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Identities and Inequalities for Tree Entropy

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The notion of tree entropy was introduced by the author as a normalized limit of the number of spanning trees in finite graphs, but is defined on random infinite rooted graphs. We give some new expressions for tree entropy; one uses Fuglede–Kadison determinants, while another uses effective resistance. We use the latter to prove that tree entropy respects stochastic domination. We also prove that tree entropy is non-negative in the unweighted case, a special case of which establishes Lück's Determinant Conjecture for Cayley-graph Laplacians. We use techniques from the theory of operators affiliated to von Neumann algebras.

1. Introduction

The enumeration of spanning trees in a finite graph is a classical subject dating to the mid-nineteenth century. Asymptotics began to play a role over 100 years later, in the 1960s. When a sequence of finite graphs converges in an appropriate, but very general, sense, Lyons [11] gave a formula for the limit of the numbers of spanning trees in that sequence of graphs, when normalized appropriately. This limit was called the tree entropy of the corresponding limit object, which was a probability measure on rooted infinite graphs.

This new concept of tree entropy allowed Lyons [11] to give simple proofs of known limits and inequalities, as well as to resolve an open question of McKay [13] and to easily calculate new limits. Here, we give some new expressions for tree entropy, in part correcting some mistakes in [11]. Tools we use from the theory of operators affiliated to von Neumann algebras were not available at the time that [11] was written. The new tools also enable us to obtain cleaner results with weaker hypotheses. Furthermore, we are

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able to extend an inequality from [11] that compares the tree entropies of two probability measures when one stochastically dominates the other.

The notion of tree entropy extends to weighted graphs, but the case of unweighted graphs is, of course, particularly interesting. Using our new representation, we prove that the tree entropy is non-negative for unweighted graphs, which is not at all obvious from the definition or from any of its representations. In fact, the special case of Cayley graphs establishes Lück's Determinant Conjecture for the graph Laplacian.

We give the details of the results of Lyons [11] referred to above and then some background on von Neumann algebras and Fuglede–Kadison determinants in Section 2. We prove that tree entropy is the logarithm of a Fuglede–Kadison determinant in Theorem 3.1. This is used to represent tree entropy with effective resistances in Theorem 3.3. Combined with Rayleigh's monotonicity principle, this representation has the immediate consequence that stochastic domination implies tree entropy domination, Theorem 3.2. This consequence is then combined with information about wired uniform spanning forests to prove in Theorem 3.4 that tree entropy is non-negative for unweighted graphs.

2. Background

In order to define the notion of convergence of finite graphs used by Lyons [11] that we referred to, we first recall the following definitions. A *rooted graph* (G, o) is a graph G with a distinguished vertex o of G, called the *root*. A *rooted isomorphism* of rooted graphs is an isomorphism of the underlying graphs that takes the root of one to the root of the other. Given a positive integer R, a finite rooted graph H, and a probability distribution ρ on rooted graphs, let $p(R, H, \rho)$ denote the probability that H is rooted isomorphic to the ball of radius R about the root of a graph chosen with distribution ρ. For a finite graph G, let $U(G)$ denote the distribution of rooted graphs obtained by choosing a uniform random vertex of G as root of G. Suppose that $\langle G_n \rangle$ is a sequence of finite graphs and that ρ is a probability measure on rooted infinite graphs. We say the *random weak limit* of $\langle G_n \rangle$ is ρ if, for any positive integer R and any finite graph H, we have $\lim_{n\to\infty} p\big(R, H, U(G_n)\big) = p(R, H, \rho)$. This notion was introduced by Benjamini and Schramm [2]. More generally, if G_n are random finite graphs, then we say the *random weak limit* of $\langle G_n \rangle$ is ρ if, for any positive integer R, any finite graph H, and any $\epsilon > 0$, we have $\lim_{n\to\infty} \mathbf{P}\big[|p\big(R, H, U(G_n)\big) - p(R, H, \rho)| > \epsilon\big] = 0$. Note that only the component of the root matters for convergence to ρ. Thus, we may and shall assume that ρ is concentrated on connected graphs.

