PREPRINT

Fritz

Best regards,

A CHARACTERIZATION OF MEASURES WHOSE FOURIER-STIELTJES TRANSFORMS VANISH AT INFINITY

Russell LYONS

UNIVERSITÉ DE PARIS-SUD

ÉQUIPE DE RECHERCHE ASSOCIÉE AU CNRS (296)
ANALYSE HARMONIQUE
MATHÉMATIQUE (Bât. 425)
91405 ORSAY CEDEX

A CHARACTERIZATION OF MEASURES WHOSE FOURIER-STIELTJES TRANSFORMS VANISH AT INFINITY

by

Russell Lyons

ABSTRACT

Let R denote the class of complex Borel measures on the circle $\mathbb{T}$ whose Fourier-Stieltjes coefficients $\hat{\mu}(n)$ tend to 0 as $|n| \to \infty$. Ju. A. Šreĭder has defined a class of subsets of $\mathbb{T}$, called W-sets, using the notion of asymptotic distribution. Our central result is that a measure μ lies in R if and only if $\mu E = 0$ for all W-sets E. This establishes a claim of Šreĭder. Rajchman conjectured that $\mu \in R$ if and only if $\mu E = 0$ for all H-sets E and Kahane and Salem made a similar conjecture about non-normal sets. Both of these conjectures are shown to be false. Similar questions are investigated for Helson sets and weak Dirichlet sets.

The characterization of R stated above extends to locally compact abelian groups. A consequence is that every measure μ may be split uniquely as $\mu = \mu_R + \mu_J$, where $\mu_R \in R$, μ_J belongs to a class J, and $\mu_R \perp \mu_J$. Any Riesz product on a compact abelian group lies purely in R or purely in J. Infinite convolutions of discrete probability measures also have this purity property, which extends the Jessen-Wintner purity law.

Dedicated to those people and things who influenced my career choices: my father, the Lexington, Massachusetts public schools, math team, the Mathematics Olympiad Program, Hampshire College Summer Studies in Mathematics, Neil Immerman, summing the reciprocal square integers by Fourier series, Royden's *Real Analysis*, a graduate student at the University of Chicago who told me about interpolation theorems, Allen Shields, and Katznelson's *An Introduction to Harmonic Analysis*.

ACKNOWLEDGEMENTS

I am very happy to be able to thank Prof. Hugh Montgomery for being a wonderful thesis advisor; there could hardly be better. He generously gave much time and deep thought to my questions. His ideas have been used to great profit throughout this work. Further suggestions have improved the presentation of results. It was a great pleasure to work so closely with such a mathematician. I have never enjoyed doing mathematics so much.

I am also very happy to thank Prof. Allen Shields for being one of the best teachers I have had. Most of my knowledge of harmonic analysis was gained with his help and guidance. My interest in harmonic analysis was strengthened and my insight deepened from his teaching. Furthermore, it was Allen who posed to me the attractive question at the center of my thesis and pointed out Šreĭder's paper.

I am indebted to Dr. Ju. A. Šreĭder for conjecturing (in 1950) the correct solution to this question, and also for not providing a proof. I am grateful to Lu Ann Custer for excellent and cheerful typing. I am very fortunate to have had the support of an NSF Graduate Fellowship and a University of Michigan Graduate Fellowship, and partial support of NSF Grant MCS-82-01602.

PREFACE

With only slight differences, the first three chapters of this work comprised a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Mathematics) in The University of Michigan, 1983. The doctoral committee was co-chaired by Professors Hugh L. Montgomery and Allen L. Shields and also included Professors Andreas R. Blass, Frederick W. Gehring, and Jens C. Zorn.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1. The Question

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the unit circle. The Riemann-Lebesgue lemma states that the Fourier coefficients

$$\hat{f}(n) = \int_{\mathbb{T}} e^{-2\pi int} f(t)dt$$

of a Lebesgue-integrable function f tend to 0 as $|n| \to \infty$. On the other hand, Wiener's theorem (Katznelson [1, p. 42]) implies that if the Fourier-Stieltjes coefficients

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-2\pi int} d\mu(t)$$

of a complex Borel measure μ tend to 0 as $|n| \to \infty$, then μ is continuous. If we use the Radon-Nikodym Theorem, we may restate the Riemann-Lebesgue lemma as follows: if μ is absolutely continuous (with respect to Lebesgue measure), then $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. We restate these results once again as

THEOREM 1.1. If μ places no mass on the sets of Lebesgue measure zero, then $\hat{\mu}(n) \to 0$ as $|n| \to \infty$. Conversely, if $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, then μ places no mass on the countable sets.

Is there some class of sets C such that $\hat{\mu}(n) \to 0$ <u>if and only if</u> μ puts no mass on the sets in C ? Such a class should be intermediate between the countable sets and the sets of Lebesgue measure zero. The question of whether C exists is the central question of this work.

Menšov [1], in 1916, was the first to construct a singular measure μ whose Fourier-Stieltjes coefficients tend to 0. It follows that we cannot take C to be the class of Lebesgue-measure-zero sets. In 1918, F. Riesz [1] constructed his "Riesz products" and showed them to be continuous measures whose Fourier-Stieltjes coefficients do not necessarily tend to 0. Thus, C cannot be taken to be the countable sets.

Let us introduce some notation to facilitate further discussion.

NOTATION. The set of all (finite) complex Borel measures on $\mathbb{T}$ is denoted $M(\mathbb{T})$. The subset of $\mu \in M(\mathbb{T})$ for which $\hat{\mu}(n) \to 0$ as $|n| \to \infty$ is denoted $M_0(\mathbb{T})$ or R; such measures are called <u>Rajchman measures</u>. The class of all Borel sets E such that $\mu E = 0$ for all $\mu \in R$ will be denoted U_0; they are called U_0-sets (see also Section IV.2).

We shall usually prefer to denote $M_o(\mathbb{T})$ by R for brevity and also because sets not in U_o are known as M_o-sets. The notation "R" is due to Zygmund [1, II, p. 143].

It is clear that if any class C characterizing R exists, then C can be taken to be U_o. Moreover, we would only have to show that if $\mu E = 0$ for all $E \in U_o$, then $\mu \in R$. However, this approach to attacking the question does not provide a useful starting point since we know nothing about U_o except the definition. Thus, the conjectures that have been made about which classes C might work are in terms of sets with certain specified structure. We begin with a conjecture of Rajchman himself.

Rajchman realized the importance of R to the study of sets of uniqueness for Fourier series (Sections III.8 and IV.2). He introduced in 1922 [1] an important new class of sets of uniqueness called H-sets (Section II.2). He also made the conjecture (see Bari [1, pp. 85-86]) that C can be taken to be the class of H-sets. He showed that H-sets do belong to U_o, but he could not show that, conversely, if $\mu E = 0$ for all $E \in H$, then $\mu \in R$.

In 1947, Šreĭder claimed to have proved that Rajchman's conjecture was correct (see Bari [1, pp. 85-86] and Šreĭder [1]), but no proof was published. Apparently, he later changed his mind, because in 1950, he published a new claim (without mentioning his earlier claim).

In Šreĭder [2], he claimed that C can be taken to be the class of W-sets (Section II.2). While he gave some indications of a proof, no full proof ever did appear.

In 1963, Kahane and Salem [1] and Kahane [1] stated that they had not been able to decide whether the class of W*-sets (Section II.2) belongs to U_o. R.C. Baker [1, p. 32] interpreted this as a conjecture and attempted to prove it (Baker [1, 2]; see also Section III.4). It turns out (Section II.4) that if $\mu E = 0$ for all W*-sets E, then $\mu \in R$. Hence, the Kahane-Salem "conjecture" is equivalent to the conjecture that C may be taken to be W*.

2. The Answers

Šreĭder's claim about W-sets is correct (Section III. 2), while Rajchman's conjecture on H-sets and Kahane-Salem's on W*-sets are incorrect (Sections III.8 and III.6). Thus Rajchman measures are indeed characterized by their common null sets (i.e. U_o) .

The proofs of these and similar results often involve some analysis of diophantine inequalities. This stems from the fact that the classes of sets themselves, H , W , and W* , are defined using notions from diophantine approximation, specifically, the notions of asymptotic distribution. Full definitions of these concepts are given in Chapter II.

One consequence of our result is a new direct sum decomposition of the measure algebra $M(\mathbb{T}) = R \oplus J$ (Section IV.1). For Riesz products and certain infinite convolutions of measures, we can prove "purity" theorems related to this decomposition (Sections IV.4 and IV.3). For example, a Riesz product must belong purely to R or purely to J .

Looking beyond the circle to the general case of locally compact abelian groups, we may ask the same question about characterizing R . The answer is again that R is characterized by its null sets and the proof involves few significant new difficulties. Thus, only our last chapter

is devoted to this case. We prefer to concentrate on $\mathbb{T}$, since it is there that most of the ideas appear and also because it is the most important case.

A list of some open questions appears in the Appendix.

3. Basic Notation

We shall usually think of $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, but sometimes it is convenient to identify $\mathbb{T}$ with the interval $[0,1)$ (including 0 but excluding 1). In Chapter IV, half the time $\mathbb{T}$ will be identified as $\mathbb{R}/\mathbb{Z}$ and the other half as $\{z \in \mathbb{C}: |z| = 1\}$, where $\mathbb{C}$ is the set of complex numbers. In all cases, m denotes normalized Lebesgue measure, so that $\|m\| = 1$ on $\mathbb{R}/\mathbb{Z}$ as well as on $\{|z| = 1\}$. In the latter case, Fourier-Stieltjes transforms are defined without the 2π in the exponent:

$$\hat{\nu}(n) = \int_{|z|=1} z^{-n}\, d\nu(z) .$$

However, this will never appear explicitly. In general, we will use the 2π and make use of the notation

$$e(t) \equiv e^{2\pi it} .$$

Other notation is as follows: $\mathbb{Q}$ denotes the rational numbers or the rational subset of $\mathbb{T}$. $\mathbb{N}$ denotes the non-negative integers and $\mathbb{Z}^+$ the positive integers. For a set E, $\# E$ denotes the cardinality of E. The complement of a set E is denoted E^c. Often mI is denoted $|I|$ if I is an arc of $\mathbb{T}$. We say that μ is concentrated on E if $|\mu|(E) = \|\mu\|$. If E is the smallest closed set on which μ is concentrated, then E is called the support of μ. We could work almost

entirely with closed sets in all that follows without significant change. However, it is unnatural to do so, and so we will rarely speak of the "support" of a measure.

The space of continuous complex-valued functions on a topological space X is denoted $C(X)$. The space of continuous functions on $\mathbb{R}$ vanishing at infinity is denoted $C_o(\mathbb{R})$. For any set E, χ_E denotes the characteristic (indicator) function of E. If $\mu \in M(\mathbb{T})$, then $\mu|E$ denotes the measure

$$(\mu|E)(F) \equiv \mu(E \cap F) .$$

4. Notes.

Rajchman's conjecture was not published. Bari [1, pp. 85-86] mentions it as his conjecture but does not give a reference. Rajchman's student Milicer-Grużewska ([3, p. 167n], see also [2, pp. 158-159]) says only that Rajchman raised the question. She does not say that he conjectured it to be true, but her papers [2, 3] are devoted in part to an attempt to prove it.

CHAPTER II

SETS OF ASYMPTOTIC DISTRIBUTION

1. Asymptotic Distribution

Consider a sequence of points $\{x_n\}_1^\infty$ on the circle $\mathbb{T}$. We wish to say something about the manner in which $\{x_n\}$ lies in $\mathbb{T}$. One obvious question is whether $\{x_n\}$ is dense in $\mathbb{T}$. This property depends on the set $\{x_n\}$, but not on the order in which the points appear. Thus, to go beyond density, we may enquire about the limiting behavior of the first N points $\{x_n\}_1^N$ as $N \to \infty$, i.e., the asymptotic behavior of $\{x_n\}_1^\infty$. Here, a fundamental question is whether the sequence is distributed uniformly in $\mathbb{T}$. This means that for any arc $I \subset \mathbb{T}$,

$$\lim_{N\to\infty} \frac{1}{N} \# \{n \leq N \mid x_n \in I\} = mI ;$$

if so, we say $\{x_n\}^\infty$ is <u>uniformly distributed</u>, or <u>equidistributed</u> (Weyl [1,2]). One way this could fail is if there is some arc I such that

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \# \{n \leq N \mid x_n \in I\} < mI ;$$

then we say $\{x_n\}$ is <u>badly distributed</u> (Kahane and Salem [1]). A special case is if $\{x_n\}$ is <u>Weyl-distributed</u> (Šreǐder [2]), i.e., there exists a probability measure $\nu \neq m$ on $\mathbb{T}$ such that for every arc I whose endpoints are not mass-points of ν (we call such arcs admissible for ν),

$$\lim_{N\to\infty} \frac{1}{N} \# \{n \le N \mid x_n \in I\} = \nu I .$$

In general, if $\{x_n\}$ is either uniformly distributed or Weyl-distributed, we say $\{x_n\}$ has an asymptotic distribution and we write $\{x_n\} \sim \nu$ if ν is the limiting distribution. These four properties of a sequence, namely, whether it is dense, uniformly distributed, badly distributed, or Weyl distributed, will correspond to the four types of sets of asymptotic distribution which we will be investigating. These are, respectively, H-sets , non-normal sets ($\check{W}^*$-sets) , abnormal sets (A-sets) , and Weyl sets (W-sets) .

Given a point $x \in \mathbb{T}$, we may form the sequence $\{nx\}_{n=1}^{\infty}$. If x is rational, then clearly $\{nx\}$ is Weyl-distributed; the limiting distribution is $\frac{1}{q}\sum_{k=0}^{q-1} \delta(k/q)$, where $\delta(y)$ is the unit mass at y and $x = p/q$ in lowest terms. If x is irrational, then $\{nx\}$ is uniformly distributed. As Weyl [1] showed, this easily follows from the following criterion for uniform distribution (Weyl [1]).

THEOREM 1.1 (Weyl's Criterion). Let $\{x_n\}_1^{\infty} \subset \mathbb{T}$. The following are equivalent:

(i) $\{x_n\}$ is uniformly distributed.

(ii) For every bounded Riemann-integrable f on $\mathbb{T}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{\mathbb{T}} f \, dm .$$

(iii) For every non-zero integer k ,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(k\, x_n) = 0 .$$

Theorem 1.1 is a special case of the following more general criterion for asymptotic distribution. Although not proved by Weyl himself, it still goes by his name.

THEOREM 1.2 (Weyl's Criterion). Let $\{x_n\}_1^{\infty} \subset \mathbb{T}$. The following are equivalent:

(i) $\{x_n\} \sim \nu$.

(ii) For every bounded function f which is Riemann-Stieltjes integrable with respect to ν ,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_{\mathbb{T}} f \, d\nu . \tag{1.1}$$

(iii) For every $k \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(k\, x_n) = \hat{\nu}(-k) .$$

Furthermore, $\{x_n\}$ has an asymptotic distribution if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(k\, x_n)$$

exists for every $k \in \mathbb{Z}$.

We are using the following

DEFINITION. Let $\nu \in M(\mathbb{T})$. A step function g on $\mathbb{T}$ is called <u>admissible for</u> ν if the points where g is discontinuous are not mass-points of ν. A bounded function f is said to be <u>Riemann-Stieltjes integrable with respect to</u> ν if for all $\varepsilon > 0$, there exist admissible step functions g and h such that

(i) $g \leq f \leq h$,

(ii) $\int (h-g)\, d|\nu| < \varepsilon$.

It is not hard to show that f is Riemann-Stieltjes integrable with respect to ν if and only if the set of points of discontinuity of f forms a set of $|\nu|$-measure 0. The proof of this fact may be modelled on the proof of the special case $\nu = m$ (see Royden [1, p. 82, problem 2]). However, we shall not need this fact for the proof of Theorem 1.2.

In the sequel, we shall not use the equivalence (ii) of Weyl's criterion.

We now outline the proof of this most important theorem.

PROOF. Note that (i) is equivalent to

(iv) For every arc I admissible for ν,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(x_n) = \nu I = \int_{\mathbb{T}} \chi_I \, d\nu .$$

Therefore, if (i) holds, then (1.1) holds for admissible

step functions f. Hence (1.1) holds for Riemann-Stieltjes integrable functions f.

Conversely, if (ii) holds and I is any admissible arc for ν, then clearly χ_I is Riemann-Stieltjes integrable, whence we deduce (iv) (hence (i)).

Now every $f \in C(\mathbb{T})$ is uniformly approximable by admissible step functions for a given ν. Therefore continuous functions are Riemann-Stieltjes integrable. Furthermore, it is easy to see that (ii) holds if and only if (1.1) holds for all $f \in C(\mathbb{T})$. Therefore, (ii) is equivalent to

(v) $\frac{1}{N} \sum_{n=1}^{N} \delta(x_n) \to \nu$

in the weak* topology of $M(\mathbb{T})$. (Recall that $M(\mathbb{T})$ is the Banach-space dual of $C(\mathbb{T})$.)

Let us denote

$$\nu_N = \frac{1}{N} \sum_{n=1}^{N} \delta(x_n) .$$

Since $\|\nu_N\| = \|\nu\| = 1$, $\nu_N \to \nu$ weak* if and only if $\hat{\nu}_N(k) \to \hat{\nu}(k)$ for all k. (This follows from (1.1) first by using $f(t) = e(-kt)$ and conversely by the fact that trigonometric polynomials are dense in $C(\mathbb{T})$.) Thus (ii) <=> (iii).

The only part of Weyl's criterion left to show is that if for all k, $\lim_{N\to\infty} \hat{\nu}_N(k)$ exists, then there is a probability measure ν which is the weak* limit of $\{\nu_N\}$.

But since the unit ball of $M(\mathbb{T})$ is weak* compact, there is some weak* limit point ν of $\{\nu_N\}$. Necessarily, then, $\hat{\nu}(k) = \lim \hat{\nu}_N(k)$ and ν is the weak* limit of $\{\nu_N\}$. In particular, $\hat{\nu}(0) = 1$. Since $\nu \geq 0$ obviously, ν is a probability measure. This completes the proof. □

2. Exceptional Sets

We define a number $x \in \mathbb{T}$ to be <u>normal</u> with respect to the base $r \geq 2$ if when $x = 0.x_1x_2\ldots$ is written to the base r , every digit $0 \leq d < r$ appears equally often:

$$\lim_{N\to\infty} \frac{1}{N} \# \{n \leq N \mid x_n = d\} = \frac{1}{r} \ ;$$

every pair of digits d_1 , d_2 appears equally often:

$$\lim_{N\to\infty} \frac{1}{N} \# \{n \leq N \mid x_n = d_1 , x_{n+1} = d_2\} = \frac{1}{r^2} \ ;$$

every triplet appears equally often; and so on. This is evidently equivalent to the following: if I is any interval of the form $\left[\frac{a}{r^k} , \frac{a+1}{r^k}\right)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I (r^{n-1} x) = m I . \tag{2.1}$$

Now if I is any interval, let A , B be finite unions of intervals of the form $\left[\frac{a}{r^k} , \frac{a+1}{r^k}\right)$ such that $A \subset I \subset B$ and $m(I\setminus A) < \varepsilon$, $m(B\setminus I) < \varepsilon$. From (2.1), it follows that for large N

$$\begin{aligned} mI - 2\varepsilon < m A - \varepsilon &< \frac{1}{N} \sum_1^N \chi_A (r^{n-1} x) \\ &\leq \frac{1}{N} \sum_1^N \chi_I (r^{n-1} x) \leq \frac{1}{N} \sum_1^N \chi_B (r^{n-1} x) \\ &< m B + \varepsilon < m I + 2\varepsilon . \end{aligned}$$

In other words, (2.1) holds for any I, so that x is normal to the base r if and only if $\{r^{n-1}x\}$ is uniformly distributed.

