Random Trees and Surfaces

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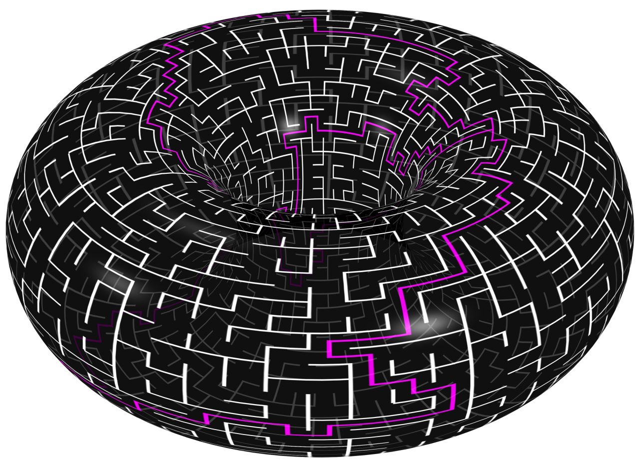
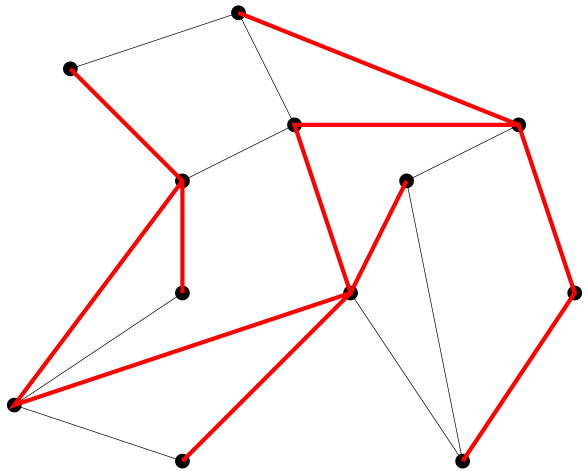
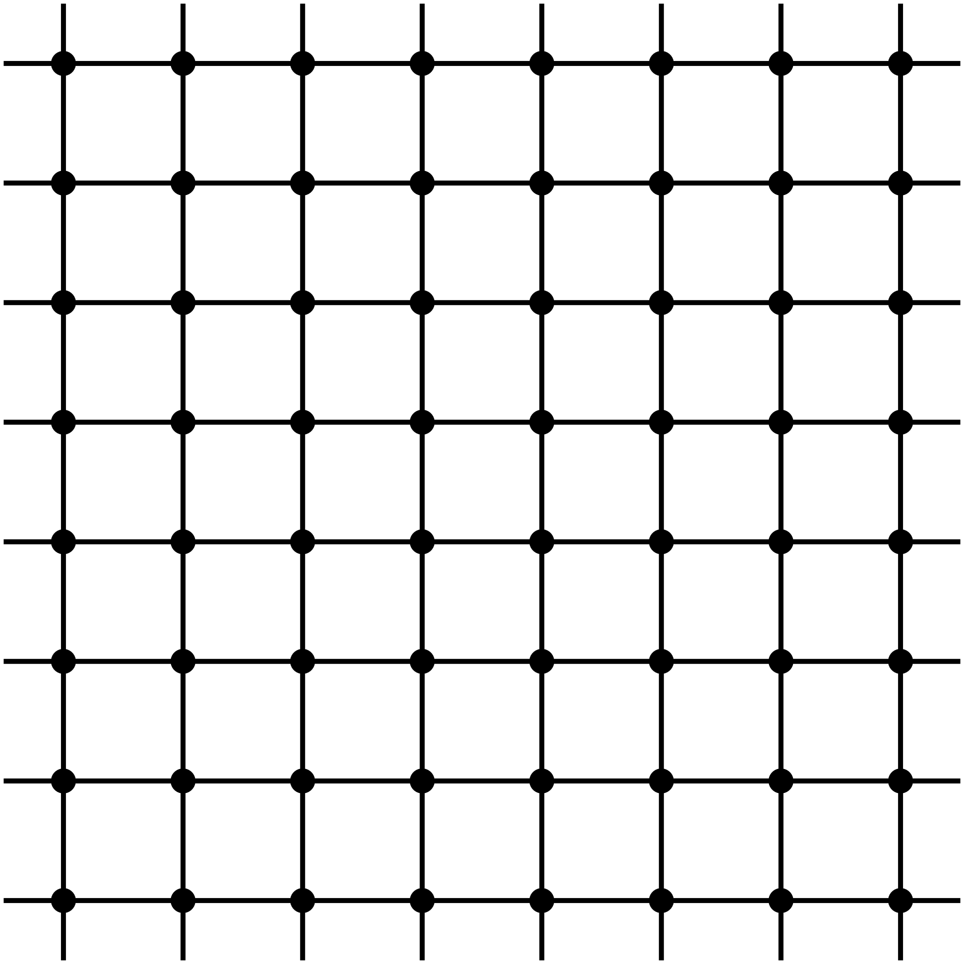