Recall from Lyons [11] that the *tree entropy* of a probability measure ρ on rooted infinite graphs is

$$\mathbf{h}(\rho) := \int \left(\log \deg_G(o) - \sum_{k \geqslant 1} \frac{1}{k} p_k(o; G) \right) d\rho(G, o), \tag{2.1}$$

where $\deg_G(o)$ is the degree of o in G and $p_k(o; G)$ is the probability that simple random walk on G started at o is again at o after k steps. One of the main theorems of [11] was Theorem 3.2, which states the following. Let $\tau(G)$ denote the number of spanning trees of a graph G.

Theorem 2.1. *If G_n are finite connected graphs with bounded average degree whose random weak limit is a probability measure ρ on infinite rooted graphs, then*

$$\lim_{n\to\infty} \frac{1}{|\mathsf{V}(G_n)|} \log \tau(G_n) = \mathbf{h}(\rho).$$

The same limit holds in probability when G_n are random with bounded expected average degree.

In the case of regular graphs G_n with girth tending to infinity, the random weak limit is a rooted regular tree (of the same degree); with additional hypotheses on G_n, McKay [14] proved what amounts to the same limit as in Theorem 2.1 and asked whether these additional hypotheses were needed. Theorem 2.1 shows that they are not. (Finer asymptotics for the maximum number of spanning trees in finite regular graphs were given by Chung and Yau [3].)

The class of probability measures ρ that arise as random weak limits of finite networks is contained in the class of unimodular ρ, which we now define. They also include each ρ that is concentrated on a single Cayley graph with a fixed root. For more details, see Aldous and Lyons [1]. Since we shall use labelled graphs, *i.e.*, networks, we make a definition that includes them.

Definition. Let ρ be a probability measure on rooted networks. We call ρ *unimodular* if

$$\int \sum_{x\in\mathsf{V}(G)} f(G,o,x)\, d\rho(G,o) = \int \sum_{x\in\mathsf{V}(G)} f(G,x,o)\, d\rho(G,o),$$

for all non-negative Borel functions f on locally finite connected networks with an ordered pair of distinguished vertices that is invariant in the sense that, for any (non-rooted) network isomorphism γ of G and any $x, y \in \mathsf{V}(G)$, we have $f(\gamma G, \gamma x, \gamma y) = f(G, x, y)$,

We need the following finite von Neumann algebra from Section 5 of Aldous and Lyons [1], to which we refer for more details. We also refer to Lyons [11] for more background and motivation. Suppose that ρ is a unimodular probability measure on (rooted isomorphism classes of) rooted (connected) networks. Consider the Hilbert space $H := \int^{\oplus} \ell^2\big(\mathsf{V}(G)\big)\, d\rho(G,o)$, a direct integral. Let $T : (G,o) \mapsto T_{G,o}$ be a measurable assignment of bounded linear operators $T_{G,o} : \ell^2\big(\mathsf{V}(G)\big) \to \ell^2\big(\mathsf{V}(G)\big)$ with finite supremum of the norms $\|T_{G,o}\|$. Then T induces a bounded linear operator $T := T^\rho := \int^{\oplus} T_{G,o}\, d\rho(G,o)$ on H via

$$T^\rho : \int^{\oplus} f_{G,o}\, d\rho(G,o) \mapsto \int^{\oplus} T_{G,o} f_{G,o}\, d\rho(G,o).$$