Borel [1] showed that almost every number (in the sense of Lebesgue measure) is normal to the base r. Weyl [2, §7] generalized this result as follows.

THEOREM 2.1. Let $\{n_k\}$ be any strictly increasing sequence of positive integers. Then for almost all x [m], the sequence $\{n_k x\}$ is uniformly distributed.

NOTATION. $n_k \uparrow \infty$ means n_k strictly increases to infinity.

Let us rephrase Theorem 2.1. We make the following definition (cf. Kahane [1]).

DEFINITION. A Borel set $E \subset \mathbb{T}$ is called a <u>non-normal set</u>, or <u>W*-set</u>, if there exists a sequence $n_k \uparrow \infty$ such that for every $x \in E$, $\{n_k x\}$ is not uniformly distributed.

Then Theorem 2.1 becomes

THEOREM 2.1'. If E is any W*-set, then $mE = 0$.

[illegible] shall later prove a stronger result on the smallness [illegible] sets (Theorem 4.4). Zygmund [1, I, pp.142-3] [illegible] different method to prove Theorem 2.1.

[illegible] similar manner, we make the following definitions.

DEFINITION (Kahane [1]). A Borel set $E \subset \mathbb{T}$ is an <u>abnormal set</u>, or <u>A-set</u>, if there exists $n_k \uparrow \infty$ such that for all $x \in E$, $\{n_k x\}$ is badly distributed.

DEFINITION. (Šreĭder [2]). A Borel set $E \subset \mathbb{T}$ is a <u>Weyl set</u>, or <u>W-set</u>, if there exists $n_k \uparrow \infty$ such that for all $x \in E$, $\{n_k x\}$ is Weyl-distributed.

DEFINITION (Rajchman [1, 2]). A Borel set $E \subset \mathbb{T}$ is a <u>Hardy-Littlewood-Steinhaus set</u>, or <u>H-set</u>, if there is a non-empty open arc I and $n_k \uparrow \infty$ such that for all $x \in E$, $n_k x \notin I$.

NOTATION. An H-set is also called a set of type H; likewise for the other types of sets. A countable union of sets of a given type, such as H, is called of type H_σ, or an H_σ-set.

Suppose that for all $x \in E$, $\{n_k x\}$ is not dense in $\mathbb{T}$. Then for each x there is an open interval I such that for all k, $n_k x \in I$. Clearly I may be taken to have rational endpoints. Since there are only a countable number of rational intervals, it follows that E is of type H_σ. We now see how the four possible properties of a sequence in $\mathbb{T}$ mentioned in Section 1 give rise to four kinds of "sets of asymptotic distribution."

3. Examples and Elementary Properties.

It is evident that every A-set is a W*-set . We write this as $A \subset W^*$. Likewise, it is evident that $H \subset A$ and $W \subset A$.

THEOREM 3.1. Every countable set is a W-set .

PROOF. Let $\{x_j\}_1^\infty \subset \mathbb{T}$ be any countable set. We shall show by an easy diagonal argument that there exists $\{m_k\}$ such that $m_k x_j \to \alpha_j$ for some α_j . Thus $\{m_k x_j\}_{k=1}^\infty \sim \delta(\alpha_j)$ is Weyl-distributed for each x_j . The details follow.

Since $\mathbb{T}$ is compact, $\{nx_1\}_{n=1}^\infty$ has a convergent subsequence $n_k^{(1)} x_1 \to \alpha_1$. Likewise $\{n_k^{(1)} x_2\}$ has a convergent subsequence $n_k^{(2)} x_2 \to \alpha_2$. In general, let $\{n_k^{(j+1)}\}_{k=1}^\infty$ be a subsequence of $\{n_k^{(j)}\}$ such that $n_k^{(j+1)} x_{j+1} \to \alpha_{j+1}$ for some α_{j+1} . Let $m_k = n_k^{(k)}$. Then $m_k x_j \to \alpha_j$. □

Along similar lines, we may exhibit a W-set of cardinality c . Note that every $x \in \mathbb{T}$ has a unique representation in the form

$$x = \sum_{n=2}^{\infty} \frac{a_n(x)}{n!} ,$$

where $a_n(x)$ is an integer, $0 \le a_n(x) \le n - 1$, and there is no n_0 for which $a_n(x) = n - 1$ for all $n \ge n_0$. Let

$$E = \{x \mid a_n(x) = o(n)\} .$$

Certainly E has cardinality c . Furthermore,

$$E = \{x \mid (n-1)!\, x \to 0\} .$$

(Recall that $(n-1)!\, x = x + \ldots + x$ is defined in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.) Thus, to show that E is a W-set , it remains only to show that E is a Borel set. This follows from the following proposition.

PROPOSITION 3.2. If $\{f_n\}$ are Borel-measurable functions and $z \in \mathbb{E}$, then $\{x: f_n(x) \to z\}$ is a Borel set.

PROOF. The set in question equals

$$\{x: \forall \varepsilon > 0 \ \exists N \ \forall n \ge N \ |f_n(x)-z| < \varepsilon \} =$$

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} f_n^{-1}(\{\zeta: |\zeta - z| < \tfrac{1}{k} \}) . \ \square$$

Similar propositions which we will use follow.

COROLLARY 3.3. If $n_k \uparrow \infty$ and

$$E = \{t: \{n_k t\} \text{ is not uniformly distributed}\} ,$$

then E is Borel. E is called the <u>maximal</u> <u>W*-set</u> corresponding to the sequence $\{n_k\}$, denoted $E = W^*(\{n_k\})$.

PROOF. Let

$$f_K^{(m)}(t) = \frac{1}{K}\sum_{k=1}^{K} e(mn_k t) .$$

Then

$$E = \{t: \exists m \neq 0 \;\; f_K^{(m)}(t) \not\to 0\}$$

$$= \bigcup_{\substack{-\infty<m<\infty \\ m\neq 0}} \{t: f_K^{(m)}(t) \to 0\}^c \quad . \square$$

PROPOSITION 3.4. If $\{f_n\}$ are Borel-measurable functions, then $\{x: f_n(x)$ converges$\}$ is a Borel set.

PROOF. The set in question is

$$\{x: f_n(x) \text{ is Cauchy}\} = \{x: \forall \varepsilon > 0 \;\; \exists N \;\; \forall n,m \geq N$$

$$|(f_n - f_m)(x)| < \varepsilon\}$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m\geq N} (f_n - f_m)^{-1}(\{z: |z| < \tfrac{1}{k}\}) . \square$$

COROLLARY 3.5. If $n_k \uparrow \infty$ and

$$E = \{t: \{n_k t\} \text{ is Weyl-distributed}\} ,$$

then E is Borel. We call $E = W(\{n_k\})$ the <u>maximal W-set</u> of $\{n_k\}$.

PROOF. Letting $f_K^{(m)}$ be as above, we have

$$E = \{t: \forall m \; f_K^{(m)}(t) \text{ converges and } \exists m \neq 0 \; f_K^{(m)}(t) \not\to 0\}$$

$$= \Big[\bigcap_{-\infty<m<\infty} \{t: f_K^{(m)}(t) \text{ converges}\}\Big] \cap W^*(\{n_k\}) . \square$$

PROPOSITION 3.6. If $n_k \uparrow \infty$ and

$$E = \{t: \{n_k t\} \text{ is badly distributed}\} ,$$

then

$$E = \{t: \exists \text{ rational arc } I \text{ such that } \overline{\lim_{K\to\infty}} \frac{1}{K}\sum_{k=1}^{K} \chi_I(n_k t) < mI\}$$

and E is Borel. $E = A(\{n_k\})$ is the <u>maximal</u> <u>A-set</u> of $\{n_k\}$.

PROOF. Given an arc I, let f_I be the Borel function

$$f_I(t) = \overline{\lim_{K\to\infty}} \frac{1}{K}\sum_{k=1}^{K} \chi_I(n_k t) .$$

Then

$$E = \{t: \exists I \;\; f_I(t) < mI\} .$$

If $f_I(t) < mI$, then there exists a rational arc $J \subset I$ with $f_I(t) < mJ$. Since $f_J(t) \leq f_I(t)$, it follows that

$$E = \{t: (\exists \text{ rational } I) \;\; f_I(t) < mI\}$$

$$= \bigcup_{\substack{\text{rational} \\ I}} f_I^{-1}([0, mI))$$

is Borel. $\square$

Note that the maximal H-set corresponding to a given sequence and a given open arc is obviously closed.

We may now generalize the above example of a W-set. Note that if $\{x_n\} \sim \nu$ and $y_n \to 0$, then $\{x_n + y_n\} \sim \nu$.

Hence if $\left\{\frac{a_n(x)}{n}\right\}_{n=1}^{\infty} \sim \nu$, then $\{(n-1)!\, x\} \sim \nu$. Thus

$$E = \{x: \left\{\frac{a_n(x)}{n}\right\}_1^{\infty} \text{ is Weyl-distributed}\}$$

is a W-set since $E = W(\{(n-1)!\})$.

In Section 2, we showed that the set $E = \{t: (n_k t)$ is not dense$\}$ is an H_σ-set . We shall call E the maximal H_σ-set corresponding to $\{n_k\}$. It is obvious that every maximal H_σ-set , hence every maximal A- or W*-set , contains the rationals $\mathbb{Q}$.

The standard Cantor middle-thirds set

$$E = \{t: \quad t = \sum_{n=1}^{\infty} c_n 3^{-n} , \quad c_n = 0,2\}$$

is the same as the maximal H-set corresponding to the sequence $\{3^k\}_0^\infty$ and the interval $(\frac{1}{3},\frac{2}{3})$. Let μ be the Cantor-Lebesgue measure supported on E : if $\phi: \mathbb{T} \to E$ is the map

$$\phi(\sum_1^\infty d_n 2^{-n}) = \sum_1^\infty (2d_n)3^{-n} ,$$

then $\mu F = m(\phi^{-1}(F))$. Now $\{2^k t\}_0^\infty \sim m$ if and only if $\{3^k \phi(t)\}_0^\infty \sim \mu$. By Theorem 2.2, it follows that $\{3^k t\} \sim \mu$ for almost every $t\,[\mu]$. In particular, μ is concentrated on a W-set. We shall later see that this is a general phenomenon: any measure supported on an H-set is concentrated on a W-set .

4. The Relation to Rajchman Measures

We discover some simple relations of W-sets and W*-sets to Rajchman measures when we integrate Weyl's criterion with respect to some measure. The meaning of this statement will become clearer in the proof of Theorem 4.2. First we prove the following well-known facts.

THEOREM 4.1. Let $\mu \in M(\mathbb{T})$, $\nu << \mu$. We have

(a) $\mu \in R \Rightarrow \nu \in R$.

(b) $\lim_{n\to\infty} \hat{\mu}(n) = 0 \Rightarrow \lim_{n\to\infty} \hat{\mu}(-n) = 0$.

(c) $\mu \in R \Leftrightarrow |\mu| \in R$.

NOTE. For complex measures μ , ν , by $\nu << \mu$ we mean that for all Borel sets E , $|\mu|\, E = 0 \Rightarrow |\nu|\, E = 0$. By the polar decomposition theorem for complex measures (Rudin [2, p. 133]), the Radon-Nikodym Theorem continues to hold for this sense of absolute continuity: if $\nu << \mu$, then $\exists f \in L^1(|\mu|)$ such that $d\nu = f\, d\mu$. We shall use $\mu \approx \nu$ to mean $\mu << \nu << \mu$.

PROOF. Trigonometric polynomials $P(t) = \sum_{-K}^{K} a_k e(kt)$ are dense in $C(\mathbb{T})$ in the uniform norm and $C(\mathbb{T})$ is dense in $L^1(|\mu|)$ in the L^1-norm . Therefore trigonometric polynomials are dense in $L^1(|\mu|)$ in the L^1-norm .

Let $\nu << \mu$ and $d\nu = f\,d\mu$, $f \in L^1(|\mu|)$. Given $\varepsilon > 0$, let $\|f-P\|_{L^1(|\mu|)} < \varepsilon$, $P(t) = \sum_{-K}^{K} a_k e(kt)$ a trigonometric polynomial. If $d\sigma = P d\mu$, then $\|\nu-\sigma\|_{M(\mathbb{T})} = \|f - P\|_{L^1(|\mu|)} < \varepsilon$. Also

$$\hat{\sigma}(n) = \int \sum_{-K}^{K} a_k e((k-n)t)\, d\mu(t)$$

$$= \sum_{-K}^{K} a_k\, \hat{\mu}(n-k),$$

whence $\sigma \in R$ if $\mu \in R$. Since $|\hat{\nu}(n) - \hat{\sigma}(n)| \leq \|\nu-\sigma\| < \varepsilon$, it follows that $\overline{\lim_{|n|\to\infty}} |\hat{\nu}(n)| < \varepsilon$. As ε is arbitrary, $\nu \in R$. This establishes (a).

In fact, the argument shows that if $\nu << \mu$ and $\lim_{n\to\infty} \hat{\mu}(n) = 0$, then $\lim_{n\to\infty} \hat{\nu}(n) = 0$ and likewise if $n \to -\infty$. Now $\mu \approx |\mu|$ and $|\mu|$ is real, so that $\widehat{|\mu|}(-n) = \overline{\widehat{|\mu|}(n)}$. Therefore

$$\lim_{n\to\infty} \hat{\mu}(n) = 0 \Rightarrow \lim_{n\to\infty} \widehat{|\mu|}(n) = 0$$

$$\Rightarrow \lim_{n\to\infty} \widehat{|\mu|}(-n) = 0 \Rightarrow \lim_{n\to\infty} \hat{\mu}(-n) = 0,$$

which is (b).

Part (c) follows from part (a) and the fact that $\mu \approx |\mu|$. □

This theorem is due to Rajchman [3] and his student [illegible]-Drużewska [1], although part (b) does not occur [illegible] in their papers.

The simplest relation between W*-sets and R is expressed in the following theorem. Note that $|\hat{\mu}(n)| \leq \|\mu\|$, so that if $\mu \notin R$, then $\{\hat{\mu}(n)\}$ has a non-zero limit point.

THEOREM 4.2. If $\mu E = 0$ for all W*-sets E, then $\mu \in R$. In fact, if $n_k \uparrow \infty$ or $-n_k \uparrow \infty$ and $\hat{\mu}(n_k) \to \alpha$, then the maximal W*-set E of the sequence $\{|n_k|\}$ has $|\mu|$-measure $\geq |\alpha|$.

PROOF. Note that for $t \notin E$,

$$\frac{1}{K} \sum_{k=1}^{K} e(-n_k t) \to 0.$$

Therefore

$$|\alpha| = \left|\lim_{K\to\infty} \frac{1}{K} \sum_{1}^{K} \hat{\mu}(n_k)\right| = \left|\lim \int_{\mathbb{T}} \frac{1}{K} \sum_{1}^{K} e(-n_k t)\, d\mu(t)\right|$$

$$\leq \lim \int_{E} 1 \quad d|\mu|(t) + \left|\lim \int_{\mathbb{T}\setminus E} \frac{1}{K} \sum_{1}^{K} e(-n_k t)\, d\mu(t)\right|$$

$$= |\mu| E. \ \square$$

In Chapter III, we shall see that the converse does not hold: it is not true that $\mu \in R \Rightarrow \mu E = 0$ for all W*-sets E. However, we do have

THEOREM 4.3. If $\mu \in R$, then $\mu E = 0$ for all W-sets E.

PROOF. Let $\mu \in R$, E be a W-set, and

$$c_m(t) = \begin{cases} \lim_{K\to\infty} \frac{1}{K}\sum_{1}^{K} e(mn_k t), & t \in E \\ 0, & t \notin E. \end{cases}$$

Let F be any Borel subset of E and let $\nu = \mu|F$, the restriction of μ to F. By Theorem 4.1(a), $\nu \in R$. Therefore for $m \neq 0$,

$$\int_F c_m(t)\, d\mu(t) = \int_F c_m\, d\nu = \int_{\mathbb{T}} c_m\, d\nu$$

$$= \lim_{K\to\infty} \int_{\mathbb{T}} \frac{1}{K}\sum_{1}^{K} e(mn_k t)\, d\nu(t)$$

$$= \lim \frac{1}{K}\sum_{1}^{K} \hat{\nu}(-mn_k) = 0 .$$

Since F is arbitrary, $c_m(t) = 0$ for $|\mu|$-almost all $t \in E$. But by definition, if $t \in E$, then $c_m(t) \neq 0$ for some $m \neq 0$. Hence $|\mu|\, E = 0$. □

The converse of Theorem 4.3 does hold, as shown in Chapter III. Thus R is characterized by W-sets, but not by W*-sets.

Let us introduce the following notation (Rajchman [3]).

DEFINITION. The <u>deviation</u> (<u>écart</u> in French) of a measure μ is $\overline{\lim}_{|n|\to\infty} |\hat{\mu}(n)|$ and is denoted $R(\mu)$.

Thus, $\mu \in R$ if and only if $R(\mu) = 0$. Also, from Theorem 4.2, we immediately obtain

COROLLARY 4.4. For $\mu \in M(\mathbb{T})$, there exists a W*-set E such that $|\mu|(E) \geq R(\mu)$.

Since $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ for some non-negative measures μ_j, it follows that $|\mu F| \geq \frac{1}{4} R(\mu)$ for some subset F of E. Note that F is a W*-set. The next proposition allows us to establish a somewhat better result.

PROPOSITION 4.5. Let $\mu \in M(\mathbb{T})$ and let E be any Borel subset of $\mathbb{T}$. There exists $F \subseteq E$ such that $|\mu F| \geq \frac{1}{\pi} |\mu|(E)$. The constant $\frac{1}{\pi}$ is best possible.

PROOF. Let $d\mu(t) = e(\phi(t))\, d|\mu|(t)$. Let

$$F_\theta = \{t \in E: \phi(t) \in (\theta - \tfrac{1}{4}, \theta + \tfrac{1}{4})\} .$$

Then

$$\int_{\mathbb{T}} |\mu F_\theta|\, d\theta \geq \left| \int_{\mathbb{T}} e(-\theta)\, \mu F_\theta\, d\theta \right|$$

and

$$\int_{\mathbb{T}} e(-\theta)\, \mu F_\theta\, d\theta = \int_{\mathbb{T}} e(-\theta) \int_{F_\theta} e(\phi(t))\, d|\mu|(t)\, d\theta$$

$$= \int_E e(\phi(t)) \int_{\phi(t)-\frac{1}{4}}^{\phi(t)+\frac{1}{4}} e(-\theta)\, d\theta\, d|\mu|(t)$$

$$= \frac{1}{\pi} \int_E d|\mu|(t) = \frac{1}{\pi} |\mu| E .$$

Hence for some θ, $|\mu F_\theta| \geq \frac{1}{\pi} |\mu| E$.

