,

$$\lim_{a \rightarrow \infty} \sup_{\mu \in \mathcal{M}} \mu(-a, a)^c = 0.$$

We do know, however, that not only is every weak* limit point of probability measures in the unit ball, but also that such a limit point is a positive measure. We call a (positive) measure a **subprobability measure** if its norm is at most 1. Here are three homework problems about $M(\mathbb{R})$:

Homework 1. Let μ_n and μ be subprobability measures. Show that $\mu_n \xrightarrow{w^*} \mu$ as $n \rightarrow \infty$ iff $\mu_n(a, b] \rightarrow \mu(a, b]$ for all $-\infty < a < b < \infty$ with $\mu\{a, b\} = 0$.

Homework 2. Let μ_n be probability measures with $\mu_n \xrightarrow{w^*} \mu$ as $n \rightarrow \infty$. Show that $\{\mu_n; n \geq 1\}$ is tight iff μ is a probability measure.

Homework 3. Let μ_n and μ be probability measures. Show that $\mu_n \Rightarrow \mu$ iff $\mu_n \xrightarrow{w^*} \mu$.

We now use these to prove a basic result about weak convergence:

Corollary. Let μ_n and μ be probability measures. If $\langle \mu_n; n \geq 1 \rangle$ is a tight sequence with only one weak sequential limit point, μ , then $\mu_n \Rightarrow \mu$ as $n \rightarrow \infty$.

Proof. Assume $\langle \mu_n; n \geq 1 \rangle$ is a tight sequence with only one weak sequential limit point. By the Banach–Alaoglu theorem, $\langle \mu_n; n \geq 1 \rangle$ has weak* sequential limit points, each of which, by Homework 2, is a probability measure. By Homework 3, μ is the only weak* sequential limit point of $\langle \mu_n; n \geq 1 \rangle$. Thus, $\mu_n \xrightarrow{w^*} \mu$ and by Homework 3 again, $\mu_n \Rightarrow \mu$. ■

Because of Skorohod’s theorem, we have easy extensions of results on limits of integrals, such as a version of Fatou’s lemma: $X_n \Rightarrow X$ implies $E[|X|] \leq \liminf_{n \rightarrow \infty} E[|X_n|]$.