The norm $\|T^\rho\|$ of T^ρ is the ρ-essential supremum of $\|T_{G,o}\|$. Let Alg be the von Neumann algebra of (ρ-equivalence classes of) such maps T that are equivariant in the sense that, for all network isomorphisms $\phi : G_1 \to G_2$, all $o_1, x, y \in \mathsf{V}(G_1)$ and all $o_2 \in \mathsf{V}(G_2)$, we have $(T_{G_1,o_1}\mathbf{1}_{\{x\}}, \mathbf{1}_{\{y\}}) = (T_{G_2,o_2}\phi\mathbf{1}_{\{x\}}, \phi\mathbf{1}_{\{y\}})$. For $T \in \mathsf{Alg}$, we have in particular that $T_{G,o}$ depends on G but not on the root o, so we simplify our notation and write T_G in place of $T_{G,o}$. Recall that if T is a self-adjoint operator on a Hilbert space H, we write $T \geqslant 0$ if

$(Tu, u) \geqslant 0$ for all $u \in H$. As shown in Section 5 of Aldous and Lyons [1], the functional

$$\mathrm{Tr}(T) := \mathrm{Tr}_\rho(T) := \mathbf{E}\big[(T_G \mathbf{1}_{\{o\}}, \mathbf{1}_{\{o\}})\big] := \int (T_G \mathbf{1}_{\{o\}}, \mathbf{1}_{\{o\}})\, d\rho(G, o)$$

is a trace on Alg, which is obviously finite. Write $\overline{\mathsf{Alg}}$ for the set of closed densely defined operators affiliated with Alg, *i.e.*, those closed densely defined operators that commute with all unitary operators that commute with Alg; see, *e.g.*, Kadison and Ringrose [9, p. 342].

The only networks we consider will be weighted graphs. Let $G = \big(\mathsf{V}(G), \mathsf{E}(G), w\big)$ be a graph with a positive weight function $w : \mathsf{E}(G) \to (0, \infty)$. For $x \neq y \in \mathsf{V}(G)$, let $\Delta_G(x, y) := -\sum_e w(e)$, where the sum is over all the edges between x and y, and $\Delta_G(x, x) := \sum_e w(e)$, where the sum is over all non-loop edges incident to x. We assume that $\Delta_G(x, x) < \infty$ for all x. An unweighted graph corresponds to $w \equiv 1$, in which case $\Delta_G(x, x)$ is the degree of x in G (not counting loops). The associated network random walk has the transition probability from x to y of $-\Delta_G(x, y)/\Delta_G(x, x)$; this is simple random walk in the case of unweighted simple graphs. Let $p_k(o; G)$ be the probability that the network random walk on G started at o is again at o after k steps. The extension from Lyons [11] of (2.1) to weighted graphs is the following: the *tree entropy* of a probability measure ρ on rooted weighted infinite graphs is

$$\mathbf{h}(\rho) := \int \left(\log \Delta_G(o, o) - \sum_{k \geqslant 1} \frac{1}{k} p_k(o; G) \right) d\rho(G, o) \tag{2.2}$$

whenever this integral converges (possibly to $\pm\infty$).

The (*graph*) *Laplacian* Δ_G defined in the preceding paragraph determines an operator

$$f \mapsto \left(x \mapsto \sum_{y \in \mathsf{V}} \Delta_G(x, y) f(y) \right)$$

on functions $f : \mathsf{V}(G) \to \mathbb{C}$ with finite support. This operator extends by continuity to a bounded linear operator on all of $\ell^2\big(\mathsf{V}(G)\big)$ when $\sup_x \Delta_G(x, x) < \infty$. When

$$\rho\text{-}\operatorname*{ess\,sup}_{(G,o)} \sup_{x \in \mathsf{V}(G)} \Delta_G(x, x) < \infty, \tag{2.3}$$

then $(G, o) \mapsto \Delta_G$ defines an operator $\Delta \in \mathsf{Alg}(\rho)$. It is self-adjoint and positive semi-definite, *i.e.*, $\Delta \geqslant 0$. However, if we do not have such a uniform bound as (2.3), we proceed as follows. Let

$$\mathcal{D}_0 := \{ f \in H \,;\, \forall (G, o)\ |\mathrm{supp} f_{G,o}| < \infty \}.$$