To show that $\frac{1}{\pi}$ is best possible, consider $d\mu(t) = e(t)\,dt$ and $E = \mathbb{T}$. For any F,

$$\mathrm{Re}[e(-\theta)\,\mu F] = \int_F \mathrm{Re}\, e(t-\theta)\,dt$$

$$\leq \int_{\{t \in \mathbb{T}:\, \mathrm{Re}\, e(t-\theta) \geq 0\}} \mathrm{Re}\, e(t-\theta)\,dt$$

$$= \mathrm{Re} \int_{-1/4}^{1/4} e(s)\,ds = \frac{1}{\pi} .$$

Hence $|\mu F| \leq \frac{1}{\pi}$. □

COROLLARY 4.6. For $\mu \in M(\mathbb{T})$, there exists a W^*-set F such that $|\mu F| \geq \frac{1}{\pi} R(\mu)$.

5. Notes

Much material on asymptotic distribution and normal numbers and many references to other work are found in Kuipers and Niederreiter [1] and Koksma and Kuipers [1].

A proof of the most important parts of Weyl's criterion (Theorem 1.2) is also given in Zygmund [1, I, p. 142]. The case when ν is continuous is proved in Brown and Duncan [1] with full details. Theorem 1.1 is treated in Kuipers and Niederreiter [1, pp. 1-3, 7-8]. Treatments similar to ours of the general case of weak* convergence of probability measures may be found in Billingsley [1, pp. 7-15, 50] and Loève [1, pp. 190-191].

In the literature, the term "normal set" refers to the sets of the form

$$E = \{x \in \mathbb{R} : \langle \lambda_n x \rangle_{n=1}^{\infty} \sim m\}$$

for some sequence $\{\lambda_n\} \subset \mathbb{R}$, where $\langle u \rangle$ denotes the fractional part of u. It is not required that $\lambda_n \to \infty$. See Rauzy [1] for a characterization of normal sets.

Rajchman [1,2] named H-sets after Hardy, Littlewood and Steinhaus because of the following theorem of theirs (see Zygmund [1, I, pp. 318-319]).

PROPOSITION 5.1. Given real numbers a_n, b_n,

$$\limsup_{n\to\infty} |a_n \cos 2\pi nt + b_n \sin 2\pi nt|$$
$$= \limsup_{n\to\infty} (a_n^2 + b_n^2)^{1/2}$$

for all t except those belonging to a set of type H_σ.

Some results generalizing Theorem 4.1 appear in Graham and McGehee [1, pp. 27-29].

CHAPTER III

CHARACTERIZATIONS OF RAJCHMAN MEASURES

1. Characterizations Other Than Through Null Sets

The oldest characterization of R is due to Rajchman and Milicer-Grużewska (Milicer-Grużewska [2], Zygmund [1, II, pp. 144-145]). We shall present an easier proof and some generalizations.

THEOREM 1.1 (Rajchman-Milicer-Grużewska Criterion). For a measure $\mu \in M(\mathbb{T})$, the following conditions are equivalent:

(i) $\mu \in R$.

(ii) For every arc $I \subset \mathbb{T}$,

$$\lim_{|n|\to\infty} \int_{\mathbb{T}} \chi_I(nx)\, d\mu(x) = |I| \cdot \hat{\mu}(0) .$$

(iii) For every $f \in C(\mathbb{T})$,

$$\lim_{|n|\to\infty} \int_{\mathbb{T}} f(nx)\, d\mu(x) = \hat{f}(0) \cdot \hat{\mu}(0) . \tag{1.1}$$

The equivalence (i) $\Leftrightarrow$ (ii) is the most important for us. Let us first indicate heuristically why (i) $\Rightarrow$ (ii). If we could write χ_I as its Fourier series, integrate term by term, and take the limit term by term, then for $\mu \in R$, we would have

$$\lim_{|n|\to\infty} \int \chi_I(nx)\, d\mu(x) = \lim_{|n|\to\infty} \int \sum_{k=-\infty}^{\infty} \hat{\chi}_I(k)\, e(knx)\, d\mu(x)$$

$$= \lim_{|n|\to\infty} \sum_{k=-\infty}^{\infty} \hat{\chi}_I(k)\, \hat{\mu}(-kn) = \hat{\chi}_I(0)\, \hat{\mu}(0) = |I| \cdot \hat{\mu}(0) .$$

We now give a rigorous proof of the theorem.

PROOF. (i) $\Rightarrow$ (iii) . Clearly the set of $f \in C(\mathbb{T})$ for which (1.1) holds is a closed linear subspace of $C(\mathbb{T})$. By (i), we see that (1.1) holds when $f(x) = e(kx)$ for any $k \in \mathbb{Z}$. Since the closed linear span of the exponentials is all of $C(\mathbb{T})$, it follows that (iii) holds.

(iii) $\Rightarrow$ (i). Take $f(x) = e(-x)$.

(i) & (iii) $\Rightarrow$ (ii). Let I be a given arc. For $\varepsilon > 0$, choose f , $g \in C(\mathbb{T})$ so that $|f - \chi_I| \le g$ and $\|g\|_{L^1(m)} < \varepsilon$. Then

$$\left| \int_{\mathbb{T}} \chi_I(nx)\, d\mu(x) - |I|\, \hat{\mu}(0) \right|$$

$$\le \left| \int (f(nx) - \chi_I(nx))\, d\mu(x) \right| + |\hat{f}(0) - \hat{\chi}_I(0)| \cdot |\hat{\mu}(0)|$$

$$\le \int g(nx)\, d|\mu|(x) + \varepsilon \cdot |\hat{\mu}(0)| .$$

Since $\mu \in R$, also $|\mu| \in R$, whence the limit of the last quantity is $\le 2\varepsilon\, \|\mu\|$. Now (ii) follows.

(ii) $\Rightarrow$ (iii). Since (ii) asserts that (1.1) holds for characteristic functions of intervals, and since the span of such functions in the sup norm includes $C(\mathbb{T})$, it follows that (1.1) holds for $f \in C(\mathbb{T})$. □

What does the Rajchman-Milicer-Grużewska criterion tell us about Rajchman measures? In other words, what intuition can be gained from it? If I is any arc, let

$$E_n = \{x\colon nx \in I\} ;$$

E_n is a union of n equally spaced arcs of length $\frac{1}{n}|I|$. If A is any other arc, then for Lebesgue measure, it is clear that

$$m(A \cap E_n) \to |I| \cdot mA$$

as $n \to \infty$. We now see that the same happens for any $\mu \in R$. Indeed, if A is any Borel set, not merely an arc, let $\nu = \mu|A$. Then $\chi_I(nx) = \chi_{E_n}(x)$ and $\nu \in R$, whence

$$\mu(A \cap E_n) = \nu(E_n) = \int \chi_{E_n}(x)\, d\nu(x) \tag{1.2}$$
$$\to |I|\, \hat{\nu}(0) = |I| \cdot \mu A .$$

Furthermore, (1.2) holds only when $\mu \in R$.

As an illustration, consider the Cantor-Lebesgue measure μ . It is supported on the Cantor middle-thirds set $A = \bigcap_{k=1}^{\infty} E^c_{3^k}$, where $I = (\frac{1}{3}, \frac{2}{3})$. Since $\mu(A \cap E_{3^k}) = \mu(\emptyset) = 0$, (1.2) fails to hold, whence $\mu \notin R$.

Let us now show that in Theorem 1.1, (i) follows from much weaker assumptions than (ii) or (iii). We denote the translate of a set by

$$E - t \equiv \{x - t : x \in E\}$$

and the translate of a function by

$$f_t(x) \equiv f(x - t) .$$

THEOREM 1.2. Let $\mu \in M(\mathbb{T})$. Then $\mu \in R$ if either of the following conditions hold:

(i) For some arc I with $0 < |I| < 1$, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} \chi_{I-t}(nx)\, d\mu(x) \tag{1.3}$$

exists for almost all $t\,[m]$ and for some t, (1.3) equals $|I| \cdot \hat{\mu}(0)$.

(ii) For some $f \in C(\mathbb{T})$ with $\hat{f}(-1) \neq 0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{T}} f_t(nx)\, d\mu(x) \tag{1.4}$$

exists for almost all $t\,[m]$ and for some t such that $f(t) \neq \hat{f}(0)$, (1.4) equals $\hat{f}(0) \cdot \hat{\mu}(0)$.

PROOF. Assume (i). Let $g(t)$ be equal to the limit (1.3) when it exists. Then

$$\hat{g}(-1) = \int_{\mathbb{T}} g(t)\, e(t)\, dm(t)$$

$$= \int \lim_{n\to\infty} \int \chi_{I-t}(nx)\, d\mu(x)\, e(t)\, dm(t) .$$

By the bounded convergence theorem and Fubini's theorem, this

$$= \lim_{n\to\infty} \iint \chi_I(nx + t)\, e(t)\, dm(t)\, d\mu(x) .$$

Setting $s = nx + t$ yields

$$\hat{g}(-1) = \lim \iint \chi_I(s)\, e(s)\, dm(s)\, e(-nx)\, d\mu(x)$$

$$= \lim \int \hat{\chi}_I(-1)\, e(-nx)\, d\mu(x)$$

$$= \lim \hat{\chi}_I(-1)\, \hat{\mu}(n) .$$

Since $\hat{\chi}_I(-1) \neq 0$, $\lim_{n\to\infty} \hat{\mu}(n)$ exists. Let the limit be α. By application of Theorem II.4.1(b) to the measure $\mu - \alpha\delta(0)$, we see that $\lim_{n\to\infty} \hat{\mu}(-n) = \alpha$, whence $\mu - \alpha\delta(0) \in R$. Theorem 1.1 now allows us to compute $g(t)$:

$$g(t) = \lim_{n\to\infty} \int \chi_{I-t}(nx)\, d(\mu - \alpha\delta(0))(x)$$

$$+ \lim_{n\to\infty} \int \chi_{I-t}(nx)\, d(\alpha\delta(0))(x)$$

$$= |I|\, (\mu - \alpha\delta(0))^{\wedge}(0) + \alpha\, \chi_I(t)$$

$$= |I|\, \hat{\mu}(0) + \alpha(\chi_I(t) - |I|) .$$

If $g(t) = |I|\ \hat{\mu}(0)$ for some t, then clearly $\alpha = 0$. In other words, $\mu \in R$.

The proof that (ii) $\Rightarrow \mu \in R$ is virtually identical. If $g(-t)$ denotes the limit (1.4), then we find that $\hat{g}(-1) = \lim_{n\to\infty} \hat{f}(-1)\ \hat{\mu}(n)$. If $\alpha = \lim_{n\to\infty} \hat{\mu}(n)$, then we find that

$$g(t) = \hat{f}(0)\ \hat{\mu}(0) + \alpha(f(t) - \hat{f}(0)) .$$

If $g(t) = \hat{f}(0)\ \hat{\mu}(0)$ and $f(t) \neq \hat{f}(0)$ for some t, then $\alpha = 0$, i.e., $\mu \in R$. □

A useful consequence is the following immediate

COROLLARY 1.3. If $\mu \notin R$, then for any arc I with $0 < |I| < 1$, there exists t for which

$$\int \chi_{I-t}(nx)\ d\mu(x) \not\to |I|\ \hat{\mu}(0)$$

as $|n| \to \infty$. Also, for any $f \in C(\mathbb{T})$ with $\hat{f}(-1) \neq 0$, there exists t for which

$$\int f_t(nx)\ d\mu(x) \not\to \hat{f}(0)\ \hat{\mu}(0)$$

as $|n| \to \infty$.

The heart of the proof of Theorem 1.2 was calculating a Fourier coefficient of (1.3). Consideration of all the Fourier coefficients yields a quantitative formulation of Theorem 1.1.

THEOREM 1.4. Let $\mu \in M(\mathbb{T})$ and $f \in L^2(m)$. Define

$$g_n(t) = \int_{\mathbb{T}} f_t(nx)\ d\mu(x) - \hat{f}(0)\ \hat{\mu}(0) ,$$

where f_t denotes the function $f_t(x) = f(x - t)$. Then $g_n(t)$ exists for m-almost all t and

$$|\hat{f}(1)|R(\mu) \leq \limsup_{|n|\to\infty} \|g_n\|_{L^2(m)} \leq \|f-\hat{f}(0)\|_{L^2(m)}\ R(\mu) .$$

PROOF. Since $L^2(m) \subset L^1(m)$, the Fubini-Tonelli theorem shows that $g_n(t)$ exists a.e. [m]. Furthermore, $\hat{g}_n(0) = 0$ and

$$\hat{g}_n(k) = \hat{f}(-k)\ \hat{\mu}(kn) , \quad k \neq 0 .$$

Hence

$$\|g_n\|^2_{L^2(m)} = \|\hat{g}_n\|^2_{\ell^2} = \sum_{k\neq 0} |\hat{f}(-k)\ \hat{\mu}(kn)|^2 .$$

Therefore

$$\overline{\lim}\|g_n\| \geq \overline{\lim}|\hat{f}(1)\ \hat{\mu}(-n)| = |\hat{f}(1)|R(\mu)$$

and

$$\overline{\lim}\|g_n\| \leq \Big(\sum_{k\neq 0} |\hat{f}(k)|^2\ R(\mu)^2\Big)^{1/2}$$

$$= \|f-\hat{f}(0)\|_{L^2(m)}\ R(\mu) . \ \square$$

A useful case of Theorem 1.4 is

COROLLARY 1.5. Let $\mu \in M(\mathbb{T})$, I be an arc of $\mathbb{T}$, and

$$g_n(t) = \int \chi_{I+t}(nx)\, d\mu(x) - |I|\, \hat{\mu}(0) .$$

Then

$$\frac{1}{\pi}(\sin \pi|I|)\, R(\mu) \le \overline{\lim_{|n|\to\infty}} \|g_n\|_{L^2(m)} \le (|I||I^c|)^{1/2} R(\mu) .$$

PROOF. We have only to observe that

$\hat{\chi}_I(0) = |I|$, $\hat{\chi}_I(1) = \frac{1}{\pi} \sin \pi |I|$, and

$$\|\chi_I - |I|\, \|_{L^2(m)} = ((1-|I|^2)|I| + |I|^2 (1-|I|))^{1/2}$$

$$= (|I|\, |I^c|)^{1/2} . \square$$

Another characterization of R, due to Goldberg and Simon [1], we merely mention without proof.

THEOREM 1.6. $\mu \in R$ if and only if $\|\hat{\mu}_t - \hat{\mu}\|_{\ell^\infty} \to 0$ as $t \to 0$, where μ_t is the translate of μ: $\mu_t(E) = \mu(E-t)$.

2. Weyl Sets

We now prove that Šreĭder's claim is correct. This is our central result.

THEOREM 2.1. A measure μ is in R if and only if $\mu E = 0$ for all W-sets E.

An immediate consequence, of course, is

COROLLARY 2.2. A measure μ is in R if and only if $\mu E = 0$ for all U_o-sets E. That is, R is characterized by its class of common null sets, U_o.

We shall in fact prove the stronger

PROPOSITION 2.3. For any $\mu \in M(\mathbb{T})$, there exists a W-set E such that $|\mu|(E) \ge R(\mu)$. There exists a W-set F such that $|\mu F| \ge \frac{1}{\pi} R(\mu)$.

REMARK. It follows that R is also characterized by the class of closed U_o-sets and by the class of closed W-sets. For if $\mu \notin R$, then there is a W-set E with $|\mu|E \ne 0$. Since μ is regular, there is a closed subset $F \subset E$ with $|\mu|F \ne 0$. Also, F is a W-set since every Borel subset of a W-set is a W-set.

We have already proved the easy half of Theorem 2.1 (see Theorem II. 4.3). The difficulty in proving the converse direction is found on examination of the proof of Theorem II.4.2 (that if $\mu \notin R$, then there is a W*-set E for which $|\mu|E > 0$). Namely, we cannot assert the existence of

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} e(-n_k t)$$

for $|\mu|$-almost all $t \notin W(\{n_k\})$, whereas we can for all $t \notin W^*(\{n_k\})$. Remarkably, however, we will be able to choose $\{n_k\}$ in such a way that the above limit does exist for $|\mu|$-almost all t! The proof of Theorem 2.1 then becomes straightforward.

In order to obtain the a.e.-existence of the above limit, we shall need to adopt the viewpoint that $e(nt) \in L^2(|\mu|)$. Let us show how this viewpoint could be used to reword the proof of Theorem II.4.3. Recall the following

DEFINITION. Let μ be a measure on any measurable space. A set E is called an <u>atom</u> of μ if $|\mu|(E) \neq 0$ and if whenever $F \subset E$ is measurable, either $\mu F = 0$ or $\mu F = \mu E$.

LEMMA 2.4. Let μ be any positive measure without atoms of infinite measure. Let $f_n \to 0$ weakly in $L^2(\mu)$; i.e., for all $g \in L^2(\mu)$, $(f_n, g) \equiv \int f_n \bar{g}\, d\mu \to 0$ as $n \to \infty$. Then for almost all t $[\mu]$, if $\lim_{n\to\infty} f_n(t)$ exists and is finite, it equals 0.

PROOF. Suppose F is a set of positive measure on which $\lim_{n\to\infty} f_n(t)$ exists and is finite. Since μ has no atoms of infinite measure, we may take F to have finite measure.

For $t \in F$, $\{f_n(t)\}_{n=1}^{\infty}$ is bounded. Hence

$$F = \bigcup_{M=1}^{\infty} \{t: (\forall n)\ |f_n(t)| \le M\}.$$

For some M, one of these sets, call it E, has positive measure. Set

$$g(t) = \begin{cases} f(t)/|f(t)| & , t \in E \\ 0 & , t \notin E. \end{cases}$$

Then the bounded convergence theorem yields

$$\int_E |f|\, d\mu = \lim \int_E f_n \bar{g}\, d\mu = \lim \int f_n \bar{g}\, d\mu$$

$$= \lim\, (f_n, g) = 0 \quad . \ \square$$

We now give the reworded

PROOF OF THEOREM II.4.3. Recall that by Theorem II.4.1(c), it suffices to consider only positive μ. Let $0 \le \mu \in R$. We wish to show that for any $n_k \uparrow \infty$, if

$$E = W(\{n_k\}) = \{t: \text{for all } m,\ \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} e(m n_k t) \text{ exists and, for some } m \neq 0, \text{ is not } 0\},$$

then $\mu E = 0$. In other words, we wish to show that for all $m \neq 0$ and for μ-almost all t, if $\frac{1}{K} \sum_{k=1}^{K} e(m n_k t)$ has a limit as $K \to \infty$, then that limit is 0. Now we claim that $e(nt) \to 0$ weakly in $L^2(\mu)$ as $|n| \to \infty$. For if $g \in L^2(\mu)$, then $\bar{g} \in L^1(\mu)$, so that $\bar{g}\, d\mu \ll d\mu$ and $\bar{g}\, d\mu \in R$ by Theorem II.4.1(a):

$$(e(nt),\ g(t)) = \int e(nt)\ \bar{g}(t)\ d\mu(t) \to 0 \ .$$

This is precisely what was claimed.

It follows that for all $m \neq 0$,

$$\frac{1}{K} \sum_{k=1}^{K} e(m\, n_k t) \to 0 \quad \text{weakly in} \quad L^2(\mu)$$

as $K \to \infty$, whence the result follows from Lemma 2.4. □

As mentioned above, the converse depends on the remarkable

LEMMA 2.5. Given $\mu \in M(\mathbb{T})$ and $\{n_k\}_1^\infty \subset \mathbb{Z}$, there exists a subsequence $\{n_k'\} \subset \{n_k\}$ such that $\{n_k' t\}$ has an asymptotic distribution for almost all t $[|\mu|]$. Furthermore, $\{n_k'\}$ can be chosen so that if $\{n_k' t\} \sim \nu_{(t)}$ for $|\mu|$-almost all t, then for any further subsequence $\{n_k''\} \subset \{n_k'\}$ and for $|\mu|$-almost all t, $\{n_k'' t\} \sim \nu_{(t)}$.