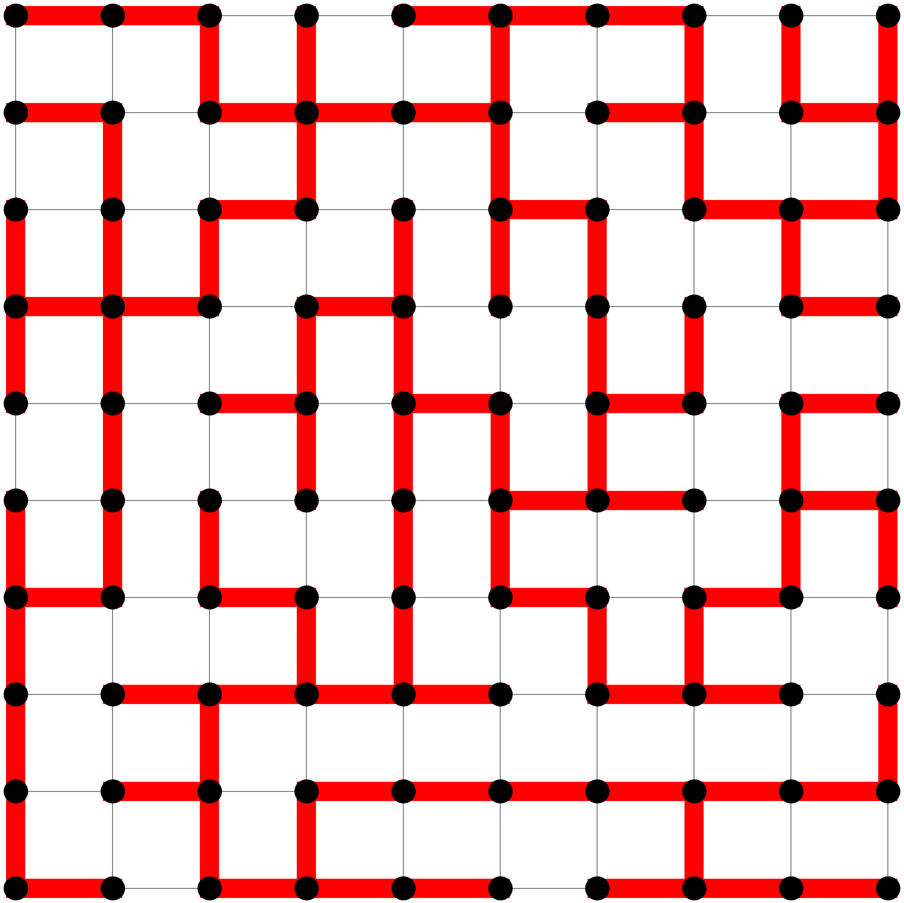
Uniform Spanning Trees



Algorithm of Aldous (1990) and Broder (1989): if you start a simple random walk at *any* vertex of a graph G and draw every edge it traverses except when it would complete a cycle (i.e., except when it arrives at a previously-visited vertex), then when no more edges can be added without creating a cycle, what will be drawn is a uniformly chosen spanning tree of G.