The operator Δ is defined on the dense subspace $\mathcal{D}_0$, where it is symmetric. Let D be the diagonal weighted degree operator on $\mathcal{D}_0$, *i.e.*, $D_G(x, x) := \Delta_G(x, x)$ and $D_G(x, y) := 0$ for $x \neq y$. Its closure $\overline{D}$ is easily seen to be self-adjoint and affiliated with Alg. Let P be the transition operator for the network random walk, which is obviously in Alg. Define $\delta := \overline{D}(I - P)$; since $\overline{D} \in \overline{\mathsf{Alg}}$ and $I - P \in \mathsf{Alg}$, it follows that $\delta \in \overline{\mathsf{Alg}}$. We claim that Δ is closeable and that $\delta = \overline{\Delta}$. First, an easy calculation shows that δ and Δ agree on $\mathcal{D}_0$, so that δ extends Δ. Since δ is closed, Δ is closeable. Therefore $\overline{\Delta} \in \overline{\mathsf{Alg}}$ and, furthermore, is self-adjoint by Lemma 16.4.1 of Murray and von Neumann [15] (which is the same as Exercise 6.9.53 of Kadison and Ringrose [10]). Since $\overline{\Delta} \subseteq \delta$, it follows that $\overline{\Delta} = \delta$ by Lemma 16.4.2 of [15] (or Exercise 6.9.54 of [10]). From now on, we omit the overlines and write more simply D and Δ for their closures, $\overline{D}$ and $\overline{\Delta}$.

Let $T \in \overline{\mathsf{Alg}}$ be a self-adjoint operator with spectral resolution E_T. We define the Borel measure $\mu_{\rho,T}$ by

$$\mu_{\rho,T}(B) := \mathrm{Tr}_\rho\big(E_T(B)\big) \tag{2.4}$$

for Borel subsets $B \subseteq \mathbb{R}$. We extend the trace by defining

$$\mathrm{Tr}_\rho(T) := \int_0^\infty \lambda \, d\mu_{\rho,T}(\lambda)$$

for positive operators $T \in \overline{\mathsf{Alg}}$ and then by linearity to all of $\overline{\mathsf{Alg}}$ when it makes sense. Write $|T| := \sqrt{T^* T}$.

As in Haagerup and Schultz [8] (though with different notation), write DetAlg for the set of $T \in \overline{\mathsf{Alg}}$ for which

$$\mathrm{Tr}_\rho(\log^+ |T|) = \int_0^\infty \log^+ \lambda \, d\mu_{\rho,|T|}(\lambda) < \infty.$$

(The equality is justified by the functional calculus; see Theorem 5.6.26 of Kadison and Ringrose [9].) For $T \in \mathsf{DetAlg}$, we define its *Fuglede–Kadison determinant* by

$$\mathrm{Det}(T) := \mathrm{Det}_\rho(T) := \exp \int_0^\infty \log \lambda \, d\mu_{\rho,|T|}(\lambda) \in [0,\infty). \tag{2.5}$$

For example, for the diagonal weighted degree operator, D, its Fuglede–Kadison determinant is the geometric-mean weighted degree of the root,

$$\mathrm{Det}_\rho D = \exp \int \log D_G(o,o) \, d\rho(G,o), \tag{2.6}$$

provided this is $< \infty$; this can be seen either from the definition by using the fact that $\mu_{\rho,D}$ is the law of $D_G(o,o)$, or alternatively by truncation of D and Fubini's theorem.

3. Tree entropy

We now give two new representations of tree entropy and two consequences. The first representation is as the logarithm of a Fuglede–Kadison determinant.

Theorem 3.1. *If ρ is a unimodular probability measure on rooted weighted connected infinite graphs with*

$$\int \log D_G(o,o) \, d\rho(G,o) \in [-\infty,\infty), \tag{3.1}$$

then

$$\mathbf{h}(\rho) = \log \mathrm{Det}_\rho \Delta \in [-\infty,\infty). \tag{3.2}$$

Proof. The hypothesis is equivalent to $D \in \mathsf{DetAlg}$. Since $I - P \in \mathsf{Alg} \subseteq \mathsf{DetAlg}$, it follows that $\Delta = D(I - P) \in \mathsf{DetAlg}$ with

$$\operatorname{Det} \Delta = \operatorname{Det} D \cdot \operatorname{Det}(I - P) \tag{3.3}$$

by Proposition 2.5 of Haagerup and Schultz [8] (which extends the fundamental theorem of Fuglede and Kadison [7] to unbounded operators, as well as to non-invertible operators).