The second part of the lemma is not necessary to the proof of Proposition 2.3. Before demonstrating Lemma 2.5, we present the

DEDUCTION OF PROPOSITION 2.3 FROM LEMMA 2.5. Let $\mu \notin R$ and let $n_k \uparrow \infty$ be such that $\hat{\mu}(n_k) \to \alpha \neq 0$. Let $\{n_k'\}$ be as in Lemma 2.5, let $f_k(t) = \frac{1}{K} \sum_{k=1}^{K} e(-n_k' t)$, and let E be the set of t for which $\{n_k' t\}$ has an asymptotic distribution and $\lim_{K\to\infty} f_K(t) \neq 0$. Since $\lim f_k$ exists a.e. $[|\mu|]$, we have

$$\int_E \lim f_K \ d\mu = \int_{\mathbb{T}} \lim f_K \ d\mu = \lim \int_{\mathbb{T}} f_K \ d\mu$$

$$= \lim \frac{1}{K} \sum_{k=1}^{K} \hat{\mu}(n_k) = \alpha \ .$$

Since $|\lim f_K| \leq 1$, it follows that the W-set E has $|\mu|$-measure $\geq |\alpha|$.

This proves the first assertion of Proposition 2.3. The second assertion follows from Proposition II.4.5. □

Lemma 2.5 is a special case of Corollary 2.9 below. That corollary allows us to extract from a bounded sequence in $L^2(|\mu|)$ a pointwise Cesàro-convergent subsequence. Corollary 2.9 in turn depends on a couple of facts about Hilbert spaces and an interpolation argument. We begin with

LEMMA 2.6. Let H be a Hilbert space and let $x_n \to y$ weakly in H. Then there exists a subsequence $\{x_n'\} \subset \{x_n\}$ such that for every subsequence $\{x_n''\} \subset \{x_n'\}$, $\frac{1}{N} \sum_{n=1}^{N} x_n'' \to y$ in norm at the following rate:

$$\left\| \frac{1}{N} \sum_{1}^{N} x_n'' \ - \ y \right\|^2 = O\left(\frac{1}{N}\right) \ .$$

PROOF. Without loss of generality, let $y = 0$. We define $x_n' = x_{r(n)}$ inductively. Let $r(1) = 1$, so that $x_1' = x_1$. Given $r(1), \ldots, r(n)$, choose $r(n+1) > r(n)$ so that

$$|(x_{r(m)}, x_{r(n+1)})| < (n+1)^{-3} \quad \text{for} \quad 1 \leq m \leq n \ .$$

By the principle of uniform boundedness, the x_n are bounded: $\|x_n\| \leq C$. If $\{x_n''\} \subset \{x_n'\}$, it follows that

$$\|\frac{1}{N}\sum_{n=1}^{N} x_n''\|^2 \leq \frac{1}{N^2}\sum_{1}^{N}(x_n'', x_n'') + \frac{2}{N^2}\sum_{1\leq m<n\leq N}|(x_m'', x_n'')|$$

$$\leq \frac{C^2}{N} + \frac{2}{N^2}\sum_{1\leq m<n\leq N} n^{-3} \leq \frac{C^2}{N} + \frac{2}{N^2}\sum_{1\leq n\leq N} n^{-2}$$

$$= O(\tfrac{1}{N}) . \ \square$$

COROLLARY 2.7. Let H be a Hilbert space. Suppose that for each $m \in \mathbb{Z}^+$, $x_{m,n} \to y_m$ weakly as $n \to \infty$. Then there exists a sequence $n_k \uparrow \infty$ such that for every subsequence $\{n_k'\} \subset \{n_k\}$ and each fixed m,

$$(2.1) \qquad \|\frac{1}{K}\sum_{1}^{K} x_{m,n_k'} - y_m\|^2 = O(\tfrac{1}{K}) .$$

PROOF. This is an easy diagonal argument. By Lemma 2.6, there exists $\{n_k^{(1)}\}$ such that (2.1) holds for $m = 1$ and every $\{n_k'\} \subset \{n_k^{(1)}\}$. Let $\{n_k^{(2)}\} \subset \{n_k^{(1)}\}$ be such that (2.1) holds for $m = 2$ and every $\{n_k'\} \subset \{n_k^{(2)}\}$. Continuing in this way for all m, we let $n_k = n_k^{(k)}$. $\square$

LEMMA 2.8. Let μ be a positive measure and let C_m be constants for $m \in \mathbb{Z}^+$. Suppose $f_{m,n}, g_m \in L^2(\mu)$ satisfy

(i) $|f_{m,n}(t)| \leq C_m$ a.e. $[\mu]$,

(ii) for each m, $f_{m,n} \to g_m$ weakly as $n \to \infty$.

Then there exists $n_k \uparrow \infty$ such that for every subsequence $\{n_k'\} \subset \{n_k\}$ and every m,

$$(2.2) \qquad \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} f_{m,n_k'} = g_m \quad \text{a.e. } [\mu] .$$

PROOF. Without loss of generality, let $g_m = 0$. By Corollary 2.7, there exist $n_k \uparrow \infty$ such that

$$\|\frac{1}{K}\sum_{k=1}^{K} f_{m,n_k'}\|^2_{L^2(\mu)} = O(\tfrac{1}{K})$$

for every $\{n_k'\} \subset \{n_k\}$ and each m. Therefore

$$\int \sum_{K=1}^{\infty} |\frac{1}{K^2}\sum_{k=1}^{K^2} f_{m,n_k'}|^2 \, d\mu = \sum_{1}^{\infty} \|\frac{1}{K^2}\sum_{1}^{K^2} f_{m,n_k'}\|^2_{L^2(\mu)} < \infty ,$$

so that the integrand is finite a.e. $[\mu]$, whence

$$\frac{1}{K^2}\sum_{1}^{K^2} f_{m,n_k'} \to 0 \quad \text{a.e. } [\mu] .$$

Having shown that (2.2) holds as $K \to \infty$ along the square integers, we shall now employ an interpolation argument used by Weyl [2,§7] to show that (2.2) holds as $K \to \infty$ along all the integers.

Given N, let $K^2 \leq N < (K+1)^2$. Then

$$\left|\frac{1}{N}\sum_{k=1}^{N} f_{m,n_k'}\right| \le \left|\frac{1}{N}\sum_{1}^{N} f_{m,n_k'} - \frac{1}{N}\sum_{1}^{K^2} f_{m,n_k'}\right| + \left|\frac{1}{N}\sum_{1}^{K^2} f_{m,n_k'}\right|$$

$$= \frac{1}{N}\left|\sum_{K^2<k\le N} f_{m,n_k'}\right| + \frac{K^2}{N}\left|\frac{1}{K^2}\sum_{1}^{K^2} f_{m,n_k'}\right|$$

$$\le C_m \frac{(K+1)^2 - K^2}{K^2} + \left|\frac{1}{K^2}\sum_{1}^{K^2} f_{m,n_k'}\right| .$$

Since this tends to 0 a.e. $[\mu]$ as $N \to \infty$, (2.2) follows. □

COROLLARY 2.9. Let μ be a positive measure and let C_m be constants for $m \in \mathbb{Z}^+$. Suppose that $f_{m,n} \in L^2(\mu)$ satisfy

(i) $|f_{m,n}(t)| \le C_m$ a.e. $[\mu]$,

(ii) $\|f_{m,n}\|_{L^2(\mu)} \le C_m$.

Then there exists $g_m \in L^2(\mu)$ and $n_k \uparrow \infty$ such that for every subsequence $\{n_k'\} \subset \{n_k\}$ and every m, (2.2) holds.

PROOF. For each m, the functions $\{f_{m,n}\}_{n=1}^{\infty}$ lie in the ball of $L^2(\mu)$ of radius C_m. Since the ball is weakly sequentially compact (Hutson and Pym [1, p. 160]), there is a subsequence of $\{f_{m,n}\}$, call it again $\{f_{m,n}\}$, such that $f_{m,n} \to g_m$ weakly for some $g_m \in L^2(\mu)$. By a diagonal argument, we can choose the subsequence so that this happens for all m simultaneously.

Lemma 2.8 now finishes the proof. □

REMARK 1. The abstract theorem that the unit ball in Hilbert space is weakly sequentially compact is not necessary for the proof. That is, a complete orthonormal basis of $L^2(\mathbb{T},|\mu|)$ is countable, so that we could directly extract a weakly convergent subsequence of $\{e(m\,n_k t)\}_{k=1}^{\infty}$. (Indeed, the same method of proof can be modified to prove sequential compactness in the general case.)

REMARK 2. Just as Lemma 2.8 is reformulated as Corollary 2.9, so could we reformulate Lemma 2.6 and Corollary 2.7 through using boundedness in norm rather than weak convergence.

PROOF OF LEMMA 2.5. If Corollary 2.9 is applied to the measure $|\mu|$ and the functions $f_{m,k}(t) = e(m\,n_k t)$ with constants $C_m = \max(1,\|\mu\|)$, then we obtain functions g_m and a subsequence $\{n_k'\} \subset \{n_k\}$ such that for every $\{n_k''\} \subset \{n_k'\}$ and for every m,

$$\frac{1}{K}\sum_{1}^{K} e(m\,n_k'' t) \to g_m(t) \quad \text{a.e. } [|\mu|] .$$

By Weyl's criterion, this is exactly what we wanted to prove. □

3. Abnormal Sets.

From the results in the previous section, we may easily show

THEOREM 3.1. For any measure μ , $\mu \in R$ if and only if $\mu E = 0$ for all A-sets E .

PROOF. By Theorem II.4.1(c), we may assume $\mu \geq 0$ in what follows. Let $\mu \in R$, I be an arc, $n_k \uparrow \infty$, and

$$E = \{x: \overline{\lim_{K\to\infty}} \frac{1}{K} \sum_{k=1}^{K} \chi_I(n_k x) < |I|\} .$$

Let $\nu = \mu|E \in R$. By the Rajchman-Milicer-Grużewska criterion (Theorem 1.1),

$$\lim_{K\to\infty} \int \frac{1}{K} \sum_{1}^{K} \chi_I(n_k x)\, d\nu(x) = |I|\, \hat{\nu}(0) = |I|\, \nu E .$$

Therefore Fatou's lemma applied to $1 - \frac{1}{K} \sum_{1}^{K} \chi_I(n_k x)$ and the definition of E give

$$|I|\, \nu E = \overline{\lim} \int_E \frac{1}{K} \sum_{1}^{K} \chi_I(n_k x)\, d\nu(x)$$

$$\leq \int_E \overline{\lim}\, \frac{1}{K} \sum_{1}^{K} \chi_I(n_k x)\, d\nu(x) \leq |I|\, \nu E .$$

Since equality holds in the last step, the definition of E shows that $\nu E = 0$, i.e., $\mu E = 0$. But all A-sets are contained in countable unions of sets of the form E (Proposition II.3.6), whence all A-sets have null

μ-measure.

The converse follows trivially from Theorem 2.1 since all W-sets are A-sets. □

The converse can also be proved directly from the Rajchman-Milicer-Grużewska criterion and Corollary 2.9. The proof follows the outlines of that of Lemma 2.5 and yields the following refinement of Theorem 3.1.

THEOREM 3.2. Given $\mu \notin R$ and any $\gamma \in (0,1)$, there exists an arc I of length γ or $1 - \gamma$ and $n_k \uparrow \infty$ such that

(i) $\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_I(n_k x)$ exists a.e. $[|\mu|]$

and

(ii) the A-set $\{x: \lim \frac{1}{K} \sum_{1}^{K} \chi_I(n_k x) < |I|\}$ has positive $|\mu|$-measure.

PROOF. By replacing μ with $|\mu|$ if necessary, we may assume that $\mu \geq 0$. By Corollary 1.3, there exists an arc J of length γ and $n_k \uparrow \infty$ such that

$$\int \chi_J(n_k x)\, d\mu(x) \to \alpha \neq |J| .$$

By Corollary 2.9, $\{n_k\}$ has a subsequence, call it again $\{n_k\}$, such that for some f ,

$$\frac{1}{K} \sum_{k=1}^{K} \chi_J(n_k x) \to f(x) \quad \text{a.e. } [\mu] .$$

Note that $\int f(x)\, d\mu(x) = \alpha$, so that

$$E = \{x: f(x) \neq |J|\}$$

has positive μ-measure. Let

$$E_1 = \{x: f(x) < |J|\} , \quad E_2 = \{x: f(x) > |J|\} .$$

Since $E = E_1 \cup E_2$, we have $\mu E_1 > 0$ or $\mu E_2 > 0$. In the first case, $I = J$ is the desired arc and E_1 the desired A-set. In the second case, $I = J^c$ is the desired arc and

$$E_2 = \{x: \lim \frac{1}{K} \sum_1^K \chi_{J^c} (n_k x) < |J^c|\}$$

the desired A-set . □

4. Non-normal Sets.

In this section we prove that W^*-sets have μ-measure zero for certain $\mu \in R$. In Section 6, we show that this does not hold for all $\mu \in R$, thus showing that the converse to Theorem II.4.2 is false. Section 5 is needed to establish some technical details in preparation for Section 6.

All the results of this section stem from the application of one idea. Before we formalize it in Theorem 4.1, let us give the motivation. In order to prove that for some measure $\mu \geq 0$, all W^*-sets have μ-measure 0 , it is necessary and sufficient to establish that

$$(4.1) \qquad \frac{1}{N} \sum_{k=1}^{N} e(n_k x) \to 0 \quad \text{a.e. } [\mu]$$

for all sequences $n_k \uparrow \infty$. If we let

$$(4.2) \qquad f_N(x) = \frac{1}{N} \sum_1^N e(n_k x) \quad ,$$

then (4.1) does <u>not</u> follow from $f_N \to 0$ in $L^2(\mu)$. However, if, say, $\|f_N\|^2_{L^2(\mu)} = O(\frac{1}{N})$, then the interpolation argument in the proof of Lemma 2.8 gives $f_{N^2} \to 0$ a.e. and then $f_N \to 0$ a.e. Since $\|f_N\|^2_{L^2(\mu)}$ is easy to compute in terms of $\hat{\mu}$, certain conditions on $\hat{\mu}$ will allow us to show (4.1) for certain $\{n_k\}$. The interpolation argument will also work in less restrictive situations than $\|f_N\|^2 = O(\frac{1}{N})$. We present this formally as

THEOREM 4.1. If $\mu \geq 0$ and

$$\sum_{N=1}^{\infty} \frac{1}{N} \left\| \frac{1}{N} \sum_{k=1}^{N} e(n_k x) \right\|_{L^2(\mu)}^2 < \infty ,$$

then

$$\frac{1}{N} \sum_{1}^{N} e(n_k x) \to 0 \quad \text{a.e. } [\mu] .$$

Davenport, Erdős and LeVeque [1] established an almost identical theorem. However, they were concerned with Lebesgue measure on the real line and used real-valued functions $\xi_k(x)$ instead of $n_k x$. Mendès France [1, p. 31] also states a very similar theorem. The proofs of all three theorems are the same and depend on a refinement (Lemma 4.3) of the principle of Cauchy condensation. Since we shall later need the latter principle as well, we include it here. Recall that a sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{Z}^+$ is said to be <u>lacunary</u> if there exists $q > 1$ such that $n_{k+1}/n_k \geq q$ for all k .

PROPOSITION 4.2 (Cauchy condensation). Suppose that $\{x_n\}_{n=1}^{\infty}$ is a weakly decreasing sequence of positive numbers: for some constant C , $n \leq m \leq 2n \Rightarrow x_m \leq C x_n$.

(i) If $\sum_{n=1}^{\infty} \frac{x_n}{n} < \infty$, then for all lacunary

$$\{n_k\}_{k=1}^{\infty} , \sum_{k=1}^{\infty} x_{n_k} < \infty .$$

(ii) If for some $\{n_k\}_{k=1}^{\infty}$ with $1 < n_{k+1}/n_k \leq Q < \infty$, we have $\sum_{k=1}^{\infty} x_{n_k} < \infty$, then $\sum_{n=1}^{\infty} \frac{x_n}{n} < \infty$.

PROOF. (i) Let $\sum x_n/n < \infty$ and let $n_{k+1}/n_k \geq q > 1$. By adding terms to the sequence $\{n_k\}$ if necessary, we may assume that $n_{k+1}/n_k \leq Q$ for some $Q < \infty$. Let $r \in \mathbb{Z}^+$ be such that $2^r \geq Q$ and let $C_1 = C^r$. Then $n \leq m \leq Qn$ implies

$$x_m \leq C_1 x_n ,$$

so that

$$\sum_{n=1}^{\infty} \frac{x_n}{n} = \sum_{k=1}^{\infty} \sum_{n_k \leq n < n_{k+1}} \frac{x_n}{n} \geq \sum_{k=1}^{\infty} \frac{n_{k+1} - n_k}{n_{k+1}} C_1^{-1} x_{n_{k+1}}$$

$$\geq (1 - q^{-1}) C_1^{-1} \sum_{k=2}^{\infty} x_{n_k} .$$

Therefore $\sum_{k=1}^{\infty} x_{n_k} < \infty$.

(ii) Let $1 < n_{k+1}/n_k \leq Q < \infty$ and let $\sum x_{n_k} < \infty$. If C_1 is as above, then

$$\sum_{n=1}^{\infty} \frac{x_n}{n} = \sum_{k=1}^{\infty} \sum_{n_k \leq n < n_{k+1}} \frac{x_n}{n} \leq \sum_{k=1}^{\infty} \frac{n_{k+1} - n_k}{n_k} C_1 x_{n_k}$$

$$\leq (Q - 1) C_1 \sum_{k=1}^{\infty} x_{n_k} .$$

Therefore $\sum_{n=1}^{\infty} x_n/n < \infty$. □

EXAMPLE. If $n_k = 2^k$ and $x_n \downarrow 0$, then

$$\sum_{n=1}^{\infty} \frac{x_n}{n} < \infty \Leftrightarrow \sum_{k=1}^{\infty} x_{2^k} < \infty .$$

The lemma we need to prove Theorem 4.1 is

LEMMA 4.3 (Davenport, Erdös and LeVeque [1]). If $x_n \geq 0$ and $\sum_{n=1}^{\infty} x_n/n < \infty$, then there exists $n_k \uparrow \infty$ such that $n_{k+1}/n_k \to 1$ and $\sum_{k=1}^{\infty} x_{n_k} < \infty$.