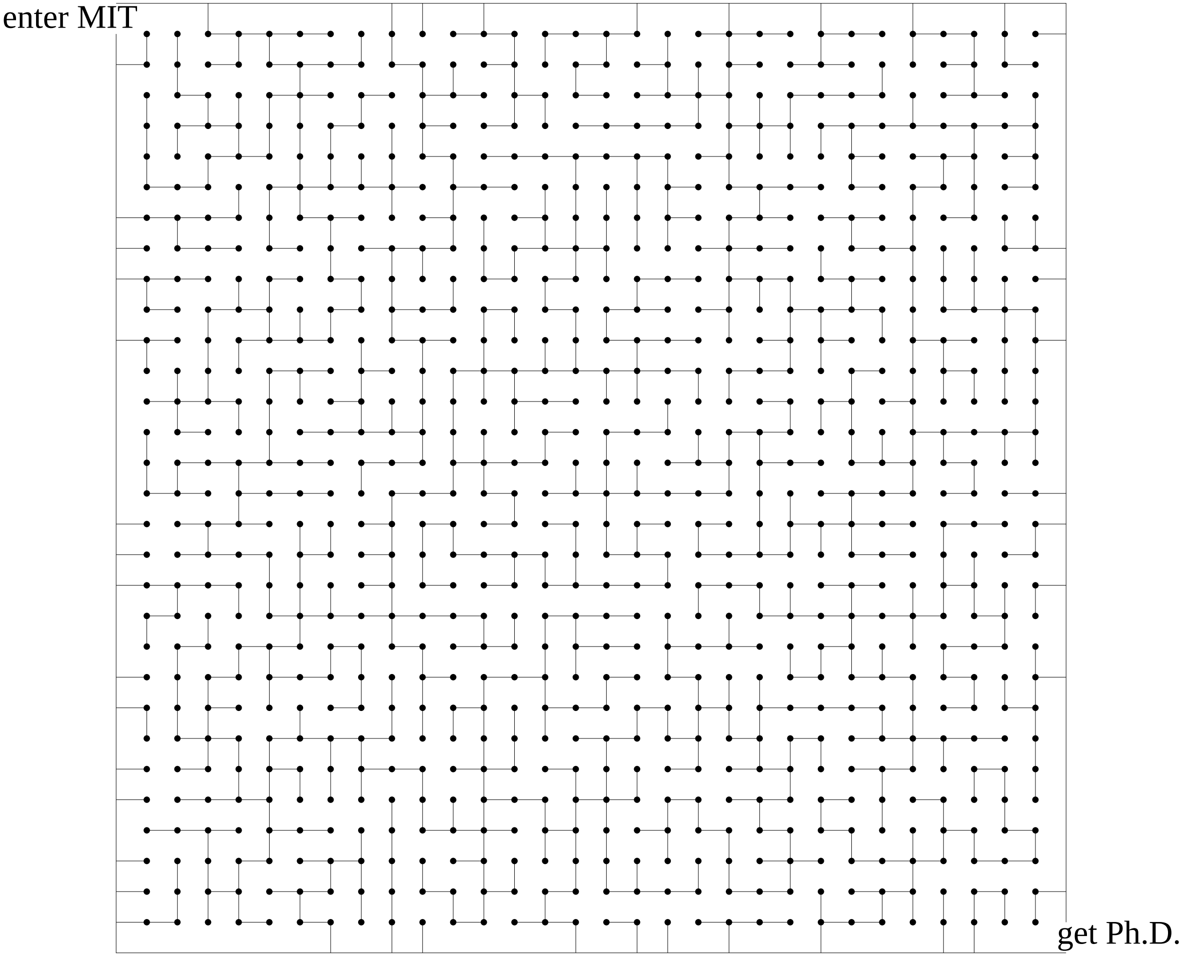
movie



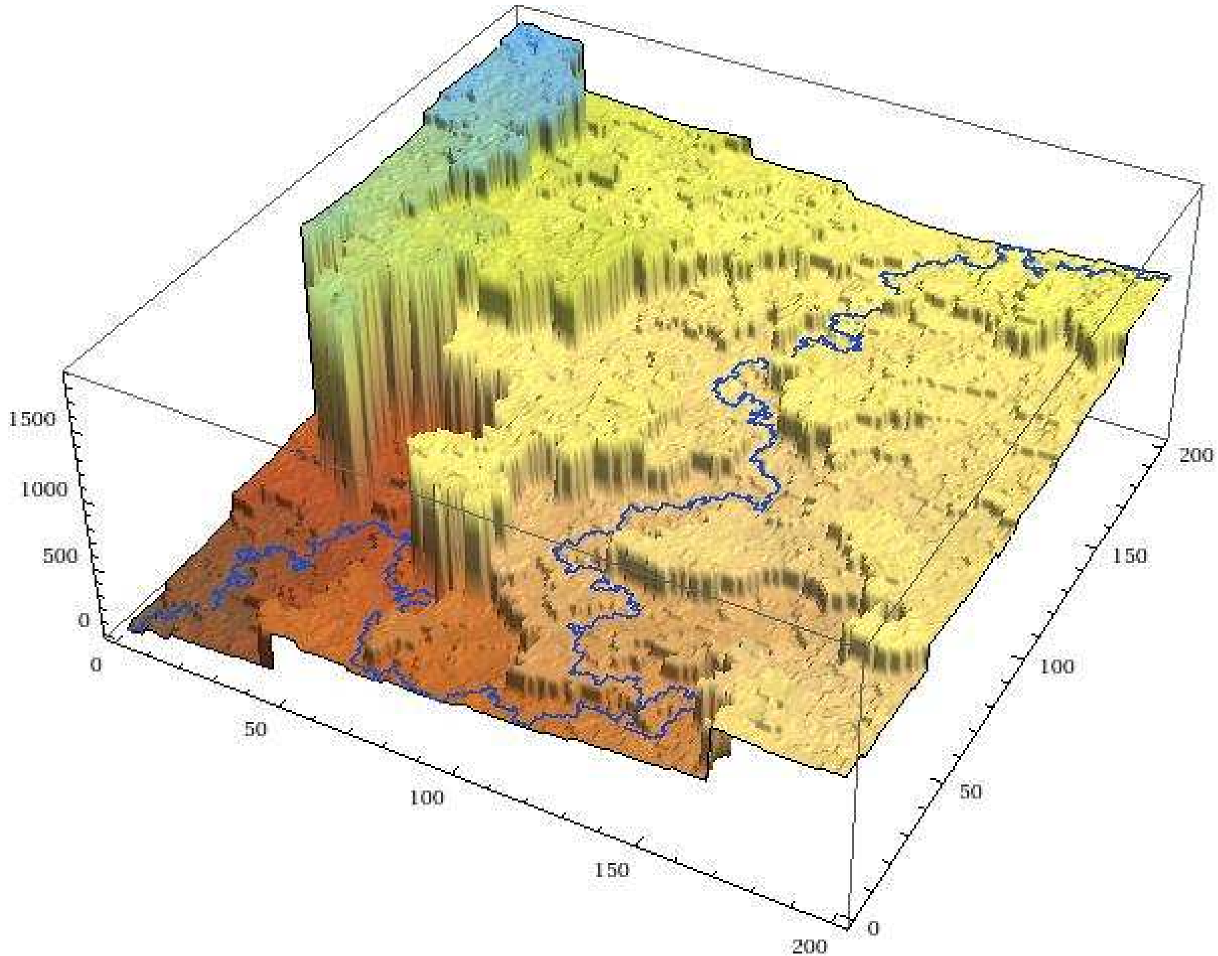


$\mathbb{Z}^3$

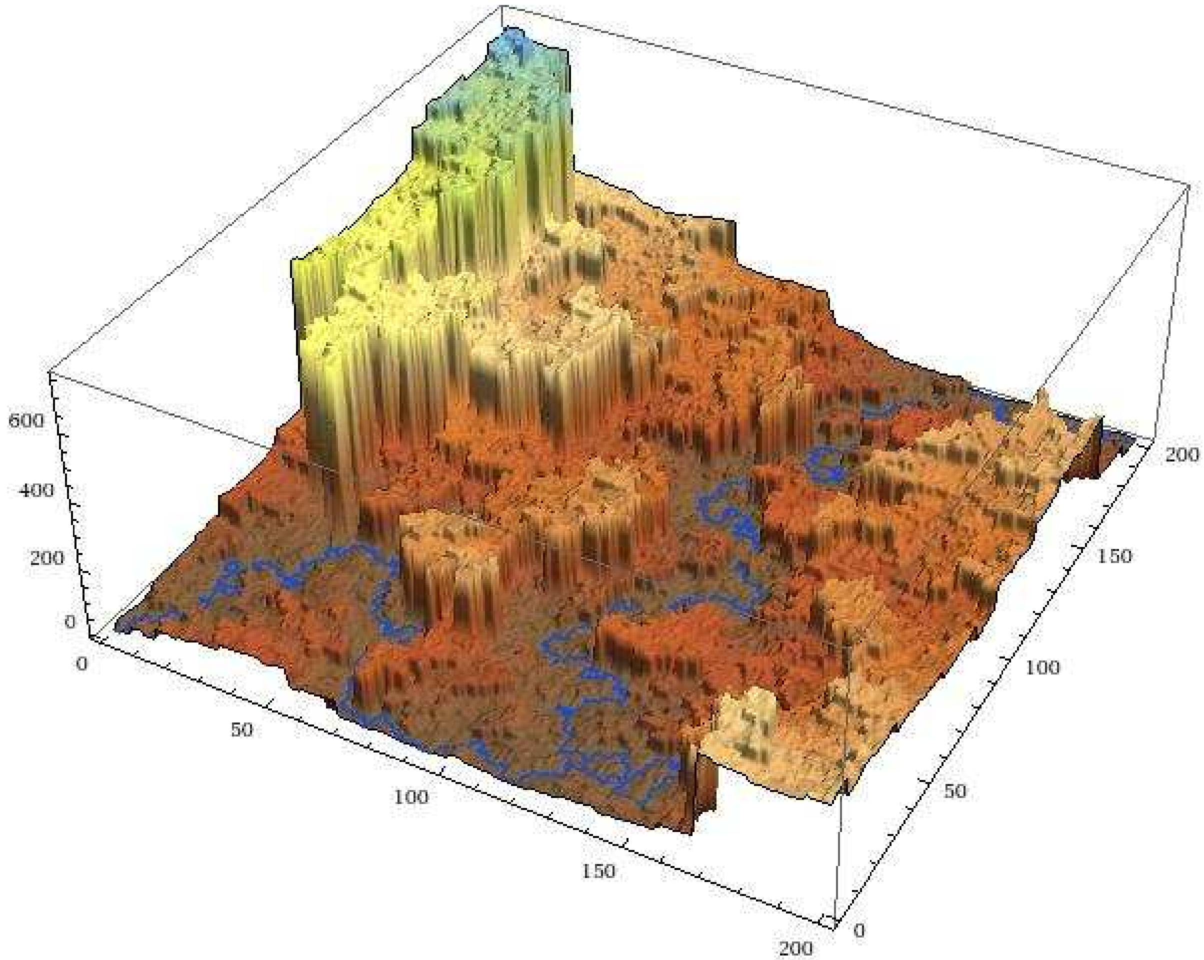
Maze and Duality



Distance to Corner



Distance to Path



Conjectures

Known: The maximum height in an $n \times n$ square is about $n^{5/4}$ by a result of Barlow and Masson (2010), which extends earlier work of Kenyon (2000) and predictions of Guttmann and Bursill (1990), Duplantier (1992) and Majumdar (1992).