Since $\|P\| \leqslant 1$, we have for $0 < c < 1$ that $\log|I - cP| \leqslant (\log 2)I$. Also, $|I - cP| \to |I - P|$ in the strong operator topology as $c \uparrow 1$, whence $\log|I - cP| \to \log|I - P|$ in the measure topology (for its definition, see Fack and Kosaki [6, §1.5]). Thus,

$$\operatorname{Det}(I - P) = \lim_{c \uparrow 1} \operatorname{Det}(I - cP)$$

by the Monotone Convergence Theorem; see, *e.g.*, Fack and Kosaki [6, Theorem 3.5(ii)]. On the other hand, for $0 < c < 1$,

$$\log \operatorname{Det}(I - cP) = \Re \operatorname{Tr} \log(I - cP)$$

by Theorem 1 (2^o) of Fuglede and Kadison [7] (or Theorem I.6.10 of Dixmier [4]) and

$$\log(I - cP) = -\sum_{k \geqslant 1} c^k P^k / k$$

(in the norm topology). Therefore,

$$\log \operatorname{Det}(I - cP) = -\sum_{k \geqslant 1} \Re \operatorname{Tr}_\rho c^k P^k / k = -\sum_{k \geqslant 1} \operatorname{Tr}_\rho c^k P^k / k,$$

whose limit as $c \uparrow 1$ is

$$-\sum_{k \geqslant 1} \operatorname{Tr}_\rho P^k / k = \int -\sum_{k \geqslant 1} \frac{1}{k} p_k(o; G)\, d\rho(G, o) \tag{3.4}$$

by the Monotone Convergence Theorem. Comparing (2.2) with equations (3.3), (2.6), and (3.4), we deduce the equality in (3.2). □

Remark. The version of this theorem given in Lyons [11] was incorrect even in the case of unweighted graphs, except when the degrees were bounded. For example, in the notation used there, whenever the degrees are unbounded, one gets $\Delta_{G_M}(o, o) = 0$ with positive probability, which means that $\operatorname{Det}_\rho(\Delta_{G_M}) = 0$. However, unbounded-degree graphs are quite natural, arising, for example, as limits of random finite graphs. In addition to that mistake, stronger hypotheses were assumed, which we now see to be superfluous, and the conclusion was less appealing, being expressed as a double limit.

An example of a unimodular probability measure ρ satisfying not only (3.1), but even the stronger

$$\int |\log D_G(o, o)|\, d\rho(G, o) < \infty, \tag{3.5}$$

yet with $\mathbf{h}(\rho) = -\infty$ is the following. We work on the nearest-neighbour graph of the integers, $\mathbb{Z}$, rooted at 0. Define the weight to be 1 of every edge of the form $(2n, 2n + 1)$

for $n \in \mathbb{Z}$. Let X be an integer-valued random variable such that $\mathbf{P}[X \geqslant m] = 1/\sqrt{m}$ for $m \geqslant 1$. Let X_n be i.i.d. copies of X for $n \in \mathbb{Z}$ and let the weight be e^{-X_n} of the edge $(2n-1, 2n)$. Define ρ to be the resulting measure on rooted weighted graphs. (In fact, ρ is defined on rooted isomorphism classes of networks, so that one does not notice the difference between 'even' and 'odd' edges.) By Theorem 3.2 of Aldous and Lyons [1], ρ is unimodular. Since

$$\int |\log D_{\mathbb{Z}}(0,0)|\, d\rho = \mathbf{E}[\log(1 + e^{-X})],$$

(3.5) is clearly satisfied. On the other hand, it is easy to see that there are constants $c_1, c_2 > 0$ such that $p_{2k}(0;\mathbb{Z}) \geqslant c_1$ for $1 \leqslant k \leqslant \exp\min\{X_0, X_1\}$, whence

$$\sum_{k\geqslant 1} \frac{1}{k} p_k(0;\mathbb{Z}) \geqslant c_2 \min\{X_0, X_1\}.$$