PROOF. We may assume that $x_n > 0$ for infinitely many n. Let $\lambda_n \uparrow \infty$ be any sequence of real numbers such that $\sum_{n=1}^{\infty} x_n \lambda_n/n < \infty$. (For example, if $R_N \equiv \sum_{n \geq N} x_n/n$, then $\lambda_n = (R_n^{1/2} + R_{n+1}^{1/2})^{-1}$ will do: since $R_n \downarrow 0$, we have $\lambda_n \uparrow \infty$ and

$$\sum \frac{x_n \lambda_n}{n} = \sum \frac{R_n - R_{n+1}}{R_n^{1/2} + R_{n+1}^{1/2}} = \sum (R_n^{1/2} - R_{n+1}^{1/2})$$

$$= R_1^{1/2} < \infty .)$$

Let $\{m_k\}_{k=1}^{\infty}$ be the sequence of positive integers defined inductively by $m_1 = 1$ and

$$m_{k+1} = \left[\frac{\lambda_{m_k}}{\lambda_{m_k} - 1} \cdot m_k \right] + 1 ,$$

where $[u]$ indicates the integer part of u. Let $n_k \in (m_k, m_{k+1}] \equiv I_k$ be such that

$$x_{n_k} = \min_{n \in I_k} x_n .$$

Then

$$x_{n_k} \leq \frac{1}{m_{k+1} - m_k} \sum_{n \in I_k} x_n \leq \frac{m_{k+1}}{m_{k+1} - m_k} \sum_{I_k} \frac{x_n}{n} .$$

Since

$$\frac{m_{k+1}}{m_{k+1} - m_k} < \lambda_{m_k} ,$$

it follows that

$$x_{n_k} \leq \sum_{I_k} \frac{x_n \lambda_n}{n} .$$

Summing both sides over k gives $\sum x_{n_k} < \infty$. Since $m_{k+1}/m_k \to 1$, we also have $n_{k+1}/n_k \to 1$. □

DEFINITION. A sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{Z}^+$ is said to be of __less-than-exponential growth__ if $\lim_{k \to \infty} n_{k+1}/n_k = 1$.

PROOF OF THEOREM 4.1. Define f_N as in (4.2). From the hypothesis

$$\sum_1^{\infty} \frac{1}{N} \|f_N\|^2 < \infty$$

and Lemma 4.3, there exists a sequence $\{N_r\}$ of less-than-exponential growth such that

$$\sum \|f_{N_r}\|^2 < \infty \quad .$$

That is,

$$\int \sum |f_{N_r}|^2 \, d\mu < \infty \quad ,$$

whence $f_{N_r} \to 0$ a.e. $[\mu]$. We now interpolate. Given M, let $N_r \leq M < N_{r+1}$. Then

$$|f_M(x)| \leq |f_M(x) - \frac{1}{M} \sum_1^{N_r} e(n_k x)| + |\frac{1}{M} \sum_1^{N_r} e(n_k x)|$$

$$= \frac{1}{M}| \sum_{N_r < k \leq M} e(n_k x)| + \frac{N_r}{M} |f_{N_r}(x)|$$

$$\leq \frac{N_{r+1} - N_r}{N_r} + |f_{N_r}(x)| \quad .$$

Since this last term $\to 0$ a.e. $[\mu]$ as $M \to \infty$, so does $f_M(x)$. □

Note that interpolation works exactly for those sequences $\{N_r\}$ of less-than-exponential growth.

Our first application of Theorem 4.1 is a slight generalization of a result of Baker [2] .

THEOREM 4.4. Suppose that $\phi(n)$ is a non-increasing function on the non-negative integers such that

$$(4.3) \qquad \sum_1^{\infty} \frac{\phi(n)}{n} < \infty \quad .$$

Then for any positive measure μ with

$$|\hat{\mu}(n)| \leq \phi(|n|) \quad ,$$

the μ-measure of every W*-set is 0.

PROOF. Let μ be as indicated and let $n_k \uparrow \infty$. Then

$$(4.4) \quad \|f_N\|^2_{L^2(\mu)} = \frac{1}{N^2} \int \sum_{k=1}^{N} e(-n_k x) \sum_{\ell=1}^{N} e(n_\ell x) \, d\mu(x)$$

$$= \frac{1}{N^2} \sum_{k,\ell} \hat{\mu}(n_k - n_\ell) = \frac{1}{N^2} \sum_{-\infty}^{\infty} r_N(m) \, \hat{\mu}(m) \quad ,$$

where

$$(4.5) \qquad r_N(m) = \#\{(k,\ell) | 1 \leq k \leq N \, , \, 1 \leq \ell \leq N \, , \, n_k - n_\ell = m\} \quad .$$

Note that $r_N(m) \leq N$ since for each k, there can be at most one ℓ such that $n_k - n_\ell = m$. By hypothesis,

$$\frac{1}{N^2} \sum_{-\infty}^{\infty} r_N(m) \, \hat{\mu}(m) \leq \frac{2}{N^2} \sum_0^{\infty} r_N(m) |\hat{\mu}(m)|$$

$$\leq \frac{2}{N^2} \sum_0^{\infty} r_N(m) \, \phi(m) \quad .$$

Now the sum of N^2 numbers chosen from $\{\phi(m)\}_0^\infty$, where each $\phi(m)$ may be chosen at most N times, is greatest when the N largest numbers are each chosen N times. Hence the last quantity above is

$$\leq \frac{2}{N} \sum_{n=1}^{N-1} \phi(n) .$$

It now follows that

$$\sum_1^\infty \frac{1}{N}\|f_N\|^2 \leq 2 \sum_1^\infty \frac{1}{N^2} \sum_0^{N-1} \phi(n)$$

$$= 2 \sum_0^\infty \phi(n) \sum_{n+1}^\infty \frac{1}{N^2} \leq 2 \sum_1^\infty \phi(n) \sum_{n+1}^\infty \frac{1}{N(N-1)} + 4\,\phi(0)$$

$$= 2 \sum_1^\infty \frac{\phi(n)}{n} + 4\,\phi(0) < \infty .$$

Theorem 4.1 applies to conclude the proof. □

REMARK. The same method of proof shows the following stronger theorem: Let $\{|\hat{\mu}|^*(n)\}_{n=0}^\infty$ denote the decreasing rearrangement of $\{|\hat{\mu}(n)|\}_{n=0}^\infty$. Then for any positive measure μ such that

$$\sum_{n=1}^\infty \frac{|\hat{\mu}|^*(n)}{n} < \infty ,$$

the μ-measure of every W*-set is 0 . We have stated Theorem 4.4 in the weaker form using $\phi(n)$ so that it is parallel to Theorem 4.5. The latter theorem probably does not extend in the same way to $|\hat{\mu}|^*(n)$.

If E is a W*-set corresponding to a lacunary sequence, we shall call E a lacunary W*-set . These are the most important W*-sets , as they include the original notion of non-normal numbers base q ; in this case, $n_k = q^{k-1}$, q an integer > 1 . A weaker hypothesis on the function ϕ in Theorem 4.3 will be enough to force all lacunary W*-sets to have measure 0 :

THEOREM 4.5. Suppose that $\phi(n)$ is a non-increasing function on the non-negative integers such that

$$(4.6) \qquad \sum_{n=2}^\infty \frac{\phi(n)}{n \log n} < \infty .$$

Then for any positive measure μ with $|\hat{\mu}(n)| \leq \phi(|n|)$, the μ-measure of every lacunary W*-set is 0 .

PROOF. Extend ϕ to be a non-increasing function on $[0,\infty)$. We have

$$\|f_N\|^2_{L^2(\mu)} = \frac{1}{N} + \frac{2}{N^2}\,\mathrm{Re} \sum_{1\leq \ell<k\leq N} \hat{\mu}(n_k - n_\ell)$$

$$\leq \frac{1}{N} + \frac{2}{N^2} \sum_{1\leq \ell<k\leq N} |\hat{\mu}(n_k - n_\ell)| .$$

Now

$$n_k - n_\ell \geq q^{k-\ell}\, n_\ell - n_\ell \geq q^{k-\ell} - 1 ,$$

whence

$$|\hat{\mu}(n_k-n_\ell)| \leq \phi(n_k-n_\ell) \leq \phi(q^{k-\ell} - 1) .$$

Therefore

$$\|f_N\|^2 \le \frac{1}{N} + \frac{2}{N} \sum_{1\le r<N} \phi(q^r - 1) ,$$

so that

$$\sum_{N=1}^{\infty} \frac{1}{N}\|f_N\|^2 \le \frac{\pi^2}{6} + 2 \sum_{N=1}^{\infty} \frac{1}{N^2} \sum_{1\le r<N} \phi(q^r - 1)$$

$$= \frac{\pi^2}{6} + 2 \sum_{r=1}^{\infty} \phi(q^r - 1) \sum_{N>n} \frac{1}{N^2}$$

$$\le \frac{\pi^2}{6} + 2 \sum_{r=1}^{\infty} \frac{\phi(q^r - 1)}{r} .$$

By the principle of Cauchy condensation (use $x_n = \phi(n)/\log n$ and $n_k = [q^{k-1}]$ in Proposition 4.2) and (4.6), this last sum is finite. □

REMARK. This type of argument shows that given any fixed condition on how slowly $\hat{\mu}$ may approach 0, such as (4.3) or (4.6), there will be a corresponding condition on how rapidly n_k must tend to ∞, such as lacunarity, so that the μ-measure of every W*-set corresponding to such a sequence $\{n_k\}$ be 0. Conversely, if the rate of growth of n_k is fixed, there will be a corresponding condition on the rate of decay of $\hat{\mu}$.

In Section 6, we shall see that Theorem 4.5 is best possible; it is not possible to weaken the hypothesis on ϕ without destroying the conclusion.

What other conditions might force every W*-set to be ν-null for some measure ν? For example, if every

W*-set is μ-null and $|\hat{\nu}| \le |\hat{\mu}|$, then is every W*-set ν-null? The answer is "no". Indeed, we know that every W*-set has Lebesgue measure 0. Let ν be any measure in R. There exists $f \in L^1(m)$ such that $|\hat{f}| \ge |\hat{\nu}|$ (Katznelson [1, pp. 22, 26]). Every W*-set is $f\,dm$-null, so if the question above had an affirmative answer, every W*-set would be ν-null, hence R-null. But as we said, this will prove to be untrue. However, as we shall see below in Corollary 4.7, if $f \in L^p(m)$, $p > 1$, and if $|\hat{\mu}| \le |\hat{f}|$ for a positive measure μ, then every W*-set is μ-null.

We begin with a result (Baker [1]) which, unlike Theorems 4.4 and 4.5, does not depend on a decreasing majorant of $|\hat{\mu}|$.

PROPOSITION 4.6. If μ is a positive measure and $\hat{\mu} \in \ell^q$ for some $q < \infty$, then every W*-set is μ-null.

PROOF. Without loss of generality, we may assume that $q > 1$. Keeping to the notation of (4.2) and (4.5), we have

$$\|f_N\|^2 = \frac{1}{N^2} \sum_{m=-\infty}^{\infty} r_N(m)\, \hat{\mu}(m) \le \frac{1}{N^2}\Big(\sum r_N(m)^p\Big)^{\frac{1}{p}} \Big(\sum|\hat{\mu}(m)|^q\Big)^{\frac{1}{q}}$$

by Hölder's inequality, where $\frac{1}{p} + \frac{1}{q} = 1$. Recall that

(4.7) $\quad \sum_m r_N(m) = N^2 , \qquad r_N(m) \le N .$

Therefore

$$\sum r_N(m)^p \le N^{p-1} \sum r_N(m) = N^{p+1}$$

and so

$$\|f_N\|^2 \le \frac{N^{1+\frac{1}{p}}}{N^2} \|\hat{\mu}\|_q = N^{-1/q} \|\hat{\mu}\|_q \quad .$$

The conclusion follows from Theorem 4.1. □

COROLLARY 4.7. If $f \in L^p(m)$, $p > 1$, and μ is a positive measure with $|\hat{\mu}| \le |\hat{f}|$, then every W*-set is μ-null .

PROOF. Without loss of generality, we may assume $p \le 2$. The Hausdorff-Young theorem says that $\hat{f} \in \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore $\hat{\mu} \in \ell^q$ and the result follows from the proposition. □

The argument of Proposition 4.6 can be modified through the introduction of weights for $\hat{\mu}$. However, the result may be a different type of theorem, since the hypothesis may no longer imply $\mu \in R$. An example of this type is

PROPOSITION 4.8 (Salem [3]). If $\mu \ge 0$,

$$\sum_1^\infty \frac{|\hat{\mu}(n)|^2}{n^{1-\alpha}} < \infty ,$$

$1 - \frac{1}{p} < \alpha \le 1$, $\{n_k\}_1^\infty$ are distinct, $n_k = O(k^p)$, then the maximal W*-set of $\{n_k\}$ is μ-null .

We complete the section by proving the following generalization of Lemma 4.3.

PROPOSITION 4.9. Let $x_n \ge 0$, $a_n \ge 0$, $S(n) = \sum_{k=1}^n a_k$. Assume that

(i) $\sum_1^\infty a_n = \infty$,

(ii) $\sum_1^\infty a_n x_n < \infty$,

and that

(iii) $a_{n+1} = o\ (S(n))$.

Then there exists n_k such that

(iv) $\sum_1^\infty S(n_k)\ x_{n_k} < \infty$

and

(v) $S(n_{k+1})/S(n_k) \to 1$.

NOTE. Lemma 4.3 (with the notation y_n in place of x_n there) is the special case $a_n = 1$, $x_n = y_n/n$. If we put $a_n = 1/n$ and $x_n = y_n/\log n$, then we get the following result: If $y_n \ge 0$ and

$$\sum_2^\infty \frac{y_n}{n \log n} < \infty ,$$

then there exists $\{n_k\}$ such that

$$\sum_1^\infty y_{n_k} < \infty \quad \text{and} \quad \log n_{k+1}/\log n_k \to 1 .$$

PROOF. We may assume that $a_1 > 0$. Choose any sequence $\lambda_n \to \infty$ such that

$$\sum a_n x_n \lambda_n < \infty .$$

(If $x_n = 0$ for all large n , take $\lambda_n = n$. Otherwise, we may choose

$$\lambda_n = (R_n^{1/2} + R_{n+1}^{1/2})^{-1} ,$$

where $R_N = \sum_{n \geq N} a_n x_n$, as in the proof of Lemma 4.3.)

Define $\{m_k\}$ inductively so that m_{k+1} is the least m for which

$$S(m) > \frac{\lambda_{m_k}}{\lambda_{m_k} - 1} S(m_k) .$$

It follows immediately that

$$(4.8) \qquad \lambda_{m_k}(S(m_{k+1}) - S(m_k)) \geq S(m_{k+1}) .$$

Now we claim that

$$(4.9) \qquad \frac{S(m_{k+1})}{S(m_k)} \to 1 .$$

Certainly the ratio is larger than 1 for all k . Also, by definition of m_{k+1} ,

$$S(m_{k+1} - 1) \leq \frac{\lambda_{m_k}}{\lambda_{m_k} - 1} S(m_k) .$$

Therefore, adding $a_{m_{k+1}}$ to both sides, we have

$$\frac{S(m_{k+1})}{S(m_k)} \leq \frac{\lambda_{m_k}}{\lambda_{m_k} - 1} + \frac{a_{m_{k+1}}}{S(m_k)} .$$

Since $\lambda_n \to \infty$, the first term $\lambda_{m_k}/(\lambda_{m_k} - 1)$ tends to 1 . Since $a_{n+1}/S(n) \to 0$, the second term tends to 0 :

$$\frac{a_{m_{k+1}}}{S(m_k)} = \frac{a_{m_{k+1}}}{S(m_{k+1} - 1)} \cdot \frac{S(m_{k+1} - 1)}{S(m_k)}$$

$$\leq \frac{a_{m_{k+1}}}{S(m_{k+1} - 1)} \cdot \frac{\lambda_{m_k}}{\lambda_{m_k} - 1} \to 0 .$$

Thus (4.9) is established.

Let $n_k \in (m_k , m_{k+1}] \equiv I_k$ be such that

$$S(n_k)\, x_{n_k} = \min\{S(n)x_n : n \in I_k\} .$$

Then for $n \in I_k$,

$$x_n \lambda_n = S(n)x_n \; S(n)^{-1} \lambda_n$$

$$\geq S(n_k)x_{n_k} \; S(m_{k+1})^{-1} \lambda_{m_k} ,$$

whence

$$\sum_{n\in I_k} a_n x_n \lambda_n \geq S(n_k)x_{n_k} \; S(m_{k+1})^{-1} \; \lambda_{m_k} \sum_{n\in I_k} a_n$$

$$= S(n_k)x_{n_k} \; S(m_{k+1})^{-1} \; \lambda_{m_k}(S(m_{k+1}) - S(m_k))$$

$$\geq S(n_k)x_{n_k}$$

by (4.8). Therefore

$$\sum_{k=1}^{\infty} S(n_k)x_{n_k} \leq \sum_{k=1}^{\infty} \sum_{n\in I_k} a_n x_n \lambda_n$$

$$= \sum_{n>m_1} a_n x_n \lambda_n < \infty \quad .$$

From (4.9) and $m_k < n_k \leq m_{k+1}$ follows (v). □

5. Infinite Product Measures and Infinite Convolutions

Some very interesting examples of measures can be most easily presented as infinite convolutions. Infinite convolutions may be defined as weak* limits of finite convolutions or as "projections" of infinite product measures. Since useful properties can be obtained from both points of view, our main task in this section is to establish the equivalence of the two definitions in certain cases of interest. Our secondary goal is to acquaint the reader with examples which pertain to Rajchman measures.

We begin with a familiar measure, the Cantor-Lebesgue measure μ . It is supported on the Cantor middle-thirds set E . The set E consists of those points $x \in [0,1]$ which have a base 3 representation using only the digits 0 and 2 . Viewing μ as a probability measure, we recall that for any $n \geq 1$, the probability that the n-th digit is 0 or 2 is $\frac{1}{2}$ in each case and that the values of the digits form independent random variables.

Now consider a random walk on $[0,\infty)$ beginning at 0 . For $n \geq 1$, the n-th step is 0 with probability $\frac{1}{2}$ or $2\cdot 3^{-n}$ with probability $\frac{1}{2}$. Let X_n be the amount of the n-th step and $X = \sum_1^{\infty} X_n$. The distribution measure of X_n is $\mu_n = \frac{1}{2}\,\delta(0) + \frac{1}{2}\,\delta(2\cdot 3^{-n})$. From probability theory, the distribution of X ought to be the infinite convolution $\tilde{\mu} = \mathop{\ast}\limits_{1}^{\infty} \mu_n$. Furthermore, it is intuively expected that

$\mu = \tilde{\mu}$. These are the ideas which we seek to clarify in the technicalities which follow.

Let $\{r_n\}_1^\infty$ be a sequence of positive real numbers such that $\sum r_n < \infty$. We denote the infinite direct product of $[-r_n, r_n]$ by

$$S = \bigtimes_{n=1}^{\infty} [-r_n, r_n] \quad .$$

On S , we define a metric as follows: if $x = \{x_n\}$, $y = \{y_n\}$, then

$$d(x,y) = |x-y| = \sum_1^\infty |x_n - y_n| \quad .$$

LEMMA 5.1. The metric topology on S coincides with the product topology.

The proof is virtually identical to that of Theorem 14 in Kelley [1, p. 122], so we omit it. We shall use the consequence that S is compact.

Let $\phi: S \to \mathbb{R}$ be the map

$$\phi(\{x_n\}) = \sum x_n \quad .$$

Then clearly

$$|\phi(x) - \phi(y)| \leq |x - y| \quad ,$$

so that ϕ is continuous.