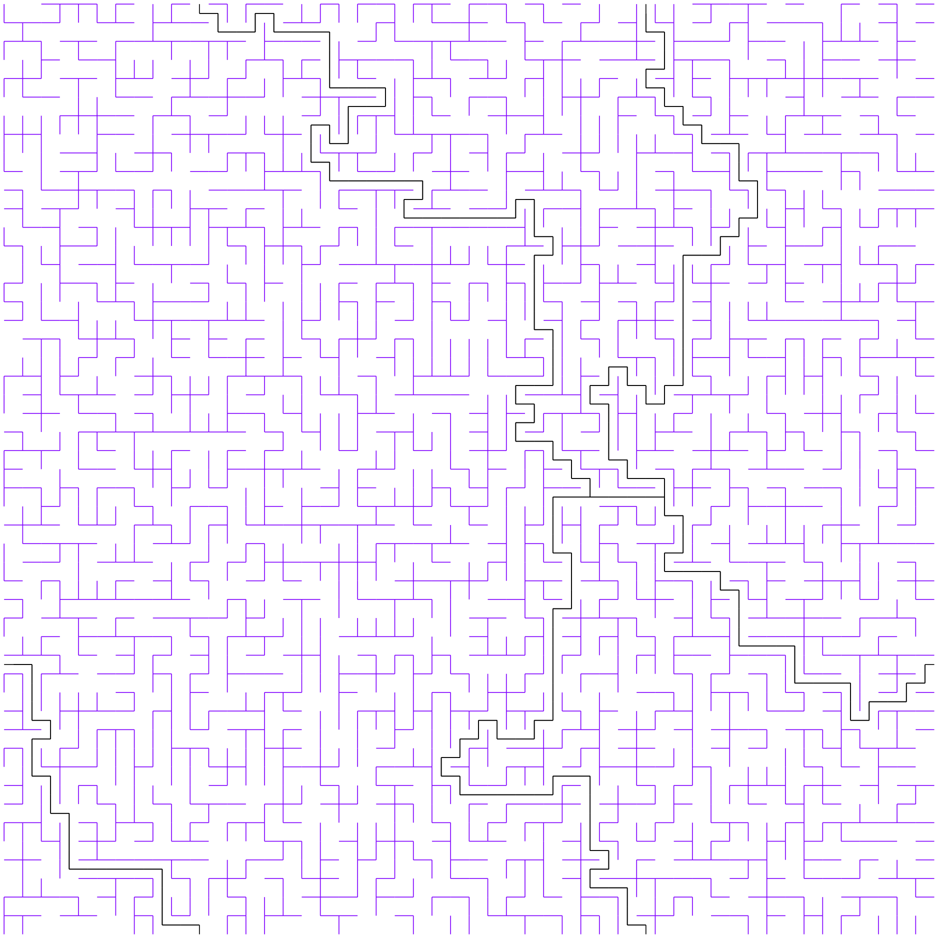
The distance in the UST between lattice neighbors is of order k (i.e., up to a constant multiple) with probability about $k^{-3/5}$ by results of Masson (2009), following a prediction of Manna, Dhar, and Majumdar (1992).

Believed: Call an edge (x, y) a **cliff** if the height difference between x and y is large, with the height of the cliff equal to that difference.

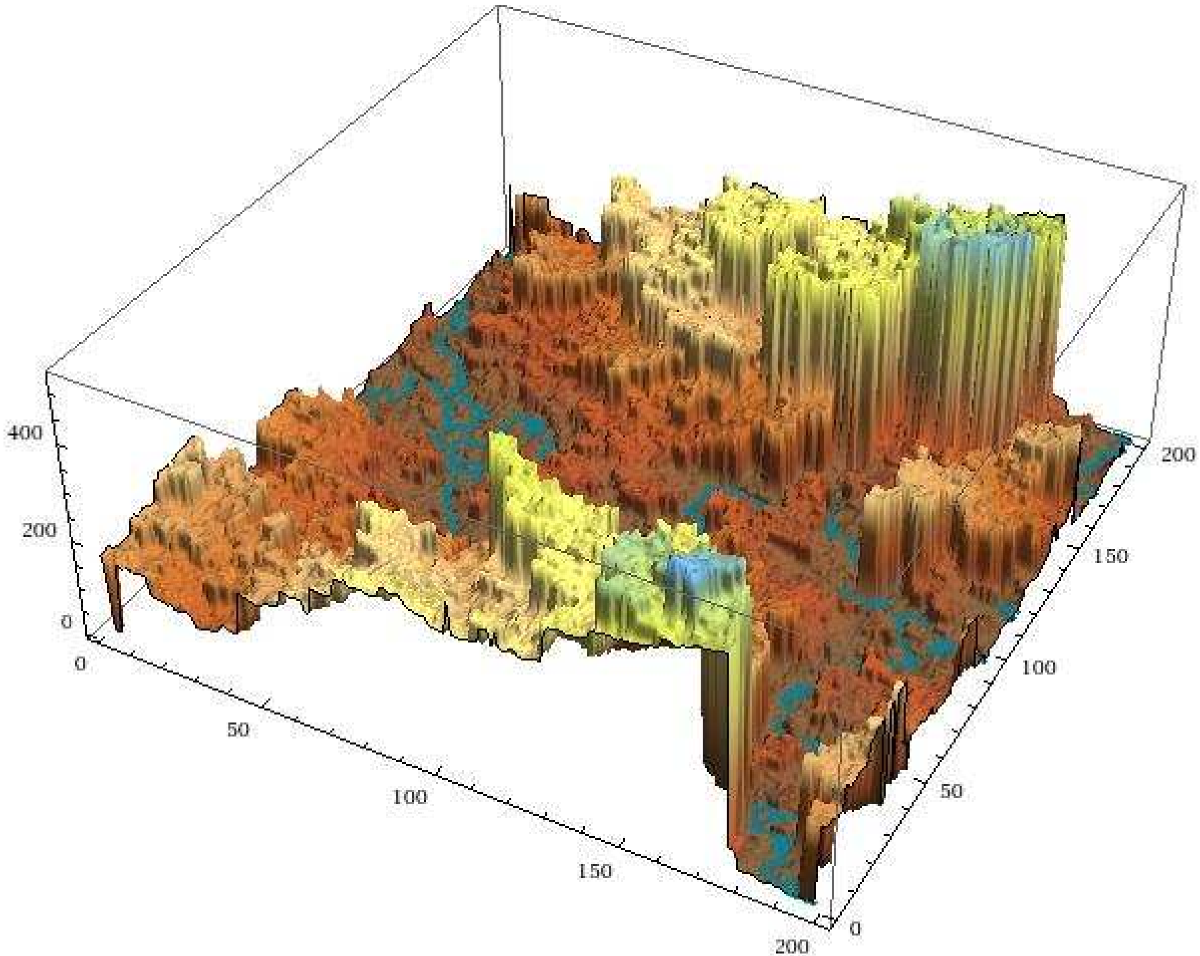
The highest cliffs should have height around $n^{5/4}$ and they should have length around $n^{5/4}$. More generally, it seems to me that cliffs of height roughly n^a occur for about $n^{2-3a/5}$ edges. Here, $0 \le a \le 5/4$.