Therefore

$$\int \sum_{k\geqslant 1} \frac{1}{k} p_k(0;\mathbb{Z})\, d\rho \geqslant c_2 \mathbf{E}[\min\{X_0, X_1\}] = c_2 \sum_{m\geqslant 1} \mathbf{P}[X_0 \geqslant m]^2 = \infty.$$

A small, but significant, extension of Theorem 4.2 of Lyons [11] is as follows. Let (G_1, o_1, w_1) and (G_2, o_2, w_2) be two rooted weighted graphs. Say that (G_1, o_1, w_1) *dominates* (G_2, o_2, w_2) if there is a graph isomorphism ϕ from G_2 to a subgraph of G_1 that takes o_2 to o_1, and such that, for all $e \in \mathsf{E}(G_2)$, we have $w_2(e) \leqslant w_1(\phi(e))$. This notion is a partial order on rooted weighted graphs and we use the usual notion of stochastic domination that corresponds to it. That is, if ρ_1 and ρ_2 are two probability measures on rooted weighted graphs, say that ρ_1 *stochastically dominates* ρ_2 if there exists a probability measure ν on pairs $((G_1, o_1, w_1), (G_2, o_2, w_2))$ such that the ν-law of (G_i, o_i, w_i) is ρ_i for $i = 1, 2$ and (G_1, o_1, w_1) dominates (G_2, o_2, w_2) ν-a.s.

Theorem 3.2. *If $\rho_1 \neq \rho_2$ are unimodular probability measures on rooted weighted connected infinite graphs that both satisfy* (3.1) *and ρ_1 stochastically dominates ρ_2, then $\mathbf{h}(\rho_1) > \mathbf{h}(\rho_2)$.*

The proof of the corresponding result, Theorem 4.2, in [11] was in fact not complete. We give a more direct proof here based on a different approach. In addition, Theorem 4.2 of [11] assumed (3.5) in place of our hypothesis (3.1) and also assumed a further bound.

The significance of our extension is that Theorem 4.2 of [11] required the two probability measures ρ_i to be coupled on the *same* graphs, differing only in their edge weights. This makes it impossible to handle naturally occurring stochastic domination situations, such as those occurring for limits of random finite graphs. Thus, the present result can answer a question of [11] concerning the giant component in the Erdős–Rényi model of random graphs, provided one can show stochastic domination of Poisson–Galton–Watson measures conditioned on survival. Indeed, this domination was proved by Lyons, Peled and Schramm [12].

To prove Theorem 3.2, we rely on an entirely new representation of tree entropy. Given a network G, one of its vertices x, and a positive number s, let $R(G, x, s)$ be the effective resistance between x and infinity in the network G^s formed from G by adding an edge of conductance s between every vertex and infinity, where ∞ is also a vertex of G^s. To be more precise, consider an exhaustion of G by finite subnetworks G_n. Let H_n be the network formed from G by identifying all vertices outside G_n to a single vertex z_n and then adding an edge of conductance s between each vertex of G_n and z_n. For large enough n, we have that $x \in \mathsf{V}(G_n)$, so that we may define the effective resistance $\mathcal{R}(x \leftrightarrow z_n; H_n)$ between x and z_n in H_n. These effective resistances have a limit, which we are calling $R(G, x, s)$.

Our second representation of tree entropy is in terms of electrical resistance.