Let μ_n be any probability measure on $[-r_n, r_n]$. Then

$$\nu_N = (\bigtimes_{n=1}^{N} \mu_n) \times (\bigtimes_{N+1}^{\infty} \delta(0)) \quad ,$$

$$\nu = \bigtimes_{n=1}^{\infty} \mu_n$$

are probability measures on S (Zaanen [1, pp. 98-99] or Hewitt and Stromberg [1, Section 22]). Recall that $\mathop{\ast}_{n=1}^{N} \mu_n$ is defined by the Riesz representation theorem to be the unique Borel measure satisfying

$$\int_{\bigtimes_1^N [-r_n, r_n]} f(\sum_1^N x_n) \, d(\bigtimes_1^N \mu_n) \, (x_1, \ldots, x_N)$$

$$= \int_{\mathbb{R}} f(t) \, d(\mathop{\ast}_1^N \mu_n) \, (t)$$

for $f \in C_o(\mathbb{R})$. This is equivalent to

(5.1) $$\int_S f \circ \phi \, d\nu_N = \int_{\mathbb{R}} f \, d(\mathop{\ast}_1^N \mu_n) \quad .$$

Note that the continuity of ϕ ensures that $f \circ \phi$ is continuous when f is. It also ensures that μ defined by $\mu(E) \equiv \nu(\phi^{-1}[E])$ is a Borel measure. Thus

(5.2) $$\int_S f \circ \phi \, d\nu = \int_{\mathbb{R}} f \, d\mu$$

for characteristic functions f, hence for all $f \in L^1(\mu)$ (cf. Royden [1, p. 318]). Note that $C_o(\mathbb{R}) \subset L^1(\mu)$.

Now by definition of ν (Zaanen [1, pp. 98-99]), if g is a simple function on S, then

(5.3) $$\int g \, d\nu_N \to \int g \, d\nu$$

as $N \to \infty$. We claim (5.3) holds also for all $g \in C(S)$. For let $g \in C(S)$, $\varepsilon > 0$, and h a simple function such that $\|g-h\|_{C(S)} < \varepsilon/3$. Such an h exists by compactness of S. If N is such that

$$\left|\int h \, d\nu_N - \int h \, d\nu\right| < \varepsilon/3 ,$$

then

$$\left|\int g \, d\nu_N - \int g \, d\nu\right| \leq \int |g-h| \, d\nu_N + \left|\int h \, d\nu_N - \int h \, d\nu\right| + \int |h-g| \, d\nu$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Thus (5.3) holds for $g \in C(S)$. In other words, $\nu_N \to \nu$ weak*. From (5.1), (5.2), and (5.3), it follows that $\mathop{\ast}\limits_{1}^{N} \mu_n \to \mu$ weak*. That is, the two possible definitions of infinite convolution coincide.

Given a function $f \in L^1(\nu)$, i.e., a random variable with finite expectation, we would like to find a function $g \in L^1(\mu)$ such that for all Borel $E \subset \mathbb{R}$,

(5.4) $$\int_E g \, d\mu = \int_{\phi^{-1}[E]} f \, d\nu .$$

To this end, set

$$\sigma(E) = \int_{\phi^{-1}[E]} f \, d\nu .$$

It is easily checked that σ is a complex Borel measure. Furthermore, if $\mu E = 0$, then $\nu(\phi^{-1}[E]) = 0$ by definition of μ, whence $\sigma E = 0$. That is, $\sigma \ll \mu$. Now the Radon-Nikodym theorem supplies the sought-for g.

Note that by (5.2),

$$\int_E g \, d\mu = \int_{\phi^{-1}[E]} g \circ \phi \, d\nu .$$

Comparing this with (5.4), we might expect that in certain cases $f = g \circ \phi$ a.e. $[\nu]$. Let us call ϕ <u>measure-copreserving</u> if for all Borel $F \subset S$, we have $\mu(\phi[F]) = \nu F$; note that for Borel F, the set $\phi[F]$ is μ-measureable since it is analytic (Arveson [1, pp. 64-67]). (In the standard terminology, our ϕ is already measure-preserving since $\nu(\phi^{-1}[E]) = \mu E$.)

We shall need the fact that if E is μ-measurable, then $\phi^{-1}[E]$ is ν-measurable. For by definition (Arveson [1, p. 67]), there exist Borel sets E_1, E_2 such that $E_1 \subset E \subset E_2$ and $\mu(E_2 \setminus E_1) = 0$. Since $\phi^{-1}[E_1] \subset \phi^{-1}[E] \subset \phi^{-1}[E_2]$ and, by definition of μ, $\nu(\phi^{-1}[E_2] \setminus \phi^{-1}[E_1]) \leq \nu(\phi^{-1}[E_2 \setminus E_1]) = \mu(E_2 \setminus E_1) = 0$, it follows that $\phi^{-1}[E]$ is ν-measurable.

PROPOSITION 5.2. If ϕ is measure-copreserving, then $f = g \circ \phi$ a.e. [ν] .

PROOF. For Borel $F \subset S$, let $\tilde{F} = \phi^{-1}[\phi[F]]$. Then, as noted above, $\phi[F]$ is μ-measurable and $\tilde{F}$ is ν-measurable. Thus ϕ is measure-copreserving if and only if $\nu(\tilde{F} \setminus F) = 0$ for all Borel F . In this case,

$$\int_F f \, d\nu = \int_{\tilde{F}} f \, d\nu = \int_{\phi[F]} g \, d\mu = \int_{\tilde{F}} g \circ \phi \, d\nu$$

$$= \int_F g \circ \phi \, d\nu .$$

Since F is arbitrary, the result follows. □

Recall that $f_1 , f_2 \in L^1(\nu)$ are said to be <u>independent</u> if for all Borel sets $D_1 , D_2 \subset \mathbb{E}$,

$$\int (\chi_1 \circ f_1)(\chi_2 \circ f_2) \, d\nu$$

$$= (\int \chi_1 \circ f_1 \, d\nu)(\int \chi_2 \circ f_2 \, d\nu) ,$$

where χ_i denotes the characteristic function of D_i .

PROPOSITION 5.3. Suppose that ϕ is measure-copreserving. Let $f_1 , f_2 \in L^1(\nu)$ be independent and let $f_i = g_i \circ \phi$ a.e. [ν] $(i = 1,2)$. Then g_1 and g_2 are independent.

PROOF. With the previous notation, we have

$$\int (\chi_1 \circ g_1)(\chi_2 \circ g_2) \, d\mu = \int [(\chi_1 \circ g_1)(\chi_2 \circ g_2)] \circ \phi \, d\nu$$

$$= \int (\chi_1 \circ g_1 \circ \phi)(\chi_2 \circ g_2 \circ \phi) \, d\nu = \int (\chi_1 \circ f_1)(\chi_2 \circ f_2) \, d\nu$$

$$= (\int \chi_1 \circ f_1 \, d\nu)(\int \chi_2 \circ f_2 \, d\nu)$$

$$= (\int \chi_1 \circ g_1 \, d\mu)(\int \chi_2 \circ g_2 \, d\mu) . \; \square$$

Of course, the same holds for any collection $\{f_i\}$ of independent random variables with finite expectation.

If for some Borel set $E \subset S$, the restriction $\phi | E$ is 1 - 1 , $\phi[E] \cap \phi[E^c] = \emptyset$, and $\nu E = 1$, we call ϕ <u>almost 1 - 1</u> . Note that if there exists a measurable such set E , then there exists a Borel one.

THEOREM 5.4. The map ϕ is almost 1 - 1 if and only if ϕ is measure-copreserving. In this case, if $E_o = \{x \in S : \forall y \in S \;\; \phi(x) \neq \phi(y)\}$, then E_o is Borel and $\nu E_o = 1$.

PROOF. Suppose that ϕ is almost 1 - 1 . Let $\phi | E$ be 1 - 1 with $\phi[E] \cap \phi[E^c] = \emptyset$ and $\nu E = 1$. Let $F \subset S$ be any Borel set. Then $\phi^{-1}[\phi[F \cap E^c]] \subset E^c$, whence

$$\mu(\phi[F \cap E^c]) = 0 \; .$$

Since $\phi^{-1}[\phi[F \cap E]] = F \cap E$, we also have

$$\mu(\phi[F \cap E]) = \nu(F \cap E) \; .$$

Hence

$$\mu(\phi[F]) = \mu(\phi[F\cap E] \cup \phi[F\cap E^c])$$
$$= \mu(\phi[F\cap E]) = \nu(F\cap E)$$
$$= \nu((F\cap E)\cup(F\cap E^c)) = \nu F \quad .$$

Therefore ϕ is measure-copreserving.

Conversely, suppose that ϕ is measure-copreserving. Let E_o be as in the theorem and $F_o = E_o^c$. Then F_o is an F_σ-set , hence Borel (Federer and Morse [1, Lemma 3.2]). By Theorem 5.1 of Federer and Morse [1], there exists a Borel set $B \subset S$ such that $\phi|B$ is 1 - 1 and $\phi[B] = \phi[S]$. If $G = F_o \setminus B$, then $\phi[G] = \phi[F_o]$ and $\phi[F_o\setminus G] = \phi[F_o]$. Hence

$$\nu G = \mu(\phi[G]) = \mu(\phi[F_o]) = \nu F_o$$

since ϕ is measure-copreserving and, likewise,

$$\nu(F_o\setminus G) = \nu F_o \ .$$

Addition of the last two equations and use of the relation $F_o = G \cup (F_o\setminus G)$ yields $\nu F_o = 2\nu F_o$, i.e., $\nu F_o = 0$. Therefore $\nu E_o = 1$. It is evident that $\phi|E_o$ is 1 - 1 and that $\phi[E_o] \cap \phi[E_o^c] = \emptyset$, so the proof is complete. □

REMARK: When E_o^c is countable, which is the only case we shall encounter, the theorem is elementary.

We are now ready to apply this theory to the example of the Cantor-Lebesgue measure. Let $\mu_n = \frac{1}{2}\,\delta(0) + \frac{1}{2}\,\delta(2\cdot 3^{-n})$. To construct a measure ν on the space S previously described, we should view μ_n as being supported in $[-2\cdot 3^{-n}, 2\cdot 3^{-n}]$. However, we may just as well view μ_n as being supported in $[0,2\cdot 3^{-n}]$ or even $\{0,2\cdot 3^{-n}\}$. Making the last choice, we define

$$S = \mathop{\times}_{n=1}^{\infty}\{0,2\cdot 3^{-n}\} \ .$$

Then ϕ is a 1 - 1 map of S into [0,1] . If f_n is the characteristic function of the set

$$\{x \in S:\ x_n = 2\cdot 3^{-n}\} \ ,$$

then $\{f_n\}$ are clearly independent, hence so are the corresponding $\{g_n\}$. The g_n are half the value of the n-th ternary digit of $t \in [0,1]$, whereas the f_n correspond to the X_n of the random walk described earlier. Since the probability that $g_n = 0$ is

$$\int (1 - g_n)\, d\mu = \int (1 - f_n)\, d\nu = \frac{1}{2} \ ,$$

our new description of the Cantor-Lebesgue measure μ as $\nu \circ \phi^{-1}$ or as $\mathop{\ast}_{1}^{\infty} \mu_n$ is indeed correct. Consequently, identifying $\mathbb{T}$ with $[0,1]$, we can immediately calculate

$$\hat{\mu}(k) = \prod_{n=1}^{\infty} \hat{\mu}_n(k) = \prod_{n=1}^{\infty} [\tfrac{1}{2} + \tfrac{1}{2} e(-2k \cdot 3^{-n})] ,$$

since the Fourier-Stieltjes coefficient of a convolution is the product of the individual Fourier-Stieltjes coefficients.

In a similar manner, the Cantor-Lebesgue measure on the Cantor set with dissection ratio $\theta^{-1} < \frac{1}{2}$ (Zygmund [1, I, pp. 194-195]) may be represented as

$$(5.5) \qquad \mu_\theta = \mathop{\ast}_{n=1}^{\infty} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta((\theta - 1)\theta^{-n})] .$$

It is well-known that μ_θ is a Rajchman measure if and only if θ is not a Pisot-Vijayaraghavan (P-V) number (Zygmund [1, II, pp. 147-152]).

In this example, if $\theta = 2$, then ϕ is no longer $1 - 1$ as a map from $S = \bigtimes_{n=1}^{\infty} \{0, 2^{-n}\}$ to $[0,1]$ because of the fact that there are countably many numbers having two binary representations. However, it is clear that ν is continuous, so that ϕ is almost $1 - 1$, hence measure-copreserving. Since $\mu = m$, we have

$$m = \mathop{\ast}_{1}^{\infty} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2^{-n})] .$$

By changing the probabilities that the n-th binary digit is 0 or 1 to p_n and q_n, respectively, where $p_n + q_n = 1$, we obtain a different useful class of measures:

$$(5.6) \qquad \mu = \mathop{\ast}_{n=1}^{\infty} [p_n \delta(0) + q_n \delta(2^{-n})] .$$

PROPOSITION 5.5. Let $S = \bigtimes_{n=1}^{\infty} \{0, 2^{-n}\}$, let $p_n + q_n = 1$, $p_n, q_n \geq 0$, let

$$\nu = \bigtimes_{n=1}^{\infty} [p_n \delta(0) + q_n \delta(2^{-n})],$$

and let μ be as in (5.6). Then $\phi:(S,\nu) \to (\mathbb{R},\mu)$ is measure-copreserving if and only if

$$(5.7) \qquad \sum_{n=1}^{\infty} p_n = \infty = \sum_{n=1}^{\infty} q_n .$$

PROOF. Let E_o be the set of $x \in S$ having infinitely many coordinates equal to 0 and infinitely many not equal to 0. If ϕ is measure-copreserving, then by Theorem 5.4, $\nu E_o = 1$. Since p_n is the ν-probability that the n-th coordinate of a point is 0 and q_n is the probability that it is not 0, the Borel-Cantelli lemma immediately implies (5.7).

Conversely, suppose (5.7) holds. Then by the Borel-Cantelli lemma, $\nu E_o = 1$. Since $\phi|E_o$ is $1 - 1$ and $\phi[E_o] \cap \phi[E_o^c] = \emptyset$, it follows by Proposition 5.4 that ϕ is measure-copreserving. □

For μ as in (5.6), Blum and Epstein [1] showed that $\mu \in R$ if and only if $p_n \to \frac{1}{2}$.

6. Non-normal Sets (continued)

Using a slight extension of the ideas of Section 5, we now show that not all Rajchman measures put zero mass on W*-sets and that, in fact, Theorem 4.5 is best possible.

THEOREM 6.1. There exists a Rajchman measure supported in the set of non-normal numbers base 2 .

We prove Theorem 6.1 by using the following construction.

THEOREM 6.2. Let $\{K_i\}_1^\infty$ be any strictly increasing sequence of integers and let $K_o = 1$. Let $\{\varepsilon_i\}_1^\infty$ be any sequence of real numbers such that

$$0 \le \varepsilon_i \le 1 , \quad \sum_1^\infty \varepsilon_i = \infty .$$

Let μ be the probability measure

$$(6.1) \quad \mu = \mathop{\ast}_{i=1}^{\infty} \{\varepsilon_i \delta(0) + (1-\varepsilon_i) \mathop{\ast}_{K_{i-1}<k\le K_i} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2^{-k})]\} .$$

Then for $2^{K_{i-1}-1} \le n < 2^{K_i-1}$,

$$(6.2) \quad |\hat{\mu}(n)| \le \varepsilon_i \varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i-1} + 2^{-(K_{i-1}-K_{i-2})}$$

and μ is concentrated on the set of x for which

$$(6.3) \quad \limsup_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_{[0,\frac{1}{2})}(2^{k-1}x) \ge 1 - \frac{1}{2} Q^{-1} ,$$

where $Q = \liminf K_{i+1}/K_i$.

If we take $\varepsilon_i \to 0$ and $K_i - K_{i-1} \to \infty$ in Theorem 6.2, then (6.2) implies $\mu \in R$. If $\{K_i\}$ is lacunary, then $Q > 1$, whence $1 - \frac{1}{2} Q^{-1} > \frac{1}{2}$. In this case, (6.3) implies that μ is concentrated on the set of non-normal numbers base 2. Therefore, if $\varepsilon_i \to 0$ and $Q > 1$, then μ is a Rajchman measure concentrated on the set of non-normal numbers base 2. A Rajchman measure ν supported in the set E of non-normal numbers base 2 may be obtained from the given μ as follows. By regularity of μ, there exists a closed set $F \subset E$ such that $\mu F > 0$. Let $\nu = \mu | F$.

Note that if $\{K_i\}$ is hyperlacunary, i.e., $K_{i+1}/K_i \to \infty$, then by (6.3),

(6.4) $$\overline{\lim_{K\to\infty}} \frac{1}{K} \sum_{k=1}^{K} \chi_{[0,\frac{1}{2})}(2^{k-1}x) = 1 \quad \text{a.e.} [\mu] .$$

Before proving Theorem 6.2, we demonstrate that Theorem 4.5 is best possible (Corollary 6.4). This requires a simple

PROPOSITION 6.3. Let $x_n \geq 0$, $y_n \geq 0$, $\Sigma x_n = \infty$, and $\Sigma y_n < \infty$. Then there exists a subsequence $N \subset \mathbb{Z}^+$ such that $\sum_{n \in N} x_n = \infty$ and $y_n \leq x_n$ for $n \in N$.

PROOF. It is trivial to check that

$$N = \{n: y_n \leq x_n\}$$

satisfies the desired conclusion. □

COROLLARY 6.4. If $\phi(n)$ is any non-increasing sequence such that

(6.5) $$\sum_{n=2}^{\infty} \frac{\phi(n)}{n \log n} = \infty ,$$

then there exists a positive Rajchman measure μ concentrated on the set of non-normal numbers base 2 with

(6.6) $$|\hat{\mu}(n)| \leq \phi(|n|) .$$

PROOF. We may assume that $\phi(0) = 1$. Let $\{K_i\}$ be any lacunary sequence such that $\overline{\lim}\, K_{i+1}/K_i < \infty$. From two uses of the principle of Cauchy condensation and from (6.5), we conclude that

(6.7) $$\sum \phi(2^{K_i}) = \infty .$$

By Proposition 6.3, there exists a subsequence of $\{K_i\}$, call it again $\{K_i\}$, such that (6.7) holds and

$$2^{-(K_{i-1}-K_{i-2})} \leq \frac{1}{2} \phi(2^{K_i}) .$$

Let $\varepsilon_i = \frac{1}{6} \phi(2^{K_{i+1}})$ in (6.1). Then (6.2) reduces to (6.6) since $\varepsilon_i \varepsilon_{i-1} \leq \varepsilon_{i-1}$. □

For the proof of Theorem 6.2, we shall use the following estimate.