In addition, based on another prediction of Manna, Dhar, and Majumdar (1992), I think that the length of a connected component of cliffs (here, connected means by using the dual edges to the cliffs) of height roughly n^a should be at least about n^a.

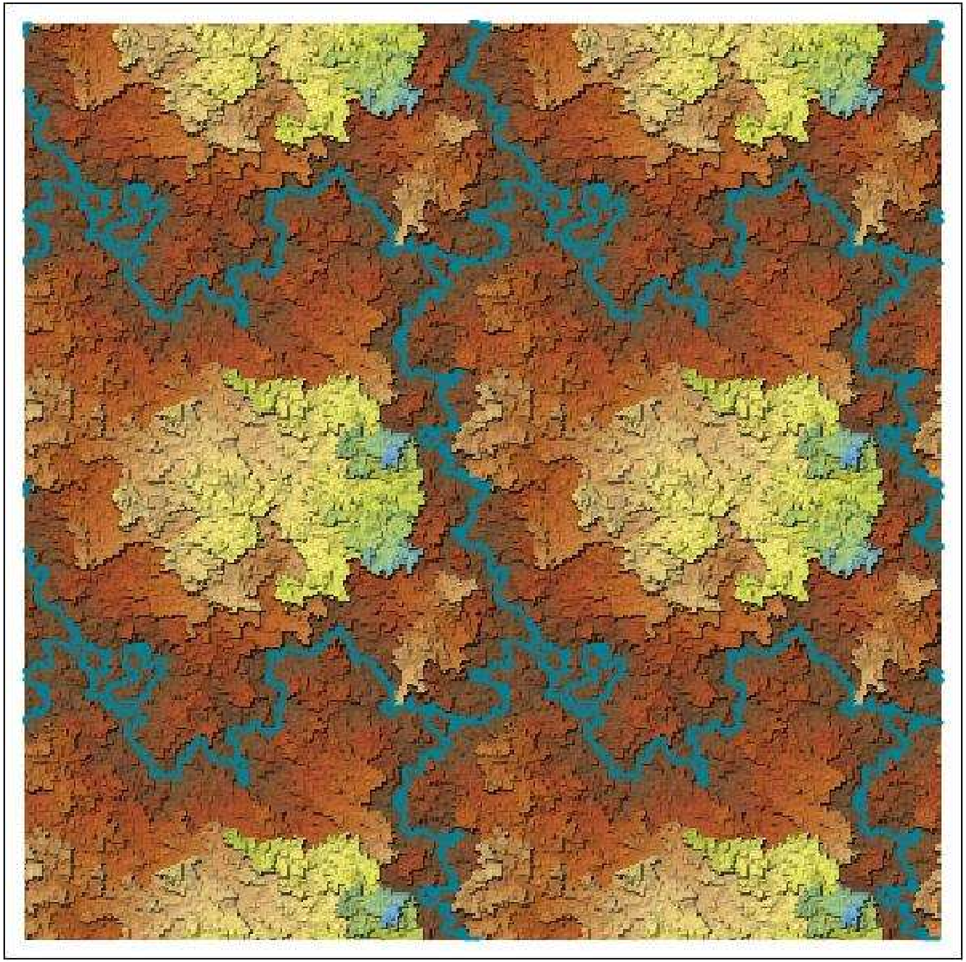
The torus:



Distance to Cycles



Distance to Cycles



duality in $\mathbb{Z}^3$

Determinantal Measures

If E is finite and $H \subseteq \ell^2(E)$ is a subspace, it defines the determinantal measure

$$\forall T \subseteq E \text{ with } |T| = \dim H \qquad \mathbf{P}^H(T) := \det[P_H]_{T,T} \, ,$$

where the subscript T, T indicates the submatrix whose rows and columns belong to T. This representation has a useful extension, namely,

$$\forall D \subseteq E \qquad \mathbf{P}^H[D \subseteq T] = \det[P_H]_{D,D} \, .$$

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite graph. Choose one orientation for each edge $e \in \mathsf{E}$. Let $\bigstar = B^1(G)$ denote the subspace in $\ell^2(\mathsf{E})$ spanned by the stars (coboundaries) and let $\diamondsuit = Z_1(G)$ denote the subspace spanned by the cycles. Then $\ell^2(\mathsf{E}) = \bigstar \oplus \diamondsuit$.