Theorem 3.3. *If ρ is a unimodular probability measure on rooted weighted infinite graphs that satisfies* (3.1)*, then*

$$\mathbf{h}(\rho) = \int_0^\infty \left(\frac{s}{1+s^2} - \int R(G, o, s)\, d\rho(G, o) \right) ds. \tag{3.6}$$

Remark. Although one might ask, from comparing (2.2) and (3.6), whether, for every network (G, o), we have

$$\log D_G(o, o) - \sum_{k \geqslant 1} \frac{1}{k} p_k(o; G) \stackrel{?}{=} \int_0^\infty \left(\frac{s}{1+s^2} - R(G, o, s) \right) ds, \tag{3.7}$$

this is not true. Thus, Theorem 3.3 depends crucially on the assumption that ρ is unimodular. One can show, however, that (3.7) does hold for every regular graph G. If $d := D_G(o, o)$, then one can show that $R(G, o, s)$ equals the expected number of visits to o divided by $d + s$, which equals $\sum_{k \geqslant 0} p_k(o; G) d^k / (d + s)^{k+1}$. This gives that

$$\int_0^\infty \big(R(G, o, s) - 1/(d + s) \big) ds = \sum_{k \geqslant 1} p_k(o; G)/k.$$

Combining this with (3.8) below gives the result.

Remark. One might also ask whether tree entropy increases under stochastic domination regardless of the unimodularity of ρ. This is not the case, however. For example, consider ρ_1 to be the measure concentrated on the fixed graph where the root has degree 1, its neighbour has degree 2, and the neighbour of the root's neighbour has attached a tree of very large degree. Let ρ_2 be the measure concentrated on the same graph to which has been adjoined a loop at the root. Then a straightforward calculation shows that $\mathbf{h}(\rho_1) > \mathbf{h}(\rho_2)$, even though $\rho_2 \succcurlyeq \rho_1$.

Proof of Theorem 3.3. For $\lambda > 0$, a well-known identity states that

$$\log \lambda = \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{\lambda + s} \right) ds. \tag{3.8}$$

Also, we have the lesser-known identity

$$\frac{1}{2} \log(1 + \lambda^2) = \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{\lambda + s} \right)^+ ds. \tag{3.9}$$

Since $\Delta \geqslant 0$, the fact that $\mathbf{h}(\rho) < \infty$ (by Theorem 3.1) implies that

$$\int_0^\infty \log(1+\lambda^2)\, d\mu_{\rho,\Delta}(\lambda) < \infty \tag{3.10}$$

by (3.2) and (2.5).

For $s > 0$, note that $(\Delta + sI)^{-1} \in \mathsf{Alg}$ since $\Delta \geqslant 0$ and define $v_s := \big(\Delta_G + sI\big)^{-1}\mathbf{1}_{\{o\}}$ on $\mathsf{V}(G)$. We claim that $\big(v_s, \mathbf{1}_{\{o\}}\big) = v_s(o) = R(G, o, s)$. Indeed, the invertibility of $\Delta + sI$ tells us that v_s is the unique function on $\mathsf{V}(G)$ that satisfies $(\Delta + sI)v_s = \mathbf{1}_{\{o\}}$. Since one such function is the limit of the voltage functions $v_{s,n}$ corresponding to the unit current flows on H_n from o to z_n, it follows that $v_s = \lim_{n\to\infty} v_{s,n}$. Since $v_{s,n}(o) = \mathcal{R}(o \leftrightarrow z_n; H_n)$, we obtain the claim. Hence

$$\mathrm{Tr}_\rho\big((\Delta + sI)^{-1}\big) = \int R(G, o, s)\, d\rho(G, o). \tag{3.11}$$

On the other hand,

$$(\Delta + sI)^{-1} = \int_0^\infty (\lambda + s)^{-1} dE_\Delta(\lambda),$$

so that

$$\mathrm{Tr}_\rho\big((\Delta + sI)^{-1}\big) = \int_0^\infty (\lambda + s)^{-1} d\mu_{\rho,\Delta}(\lambda). \tag{3.12}$$