LEMMA 6.5. Let $n \in \mathbb{Z}^+$, let $0 < \delta < \frac{1}{2}$, and let $K = [(\frac{1}{2} - \delta)n]$, where $[u]$ denotes the integer part of u. Then

$$2^{-n} \sum_{k=0}^{K} \binom{n}{k} \le e^{-2\delta^2 n} .$$

PROOF. For $0 \le x \le 1$, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \ge x^K \sum_{k=0}^{K} \binom{n}{k} ,$$

whence

$$\sum_{k=0}^{K} \binom{n}{k} \le (1+x)^n x^{-K} .$$

Choosing $x = K/(n-K)$, we obtain

$$I \equiv 2^{-n} \sum_{k=0}^{K} \binom{n}{k} \le 2^{-n} \left(\frac{n}{n-K}\right)^n \left(\frac{n-K}{K}\right)^K .$$

Define δ_1 so that $K = (\frac{1}{2} - \delta_1)n$. Then $n - K = (\frac{1}{2} + \delta_1)n$.
Also define

$$f(x) = (1+x) \log (1+x) + (1-x) \log(1-x)$$

for $|x| \le 1$. Then

$$I \le 2^{-n} \left\{ (\tfrac{1}{2} + \delta_1)^{\frac{1}{2}+\delta_1} (\tfrac{1}{2} - \delta_1)^{\frac{1}{2}-\delta_1} \right\}^{-n}$$
$$= \exp(-\tfrac{n}{2} f(2\delta_1)) .$$

Now

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n(2n-1)} \ge x^2 ,$$

whence

$$I \le e^{-2n\delta_1^2} \le e^{-2n\delta^2}$$

since $\delta_1 \ge \delta$. □

PROOF OF THEOREM 6.2 We begin by describing some properties of μ . The measure μ is not as simple an infinite convolution as the Cantor-Lebesgue measure of Section 5. There, each binary digit was independent. Here, only blocks of digits are independent (cf. (6.8) below), where the i-th block consists of the k-th digits for $K_{i-1} < k \le K_i$.
Let us denote the k-th digit of x by

$$r_k(x) = \chi_{[\frac{1}{2},1)}(2^{k-1}x) .$$

Thus

$$x = \sum_{k=1}^{\infty} r_k(x)\, 2^{-k} .$$

Let

$$\bar{r}_k(x) = 1 - r_k(x) = \chi_{[0,\frac{1}{2})}(2^{k-1}x) ,$$

$$B_i = \{k: K_{i-1} < k \le K_i\} , \quad b_i = K_i - K_{i-1} ,$$

$$R_i(x) = b_i^{-1} \sum_{k \in B_i} r_k(x) ,$$

and

$$\bar{R}_i(x) = 1 - R_i(x) .$$

Using the random-walk interpretation of μ, say, from Section 5, we see that μ may be alternatively described as the (continuous) probability measure satisfying (6.8)-(6.10):

(6.8) Given any $a_k = 0,1 \ (k \in B_i)$,

$b_j = 0,1 \ (j \in B_{i'})$, $i \neq i'$, the events

$\{x: r_k(x) = a_k, \ k \in B_i\}$,

$\{x: r_j(x) = b_j, \ j \in B_{i'}\}$

are independent.

(6.9) $$\mu\{x: R_i = 0\} = \varepsilon_i + (1 - \varepsilon_i)\, 2^{-b_i} .$$

(6.10) Given $a_k = 0,1 \ (k \in B_i)$ not all 0,

$$\mu\{x: r_k(x) = a_k, \ k \in B_i\} = (1 - \varepsilon_i)\, 2^{-b_i} .$$

Now by (6.8) and (6.9), the events $\{x: R_i(x) = 0\}$ are independent and the sum of their probabilities is infinite. Thus, the Borel-Cantelli lemma yields

$$\mu(\limsup_i \{x: R_i(x) = 0\}) = 1 .$$

Hence

(6.11) $$\limsup_{i\to\infty} \overline{R}_i(x) = 1 \quad \text{a.s.}$$

It follows that

$$\limsup_{i\to\infty} \frac{1}{K_i - K_{i-1}} \sum_{k=1}^{K_i} \overline{r}_k(x) \geq 1 \quad \text{a.s.,}$$

whence

$$\limsup_{i\to\infty} \frac{1}{K_i} \sum_{k=1}^{K_i} \overline{r}_k(x) \geq 1 - Q^{-1} \quad \text{a.s.}$$

If $Q > 2$, then we see already that μ is concentrated on $W^*(\{2^{k-1}\})$. The argument which follows is necessary only to prove the better estimate (6.3), which yields such a result for all $Q > 1$.

We now claim that

(6.12) $$\liminf_{i\to\infty} \overline{R}_i(x) \geq \tfrac{1}{2} \quad \text{a.s.}$$

For let $0 < \delta < 1/2$ and define

$$F_i = \{x: \overline{R}_i(x) \leq \tfrac{1}{2} - \delta\} .$$

Then

$$\mu F_i = (1 - \varepsilon_i)\, 2^{-b_i} \sum_{m \leq (\frac{1}{2} - \delta) b_i} \binom{b_i}{m} .$$

By Lemma 6.5,

$$\mu F_i \leq (1 - \varepsilon_i)\, e^{-2\delta^2 b_i} ,$$

whence $\Sigma \mu F_i < \infty$. By the Borel-Cantelli lemma, this implies

$$\mu(\limsup_i F_i) = 0 ,$$

whence

$$\liminf_{i\to\infty} \overline{R}_i(x) \geq \frac{1}{2} - \delta \quad \text{a.s.}$$

Since this is true for every $\delta > 0$, we deduce (6.12).

Therefore

$$(6.13)\quad \liminf_{i\to\infty} \frac{1}{K_i} \sum_{k=1}^{K_i} \overline{r}_k(x) = \liminf_{i\to\infty} \frac{\sum_{j=1}^{i} b_j \overline{R}_j(x)}{\sum_{j=1}^{i} b_j} \geq \frac{1}{2} \quad \text{a.s.}$$

Consider

$$\frac{1}{K_i} \sum_{k=1}^{K_i} \overline{r}_k(x) = \frac{K_{i-1}}{K_i}\left(\frac{1}{K_{i-1}} \sum_{k=1}^{K_{i-1}} \overline{r}_k(x)\right) + \frac{K_i - K_{i-1}}{K_i} \overline{R}_i(x) .$$

Let x be any point for which (6.11) and (6.13) hold. Given $\delta > 0$, take i so large that $\overline{R}_i(x) \geq 1 - \delta$ and

$$\frac{1}{K_{i-1}} \sum_{k=1}^{K_{i-1}} \overline{r}_k(x) \geq \frac{1}{2} - \delta .$$

Then

$$\frac{1}{K_i} \sum_{k=1}^{K_i} \overline{r}_k(x) \geq \frac{K_{i-1}}{K_i}\left(\frac{1}{2} - \delta\right) + \left(1 - \frac{K_{i-1}}{K_i}\right)(1 - \delta)$$

$$= 1 - \frac{1}{2}\frac{K_{i-1}}{K_i} + O(\delta) .$$

Letting $i \to \infty$ and $\delta \to 0$, we conclude that (6.3) holds for this x . Since (6.11) and (6.13) hold almost surely, so does (6.3).

Before turning to the proof of (6.2), we indicate why, if $\varepsilon_i \to 0$, we should expect from heuristic considerations that $\mu \in R$. Note that for any dyadic interval

$$I = \left[\sum_{k=1}^{K} a_k 2^{-k} , \sum_{k=1}^{K} a_k 2^{-k} + 2^{-K}\right) ,$$

we have

$$\int \chi_I(2^j x)\, d\mu(x) = \mu\{x: r_{j+k}(x) = a_k \text{ for } 1 \leq k \leq K\}$$

$$\to 2^{-K} = |I| \quad \text{as} \quad j \to \infty .$$

That is, the Rajchman-Milicer-Grużewska criterion is satisfied for dyadic intervals and the sequence $\{2^j\}$. Heuristically, this is all that should matter in determining for this measure whether the criterion is satisfied in full. In other words, it should follow that $\mu \in R$.

We now demonstrate (6.2). First, it is clear that

$$(6.14)\quad \hat{\mu}(n) = \prod_{i=1}^{\infty} \{\varepsilon_i + (1-\varepsilon_i) \prod_{k\in B_i} (\tfrac{1}{2} + \tfrac{1}{2} e(-n2^{-k}))\} .$$

If we multiply $\prod_{k\in B_i} (1+e(-n2^{-k}))$ by $(1-e(-n2^{-K_i}))$ and use the fact that

$$(1 - e(u))\,(1 + e(u)) = 1 - e(2u)$$

repeatedly, we find that the product telescopes:

$$\prod_{k\in B_i} (1+e(-n2^{-k})) = \frac{1-e(-n2^{-K_{i-1}})}{1-e(-n2^{-K_i})}$$

$$= e(-n\gamma_i) \frac{\sin \pi n 2^{-K_{i-1}}}{\sin \pi n 2^{-K_i}} ,$$

where $\gamma_i = 2^{-K_{i-1}-1} - 2^{-K_i-1}$. Substitution of this into (6.14) yields

$$\hat{\mu}(n) = \tag{6.15}$$

$$\prod_{i=1}^{\infty} \left\{ \varepsilon_i + (1-\varepsilon_i)\, 2^{-(K_i-K_{i-1})} e(-n\gamma_i) \frac{\sin \pi n 2^{-K_{i-1}}}{\sin \pi n 2^{-K_i}} \right\}.$$

Since the modulus of the i-th factor is at most 1 by (6.14), we may estimate $|\hat{\mu}(n)|$ by the product of the i-th and (i-1)-th factors only. Doing this, and noting that the second term of the i-th factor also has modulus ≤ 1 by (6.14), we see that

$$|\hat{\mu}(n)| \leq \varepsilon_i \varepsilon_{i-1} + \varepsilon_i + \varepsilon_{i-1}$$

$$+ 2^{-(K_i-K_{i-2})} \left| \frac{\sin \pi n 2^{-K_{i-2}}}{\sin \pi n 2^{-K_i}} \right| . \tag{6.16}$$

For $2^{K_{i-1}-1} \leq n < 2^{K_i-1}$,

$$\pi\, 2^{K_{i-1}-K_i-1} \leq \pi\, n\, 2^{-K_i} < \frac{\pi}{2} .$$

Therefore

$$\sin \pi n 2^{-K_i} \geq \sin \pi 2^{K_{i-1}-K_i-1} \geq 2^{K_{i-1}-K_i} . \tag{6.17}$$

Here we have used the well-known fact that $\sin \pi\theta \geq 2\theta$ for $\theta \in [0,\frac{1}{2})$. Substituting (6.17) into (6.16) and estimating the sine in the numerator by 1 , we obtain (6.2). □

We remark that the notion of infinite convolution could, if desired, be avoided by defining μ via (6.8)-(6.10). However, we would then be led to a rather ugly proof of (6.14). This would follow the lines of the calculation of the Fourier-Stieltjes coefficients of the Cantor-Lebesgue measure appearing, for example, in Zygmund [1, I, p. 195].

In Section IV.3, we shall see that a measure of the form (5.6) would not suffice for Theorem 6.1. This is why the more complicated measure (6.1) was used.

It is interesting that (6.4) can be achieved for any lacunary $\{K_i\}$ by a suitable choice of $\{\varepsilon_i\}$. To see this, let $i_r \uparrow \infty$ and $\varepsilon_{i_r} \to 0$ be sequences such that $i_{r+1} - i_r \to \infty$ and

$$\sum_{r=1}^{\infty} \varepsilon_{i_r}^{\,i_{r+1}-i_r} = \infty \ ;$$

for example, let j_r be the integer part of $(\log r)^{1/2}$,

$$i_R = \sum_{r<R} j_r ,$$

$$\varepsilon_{i_r} = r^{-(1/j_r)} \le e^{-j_r} .$$

Define $\varepsilon_i = \varepsilon_{i_r}$ for $i_r \le i < i_{r+1}$. Then

$$\sum_{r=1}^{\infty} \varepsilon_{i_r}\varepsilon_{i_r+1} \cdots \varepsilon_{i_{r+1}-1} = \infty ,$$

so that the sum of the probabilities of the independent events

$$\{x: R_{i_r}(x) = R_{i_r+1}(x) = \dots = R_{i_{r+1}-1}(x) = 0\}$$

is infinite. As in the proof of (6.11), it follows that

$$\limsup_{r\to\infty} \frac{1}{K_{i_{r+1}} - K_{i_r}} \sum_{k=K_{i_r}}^{K_{i_{r+1}}-1} \overline{r}_k(x) = 1 \quad \text{a.s.}$$

Since $K_{i_r}/K_{i_{r+1}} \to 0$, we deduce (6.4).

7. Helson Sets and Weak Dirichlet Sets

We shall not be concerned with sets of asymptotic distribution in this section. Instead, we shall consider some types of sets whose definitions are very natural from the perspective of our subject matter. These types of sets also arise in other important areas of harmonic analysis, where they are usually given a different, though equivalent, definition. In Section 8, the essential results of this section will be used again in discussing H-sets . Other intimate connections to H-sets will be exposed in Chapter IV.

NOTATION. Let $E \subset \mathbb{T}$ be a Borel set. We denote the set of measures $\mu \in M(\mathbb{T})$ which are concentrated on E (i.e., $|\mu|(E) = \|\mu\|$) by $M(E)$. The subset of positive measures on E is denoted $M^+(E)$.

This definition is slightly non-standard. One usually defines $M(E)$ and $M^+(E)$ only for closed sets E , in which case one can say that $\mu \in M(E)$ is supported on E , not merely concentrated on E . However, in our context, it is unnatural to restrict attention to the closed sets.

DEFINITION. Given a Borel set $E \subset \mathbb{T}$, the number

$$s(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|} : 0 \ne \mu \in M(E)\right\}$$

is called the Helson constant of E . If $s(E) = s > 0$, then E is said to be a Helson set or, if we wish to specify the constant, a Helson-s set.

This definition is not the ordinary one. Again, it is usual to consider only closed sets. In that case, our definition of Helson set is equivalent to the usual one (Lindahl and Poulsen [1, p. 16]), although the Helson constant is different. Körner (see, e.g., [1, p. 225]) also prefers our definition. Since we shall have no need for the classical definition of Helson sets, we shall not present it here.

DEFINITION. For a Borel subset $E \subset \mathbb{T}$, let

$$s^+(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|} : 0 \neq \mu \in M^+(E)\right\} .$$

Clearly $s(E) \leq s^+(E)$, so that Helson sets are included among the class of sets for which $s^+(E) > 0$. In particular, the class of sets for which $s^+(E) = 1$ includes the Helson-1 sets. We shall now show that the sets E with $s^+(E) = 1$ are precisely the weak Dirichlet sets, which are defined as follows.

DEFINITION. A Borel set $E \subset \mathbb{T}$ is called a Dirichlet set if

$$\liminf_{|n|\to\infty} \|e(nx)-1\|_{L^\infty(E)} = 0 .$$

A Borel set E is a weak Dirichlet set if for all $\mu \in M^+(E)$ and all $\varepsilon > 0$, there exists $E_1 \subset E$ which is a Dirichlet set with $\mu(E \setminus E_1) < \varepsilon$.

Again, it is customary in these definitions to require E to be closed (Lindahl and Poulsen [1, pp. 1, 148]), but this is unnecessary.

Note that since $|e(nx)| = 1$, each of the following conditions is equivalent to E being a Dirichlet set:

(i) $\liminf_{|n|\to\infty} \|1 - \cos 2\pi nx\|_{L^\infty(E)} = 0.$

(ii) $\liminf_{|n|\to\infty} \|\sin 2\pi nx\|_{L^\infty(E)} = 0.$

Appropriate use of the following lemma is key to our proof that E is weak Dirichlet if and only if $s^+(E) = 1$.

LEMMA 7.1. If μ is a probability measure and θ_n is defined by $|\hat{\mu}(n)| = e(\theta_n)\,\hat{\mu}(n)$, then

(7.1) $$\mathrm{Re}(e(\theta_n-\theta_m)\,\hat{\mu}(n-m)) \geq \tfrac{1}{2}(|\hat{\mu}(n)| + |\hat{\mu}(m)|)^2 - 1$$

for all n , m .

PROOF. The arithmetic-quadratic mean inequality (or the Cauchy-Buniakowski-Schwarz inequality) yields

$$(|\hat{\mu}(n)| + |\hat{\mu}(m)|)^2 = \left|\int (e(\theta_n)\,e(-nt) + e(\theta_m)\,e(-mt))\,d\mu(t)\right|^2$$

$$\leq \int |(\cdots)|^2\,d\mu(t) = 2(1+\mathrm{Re}(e(\theta_n-\theta_m)\,\hat{\mu}(n-m))) . \quad \square$$

REMARK. For an improved inequality to (7.1), see Section IV.7.

We shall use the following corollary of (7.1).

PROPOSITION 7.2. If $\mu \in M^+(\mathbb{T})$ and $R(\mu) = \|\mu\|$, then there exists $n_k \uparrow \infty$ such that $\hat{\mu}(n_k) \to \|\mu\|$.

PROOF. We may assume that $\|\mu\| = 1$. Let $|\hat{\mu}(m_k)| \to 1$, $m_k \uparrow \infty$. If $|\hat{\mu}(m_k)| = e(\theta_k)\,\hat{\mu}(m_k)$, then we may assume, by taking a subsequence if necessary, that θ_k converges and that $(m_{k+1} - m_k) \uparrow \infty$. From (7.1), we have

$$\varlimsup_{k\to\infty} \operatorname{Re}(e(\theta_{k+1} - \theta_k)\,\hat{\mu}(m_{k+1} - m_k)) = 1 .$$

Since $e(\theta_{k+1} - \theta_k) \to 1$, it follows that

$$\varlimsup_{k\to\infty} \operatorname{Re}\,\hat{\mu}(m_{k+1} - m_k) = 1 .$$

Let $n_k = m_{k+1} - m_k$. □

PROPOSITION 7.3. If E is a Dirichlet set, then $s^+(E) = 1$.

PROOF. Let $\mu \in M^+(E)$ and $\varepsilon > 0$. Choose n such that $\|e(nx)-1\|_{L^\infty(E)} < \varepsilon$. Then

$$|\hat{\mu}(n) - \|\mu\|\,| = |\int (e(-nx) - 1)\, d\mu|$$

$$\leq \int |e(nx) - 1|\, d\mu < \varepsilon \cdot \|\mu\| .$$

Therefore $R(\mu) = \|\mu\|$ and $s^+(E) = 1$. □

THEOREM 7.4. Let $E \subset \mathbb{T}$ be Borel. The following are equivalent.

(i) $s^+(E) = 1$.

(ii) E is weak Dirichlet.

PROOF. Assume (i) and let $\mu \in M^+(E)$. For convenience, take $\|\mu\| = 1$. Let $f_n(x) = 1 - \cos 2\pi nx$. Then

$$\int |f_n|\, d\mu = |\int f_n\, d\mu| = |\operatorname{Re}\int (1-e(-nx))\, d\mu(x)|$$

$$\leq |\int (1-e(-nx))\, d\mu(x)| = |1 - \hat{\mu}(n)| .$$

By Proposition 7.2, there exists $n_k \uparrow \infty$ such that $\hat{\mu}(n_k) \to 1$. By the above inequality, we conclude $f_{n_k} \to 0$ in $L^1(\mu)$. Hence, there exists a subsequence $f_{n_k'} \to 0$ a.e. $[\mu]$. By Egorov's theorem, $f_{n_k'} \to 0$ uniformly except on a set of arbitrarily small measure. Therefore E is a weak Dirichlet set.

Conversely, assume (ii). Let $\mu \in M^+(E)$, $\varepsilon > 0$, and $E_1 \subset E$ be a Dirichlet set with $\mu(E \backslash E_1) < \varepsilon$. Set $\nu = \mu|E_1$. By Proposition 7.3, $R(\nu) = \|\nu\|$. Since $|\,\|\mu\| - \|\nu\|\,| < \varepsilon$ and $|R(\mu) - R(\nu)| < \varepsilon$, it follows that $|R(\mu) - \|\mu\| \,| < 2\varepsilon$. Therefore $R(\mu) = \|\mu\|$ and (i) holds. □

Additional information on weak Dirichlet sets and on sets with $s^+(E) > 0$ is given in Section IV.5.