For a finite graph, Burton and Pemantle (1993) showed that the uniform spanning tree is the determinantal measure corresponding to orthogonal projection on $\bigstar = \diamondsuit^\perp$. (Precursors due to Kirchhoff (1847) and Brooks, Smith, Stone, and Tutte (1940).)

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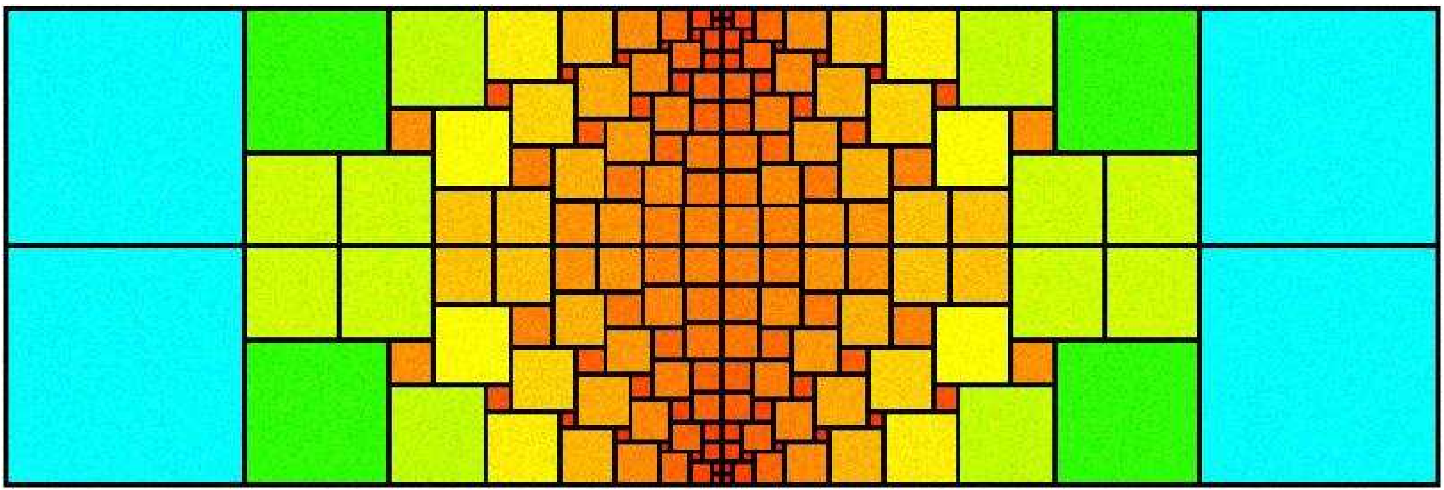
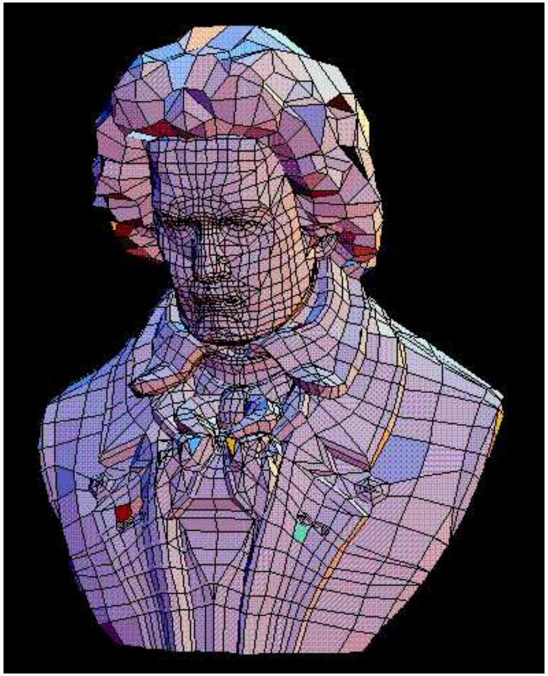
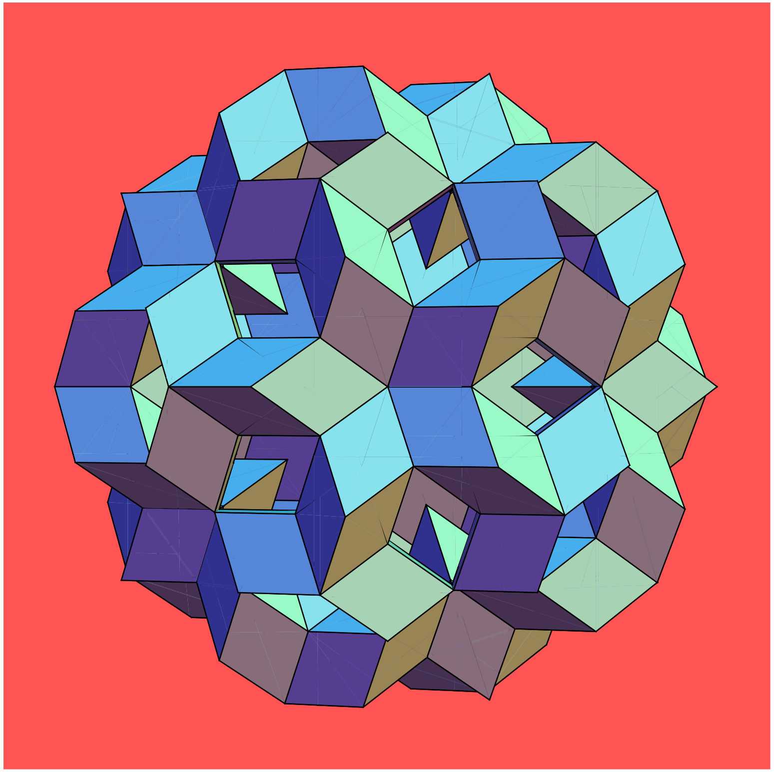