Therefore, we have

$$\begin{aligned}
\mathbf{h}(\rho) &= \int_0^\infty \log\lambda\, d\mu_{\rho,\Delta}(\lambda) = \int_0^\infty \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{\lambda+s}\right) ds\, d\mu_{\rho,\Delta}(\lambda) \\
&= \int_0^\infty \int_0^\infty \left(\frac{s}{1+s^2} - \frac{1}{\lambda+s}\right) d\mu_{\rho,\Delta}(\lambda)\, ds \\
&= \int_0^\infty \left(\frac{s}{1+s^2} - \mathrm{Tr}_\rho(\Delta + sI)^{-1}\right) ds \\
&= \int_0^\infty \left(\frac{s}{1+s^2} - \int R(G, o, s)\, d\rho(G, o)\right) ds;
\end{aligned}$$

we have used (3.2) and (2.5) in the first equality; (3.8) in the second; (3.9), (3.10), and the Fubini–Tonelli theorem in the third; (2.4) and (3.12) in the fourth; and (3.11) in the fifth. □

Theorem 3.2 follows immediately by Rayleigh's monotonicity principle. Indeed, that principle gives us that when (G_1, w_1, o_1) dominates (G_2, w_2, o_2), then

$$R(G_1, o_1, s) \leqslant R(G_2, o_2, s)$$

for all $s > 0$, where the edge conductances are understood but not notated in this inequality.

Theorem 3.4. *If ρ is a unimodular probability measure on rooted infinite (unweighted) graphs that satisfies* (3.1), *then $\mathbf{h}(\rho) \geqslant 0$, with equality if and only if $\int D_G(o, o)\, d\rho(G, o) = 2$ if and only if ρ-a.s. G is a locally finite tree with 1 or 2 ends.*

Proof. By Proposition 7.1 of Aldous and Lyons [1], the root in the wired uniform spanning forest of ρ, denoted $\mathsf{WUSF}(\rho)$, has expected degree 2, whence, by Theorem 6.2 of [1], the unimodular probability measure $\mathsf{WUSF}(\rho)$ is concentrated on trees with at most 2 ends. This implies that $\mathsf{WUSF}(\rho)$ is amenable by Corollary 8.9 of [1], whence is the random weak limit of finite trees. Of course, finite trees have average degree less than 2. By Theorem 3.2 of Lyons [11], this means that $\mathbf{h}(\mathsf{WUSF}(\rho)) = 0$. Since ρ clearly stochastically dominates $\mathsf{WUSF}(\rho)$, it follows by Theorem 3.2 that $\mathbf{h}(\rho) \geqslant 0$. The equality condition also follows from Theorem 3.2 and the above argument, combined with Theorem 6.2 of [1] again. □

Remark. Proposition 4.3 and Theorem 4.4 in Lyons [11] stated the same results as Theorem 3.4, though with an hypothesis far stronger than (3.1). However, the proofs relied on a result in a preliminary version of Aldous and Lyons [1], whose proof was incorrect.

In the special case that ρ is concentrated on a fixed Cayley graph G, then Theorem 3.4 says that $\operatorname{Det} \Delta_G \geqslant 1$. This establishes a special case of Lück's Determinant Conjecture, which says that for every group Γ and for every positive self-adjoint finite matrix over the group ring $\mathbb{Z}\Gamma$, its Fuglede–Kadison determinant is at least 1; see, *e.g.*, Elek and Szabó [5].

A consequence of Theorem 3.2 is that the set of measures of fixed tree entropy and satisfying (3.1) form an anti-chain (no two are comparable in the stochastic domination order). In the special case of tree entropy 0, if we combine this with Theorem 3.4, then we obtain that the measures on trees with at most 2 ends and satisfying (3.1) form an anti-chain.

Corollary 3.5. *If ρ_1 and ρ_2 are unimodular probability measures on rooted unweighted infinite trees with at most two ends, both measures satisfy* (3.1)*, and ρ_1 stochastically dominates ρ_2, then $\rho_1 = \rho_2$.*

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