CW-Complexes

How do we extend the foregoing to higher dimensions? The higher-dimensional analogue of a graph is a CW-complex. A CW-complex is formed by sticking together cells:



(T. Robb via Mathematica)



Sándor Kabai and Lajos Szilassi via Mathematica

From Cayley to Kalai

What is the analogue of a spanning tree?

Cayley (1889) showed that the number of spanning trees in a complete graph on n vertices is n^{n-2}. Cayley's theorem was extended to higher dimensions by Kalai (1983), who showed that a certain enumeration of k-dimensional subcomplexes in a simplex on n vertices resulted in

$$n^{\binom{n-2}{k}}.$$

Kalai did not look at it this way, but we take the defining property of a spanning tree to be its property as a base of the graphical matroid, i.e., maximal without cycles.

Chain Groups and Bases for Finite CW-Complexes

Consider each cell of a CW-complex X to be oriented (except the 0-cells). Write $\Xi_k X$ for the set of k-cells of X. Identify cells with the corresponding basis elements of the chain and cochain groups, so that $\Xi_k X$ forms a basis of $C_k(X;\mathbb{R})$ and $C^k(X;\mathbb{R})$. The boundary map

$$\partial_k : C_k(X;\mathbb{R}) \to C_{k-1}(X;\mathbb{R})$$

has kernel $Z_k(X;\mathbb{R})$ and image $B_{k-1}(X;\mathbb{R})$, while the coboundary map

$$\delta_k = \partial_{k+1}^* : C^k(X;\mathbb{R}) \to C^{k+1}(X;\mathbb{R})$$

has kernel $Z^k(X;\mathbb{R})$ and image $B^{k+1}(X;\mathbb{R})$.

Given a finite CW-complex X and a subset $T \subseteq \Xi_k X$ of its k-cells, write X_T for the subcomplex

$$X_T := T \cup \bigcup_{j=0}^{k-1} \Xi_j X$$

We call T a **k-base** if it is maximal with $Z_k(X_T) = 0$.

Let X be a finite CW-complex. The determinantal probability measure $\mathbf{P}_k$ on the set of k-bases defined by orthogonal projection of $C_k(X)$ onto the space of coboundaries $B^k(X) = Z_k(X)^\perp$ is called the **kth lower matroidal measure on** X. If X is connected, then $\mathbf{P}_1$ is the law of the uniform spanning tree of the 1-skeleton of X. Let $t_j(T)$ denote the order of the torsion subgroup of $H_j(X_T;\mathbb{Z}) := Z_j(X_T;\mathbb{Z})/B_j(X_T;\mathbb{Z})$.

Proposition. *Let X be a finite CW-complex. For each k, there exists a_k such that for all k-bases T of X,*

$$\mathbf{P}_k(T) = a_k t_{k-1}(T)^2 \,.$$

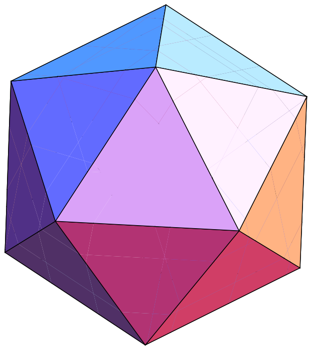
The theorem of Kalai (1983) is that when X is an $(n-1)$-dimensional simplex and $1 \le k \le n-1$,

$$\sum_T t_{k-1}(T)^2 = n^{\binom{n-2}{k}} \,,$$

where the sum is over all k-bases of X.

Example

Simplex on 6 vertices contains the projective plane, whose first homology group is $\mathbb{Z}_2$:



The projective plane can be embedded in $\mathbb{R}^4$.

Upper Matroidal Measures

Another natural probability measure $\mathbf{P}^k$ on subsets of $\Xi_k X$ is the determinantal probability measure corresponding to the subspace of k-cocycles, $Z^k(X) = B_k(X)^\perp$. We call this measure the **kth upper matroidal measure on X**. Since $B^k(X) \subseteq Z^k(X)$, it follows that the upper measure $\mathbf{P}^k$ stochastically dominates the lower measure $\mathbf{P}_k$, with equality iff $H^k(X;\mathbb{R}) = 0$. As usual, let $b_k(X)$ denote the kth Betti number of X, the dimension of $H_k(X;\mathbb{R})$. One can add $b_k(X)$ k-cells to a sample from $\mathbf{P}_k$ to get a sample from $\mathbf{P}^k$.

Topological invariants for X reside in the *difference* between the measures.

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