We now show that if $s^+(E) > 0$, then $E \in U_o$. On the other hand, we shall also show that given $s > 0$, there exists $\mu \notin R$ such that $\mu E = 0$ for all E with $s^+(E) \geq s$. Thus, if C is the class of sets E with $s^+(E) \geq s$, it follows that C is contained in U_o , but C does not characterize R . The same is therefore true of Helson sets of constant $\geq s$ and of weak Dirichlet sets, since these are but particular subclasses.

We remark that countable sets are weak Dirichlet sets. This follows from the fact that finite sets are Dirichlet sets (Lindahl and Poulsen [1, p.3]). That countable sets are weak Dirichlet sets also follows from the following two facts: countable sets are so-called N-sets (Zygmund [1, I, p. 236]) and N-sets are weak Dirichlet sets (Theorem IV.5.20). Although countable sets are not necessarily Helson sets (Graham and McGehee [1, p. 340]), clearly singletons (and finite sets) are Helson sets.

PROPOSITION 7.5. If $s^+(E) > 0$ and $\mu \in R$, then $\mu E = 0$.

PROOF. If $\mu \in R$ and $\mu E \neq 0$, let $\nu = |(\mu|E)|$. Then $\nu \in R$ and $0 \neq \nu \in M^+(E)$. It follows that $s^+(E) = 0$. □

NOTATION. For $\mu \in M(\mathbb{T})$, we denote

$$\|\hat{\mu}\|_\infty = \|\hat{\mu}\|_{\ell^\infty} = \sup_{n\in\mathbb{Z}} |\hat{\mu}(n)| .$$

THEOREM 7.6. Let μ be the Riesz product

$$(7.2) \qquad d\mu = \prod_{k=1}^{\infty} (1 + \alpha_k \cos 2\pi(n_k x + \phi_k))\, dm$$

with $-1 \leq \alpha_k \leq 1$ and

$$(7.3) \qquad n_{k+1}/n_k \geq q > 3 .$$

Then for all $\nu \ll \mu$,

$$(7.4) \qquad R(\nu) = R(\mu)\, \|\hat{\nu}\|_\infty .$$

In particular, if $\nu \geq 0$, then

$$(7.5) \qquad R(\nu) = R(\mu)\, \|\nu\| .$$

Furthermore, if $0 \leq \nu \ll \mu$ and $\{n_k'\}$ is any sequence for which $|\hat{\mu}(n_k')| \to R(\mu)$, then $|\hat{\nu}(n_k')| \to R(\nu)$ and $\{n_k'\}$ is a subsequence of $\{n_k\}$.

Background material on Riesz products is presented in Katznelson [1, Section 5.1.3, pp. 106-107] and Zygmund [1, I, Chap. V, §7, pp. 208-209]. Recall that Riesz products are probability measures.

COROLLARY 7.7. If μ is as in Theorem 7.6 with $\limsup_{k\to\infty} |\alpha_k| = s > 0$, then $\mu \notin R$ yet $\mu E = 0$ for all E with $s^+(E) > s/2$.

PROOF. The hypotheses imply that $R(\mu) = s/2$. If $\mu E > 0$, let $\nu = \mu|E$. Then by (7.5), $R(\nu)/\|\nu\| = s/2$, whence $s^+(E) \leq s/2$. □

Note that Corollary 7.7 leaves open the possibility that if $\mu E = 0$ for all E with $s^+(E) > 0$, then $\mu \in R$. In particular, the class of Helson sets could characterize R.

Before presenting the proof of Theorem 7.6, we give a sketch of the proof. It will suffice to consider ν of the form $d\nu = P\, d\mu$, where P is a trigonometric polynomial. If N is the degree of P, we easily calculate

$$\hat{\nu}(m) = \sum_{|r| \leq N} \hat{P}(-r)\, \hat{\mu}(r + m)\ .$$

Now $\hat{\mu}(r+m) = 0$ unless $r + m$ has the form

$$r + m = \sum_{k=1}^{k_o} \epsilon_k n_k\ , \qquad \epsilon_k = -1, 0, 1\ ,$$

where $\epsilon_{k_o} \neq 0$ and k_o depends on $r + m$. For sufficiently large m, the leading term is $\pm n_{k_o}$. Furthermore, we shall show that for large m, the index k_o of the leading term is the same for all $r \in [-N,N]$ such that $r + m$ has the form above. This is due to (7.3) and is the key observation, since it will lead to

$$\hat{\mu}(r+m) = \hat{\mu}(n_{k_o})\, \hat{\mu}(r+m-n_{k_o})$$

for $|r| \leq N$ and large m. From the previous calculation of $\hat{\nu}(m)$, we shall conclude that

$$\hat{\nu}(m) = \hat{\mu}(n_{k_o})\, \hat{\nu}(m - n_{k_o})\ ,$$

whence $R(\nu) \leq R(\mu)\, \|\hat{\nu}\|_\infty$. On the other hand, by using $m = n_k$, we shall see that

$$\hat{\nu}(n_k) = \hat{\mu}(n_k)\, \hat{\nu}(0)\ ,$$

whence $R(\nu) \geq R(\mu)|\hat{\nu}(0)|$. Substituting $e(mt)\, d\nu(t)$ for $d\nu(t)$ yields (7.4).

We now turn to the details.

LEMMA 7.8. Let μ be as in Theorem 7.6. Define $n_{-k} = n_k$ for $k \in \mathbb{Z}^+$ and define $n_o = 0$. There exist functions $K_o\colon \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ and $M_o\colon \mathbb{N} \to \mathbb{N}$ having the following properties:

(i) If P is a trigonometric polynomial of degree at most N, if $d\nu = P\, d\mu$, and if $|m| \geq M_o(N)$, then

$$\hat{\nu}(m) = \hat{\mu}(n_{K_o})\, \hat{\nu}(m - n_{K_o})\ , \tag{7.6}$$

where $K_o = K_o(N,m)$. If $K_o = 0$, then $\hat{\nu}(m) = 0$.

(ii) For each N,

$$\lim_{\substack{|m| \to \infty \\ K_o(N,m) \neq 0}} K_o(N,m) = \infty\ . \tag{7.7}$$

(iii) For $k \geq 1$ and all N,

(7.8) $$K_o(N, n_k) = k \quad .$$

PROOF. Fix N. Let Σ be the set of $m \in \mathbb{Z}$ having the form

(7.9) $$m = \sum_{k=1}^{k_o(m)} \varepsilon_k(m)\, n_k \ ; \ \varepsilon_k(m) = -1, 0, 1 \ ; \ \varepsilon_{k_o(m)}(m) \neq 0 \ .$$

It is well-known that because of (7.3), any m has at most one representation of the form (7.9). Given m, let r be an integer with least absolute value satisfying $r + m \in \Sigma$. Let

$$K_o(N,m) = \begin{cases} k_o(r+m) & \text{if } |r| \leq N , \\ 0 & \text{if } |r| > N . \end{cases}$$

Note that

(7.10) $$K_o(N,-m) = K_o(N,m) \ .$$

That (7.7) and (7.8) are satisfied is clear. Now if $0 < m \in \Sigma$, then $\varepsilon_{k_o(m)}(m) = 1$. Also, we have $\lim_{m \to \infty} k_o(m) = \infty$. Hence, choose $M_o(N)$ sufficiently large that if $m \geq M_o(N)$, $|r| \leq N$, $r + m \in \Sigma$, and $k_o = k_o(r+m)$, then

(7.11) $$\varepsilon_{k_o}(r + m) = 1 ,$$

(7.12) $$n_{k_o} \geq 2N(q - 1)/(q - 3) \ .$$

We claim that if $|m| \geq M_o(N)$, $|r| \leq N$, and $K_o = K_o(N,m)$, then

(iv) if $r + m \in \Sigma$, then $k_o(r+m) = K_o$;

(v) we have

(7.13) $$\hat{\mu}(r+m) = \hat{\mu}(n_{K_o})\, \hat{\mu}(r+m-n_{K_o}) \ .$$

Given these claims, we may easily deduce (i). For let P, ν, and m be as in (i). Then

$$\hat{\nu}(m) = \int P(t)\, e(-mt)\, d\mu(t)$$

$$= \int \sum_{|r| \leq N} \hat{P}(-r)\, e(-rt)\, e(-mt)\, d\mu(t)$$

$$= \sum_{|r| \leq N} \hat{P}(-r)\, \hat{\mu}(r+m) \ .$$

By (7.13), this

$$= \hat{\mu}(n_{K_o}) \sum_{|r| < N} \hat{P}(-r)\, \hat{\mu}(r+m-n_{K_o})$$

$$= \hat{\mu}(n_{K_o})\, \hat{\nu}(m - n_{K_o}) \ ,$$

which is (7.6). If $K_o = 0$, then $\hat{\mu}(r+m) = 0$ for $|r| \leq N$, whence $\hat{\nu}(m) = 0$. Thus (i) is proved.

Now if (iv) were not true, then there would exist $r' \in [-N, N]$ with $r' + m \in \Sigma$ and $K_o = k_o(r'+m) \neq k_o(r+m)$. Let $k_o = \max\{k_o(r+m), K_o\}$. For some $\delta_k \in [-2, 2]$, we may write

$$2N \geq |r - r'| = |(r+m) - (r'+m)|$$

$$= n_{k_o} + \sum_{1\leq k<k_o} \delta_k n_k$$

$$\geq n_{k_o} - 2 \sum_{1\leq k<k_o} n_k \ .$$

By (7.3), $n_{k_o} \geq q^{k_o - k} n_k$ for $k < k_o$, whence the above is

$$\geq n_{k_o} - 2 \sum_{1\leq k<k_o} n_{k_o} q^{k-k_o}$$

$$> n_{k_o} (1 - 2 \sum_{j=1}^{\infty} q^{-j}) = n_{k_o}(q-3)/(q-1) \ .$$

This contradicts (7.12), establishing (iv).

Now (v) is clear if $K_o = 0$. Suppose $K_o \neq 0$. Since μ is a Riesz product, $\hat{\mu}$ is multiplicative on Σ in the sense that if $n, n' \in \Sigma$ and $\varepsilon_k(n) \cdot \varepsilon_k(n') = 0$ for all k, then $\hat{\mu}(n+n') = \hat{\mu}(n)\,\hat{\mu}(n')$. If $r + m \in \Sigma$, then by (7.11) and (iv), $r + m - n_{K_o} \in \Sigma$, whence (7.13) follows from multiplicativity of $\hat{\mu}$. If $r + m \notin \Sigma$ and $r + m - n_{K_o} \notin \Sigma$, then both sides of (7.13) are zero. The final case to consider in proving (v) is if $K_o \neq 0$, $r + m \notin \Sigma$, and $r + m - n_{K_o} \in \Sigma$. We claim that this is impossible, however. For since $r + m \notin \Sigma$ but $r + m - n_{K_o} \in \Sigma$, it follows that $k_o \equiv k_o(r+m-n_{K_o}) \geq K_o$. Now

$$r + m = n_{K_o} + \sum_{1\leq k\leq k_o} \varepsilon_k(r+m-n_{K_o})\, n_k \ .$$

Since $K_o \neq 0$, there exists $r' \in [-N,N]$ such that

$$r' + m = n_{K_o} + \sum_{1\leq k<K_o} \varepsilon_k(r'+m)\, n_k \ .$$

Subtraction of these two equations yields

$$2N > n_{k_o} (q-3)/(q-1)$$

as in the proof of (iv), contradicting (7.12). □

PROOF OF THEOREM 7.6. First consider the case $d\nu = P\, d\mu$, where P is a trigonometric polynomial. Let N be the degree of P. Let K_o, M_o be as in Lemma 7.8. If $|m| \geq M_o(N)$, then

$$|\hat{\nu}(m)| \leq \begin{cases} \frac{1}{2}|\alpha_{K_o}| \cdot \|\hat{\nu}\|_\infty, & K_o \neq 0, \\ 0, & K_o = 0, \end{cases}$$

by Lemma 7.8(i). Letting $|m| \to \infty$ and using (7.7), we infer $R(\nu) \leq R(\mu) \cdot \|\hat{\nu}\|_\infty$. Furthermore, if $|\hat{\mu}(n_k')| \to R(\mu)$, then clearly $\{n_k'\}$ is a subsequence of $\{n_k\}$, say, $\{n_{k_i}\}_{i=1}^{\infty}$. By (7.8), $K_o(N,n_{k_i}) = k_i$, whence (7.6) reduces to

$$\hat{\nu}(n_{k_i}) = \hat{\mu}(n_{k_i})\, \hat{\nu}(0) \ .$$

Therefore $|\hat{\nu}(n_{k_i})| \to R(\mu)|\hat{\nu}(0)|$ and

(7.14) $$R(\nu) \geq R(\mu)|\hat{\nu}(0)| \ .$$

Substitute $e(-mt)\,d\nu(t)$ for $d\nu(t)$ in (7.14):

$$R(\nu) \geq R(\mu)|\hat{\nu}(m)| .$$

Since this holds for all m, (7.4) follows. Since $\|\hat{\nu}\|_\infty = \hat{\nu}(0) = \|\nu\|$ for $\nu \geq 0$, (7.5) also follows. Thus, the theorem is proved for the case where $d\nu = P\,d\mu$.

In general, if $\nu << \mu$, let $d\nu = f\,d\mu$, $f \in L^1(\mu)$. Let $\varepsilon > 0$ and let P be a trigonometric polynomial such that $\|f - P\|_{L^1(\mu)} < \varepsilon$. Let $d\nu_1 = P\,d\mu$. Then $|\hat{\nu}_1(m) - \hat{\nu}(m)| < \varepsilon$ for all m. Lemma 7.8 applied to ν_1 gives for $|m| \geq M_o(N)$,

$$\begin{aligned} \big|\,|\hat{\nu}(m)| - |\hat{\mu}(n_{K_o})\,\hat{\nu}(m-n_{K_o})|\,\big| &\leq |\hat{\nu}_1(m) - \hat{\mu}(n_{K_o})\,\hat{\nu}_1(m-n_{K_o})| \\ &\quad + |\hat{\nu}(m)-\hat{\nu}_1(m)| + |\hat{\mu}(n_{K_o})|\cdot|\hat{\nu}(m)-\hat{\nu}(m-n_{K_o})| \\ &< 2\varepsilon . \end{aligned}$$

Letting $|m| \to \infty$ and then $\varepsilon \to 0$, we deduce the same results as in the first case. □

REMARK. Instead of (7.3), it is evidently sufficient to assume only that

(7.15) $$\lim_{k\to\infty} [n_{k+1} - 2 \sum_{1\leq j\leq k} n_j] = \infty .$$

8. H-sets.

Recall that E is an H-set if there is a non-empty open arc I and a sequence $m_j \uparrow \infty$ such that $E \subset \bigcap_{j=1}^{\infty} E^c_{m_j}$, where

$$E_m = \{x: mx \in I\} .$$

If μ is a Rajchman measure, then the Rajchman-Milicer-Grużewska criterion in the form (1.2) gives

$$|I|\,\mu E = \lim \mu(E \cap E_{m_j}) = 0 ,$$

i.e., $\mu E = 0$. We have shown

THEOREM 8.1. If $\mu \in R$, then $\mu E = 0$ for all H-sets E.

Our aim is to prove that the converse (Rajchman's conjecture) fails. First, we indicate heuristically why one might already expect this in view of our earlier methods. Let $0 \leq \mu \in R$. In order to find an H-set of positive μ-measure, we must find a sequence $m_j \uparrow \infty$ and a non-empty open arc I such that $\chi_I(m_j x) = 0$ for x belonging to a set of positive μ-measure, i.e., for μ-many x. Suppose we begin, as in the proof of Theorem 3.2, by taking an arc I with $0 < |I| < 1$ and a sequence $m_j \uparrow \infty$ such that

$$\int \chi_I(m_j x)\,d\mu(x) \to \alpha \neq |I| .$$

In proving Theorem 3.2, we found that our methods enabled us to find a subsequence $\{m_j'\} \subset \{m_j\}$ such that

$$\frac{1}{K}\sum_{j=1}^{K} \chi_I(m_j' x)$$

tended to a limit a.e. $[\mu]$ which, for μ-many x, was not equal to $|I|$. This gave us an A-set with positive μ-measure. However, to get such an H-set, we need a subsequence $\{m_j'\}$ such that $\chi_I(m_j' x) = 0$ for μ-many x. Now, if we suspect that our methods are somehow "best possible," then we would not expect even to find μ-many x for which

$$\frac{1}{K}\sum_{j=1}^{K} \chi_I(m_j' x) \to 0 .$$

Indeed, we shall verify this expectation as well later in the section when we discuss asymptotic H-sets.

We shall disprove Rajchman's conjecture by finding a Riesz product $\mu \notin R$ which annihilates all H-sets. The method we shall use for proving that H-sets have μ-measure 0 is an elaboration of the standard method for showing that H-sets have Lebesgue measure 0 (see Zygmund [1, I, p. 318] for such a standard proof).

LEMMA 8.2. Let $E_j \subset \mathbb{T}$ be Borel sets. Let $E = \bigcap_1^\infty E_j$ and μ be any finite positive Borel measure. If there exists a constant $d < 1$ such that for any open arc A,

$$\text{(8.1)} \qquad \overline{\lim_{j\to\infty}}\, \mu(A \cap E_j) \le d \cdot \mu A ,$$

then $\mu E = 0$.

Note that Theorem 8.1 is a corollary of Lemma 8.2 and (1.2); d can be taken to be $|I^c|$ with $E_j = \{x: m_j x \in I^c\}$. We shall only use Lemma 8.2 in the case, such as this one, that E_j is a finite union of arcs, but the general statement is not essentially more difficult to prove.

PROOF. We only have to show that (8.1) holds for all Borel sets A. For if so, simply put $A = E$.

Now if (8.1) holds for open arcs, it certainly holds for finite unions of arcs. Let B be any Borel set. Let $\varepsilon > 0$. By regularity of μ (Rudin [2, pp. 49-50]), there is a finite union of arcs, A, such that $\mu(B \,\Delta\, A) < \varepsilon$, where $B \,\Delta\, A = (B\setminus A) \cup (A \setminus B)$. Since (8.1) holds for A, it follows that

$$\overline{\lim_{j\to\infty}}\, \mu(B \cap E_j) \le d(\mu B + \varepsilon) + \varepsilon .$$

Therefore (8.1) holds for B, as desired. □

THEOREM 8.3. Let μ be the Riesz product

$$\text{(8.2)} \qquad d\mu = \prod_{k=1}^{\infty} (1 + \alpha_k \cos 2\pi(n_k x + \phi_k))\, dm,$$

with $-1 \le \alpha_k \le 1$, $\alpha_k \not\to 0$, and

(8.3) $$n_{k+1}/n_k \to \infty \quad .$$

Then $\mu \notin R$, yet μ puts no mass on any H-set .

PROOF. Since $n_{k+1}/n_k < 4$ for at most finitely many k , it suffices to assume $n_{k+1}/n_k \ge 4$ for all k . Let $E \subseteq \bigcap_1^\infty E_j$ be an arbitrary H-set given by

$$E_j = \{x: m_j x \in I^c\} ,$$

where I is a non-empty open arc and $m_j \uparrow \infty$. We shall establish (8.1), which implies that $\mu E = 0$. We shall make the necessary calculations of the μ-measures of intervals in terms of $\hat{\mu}$. To make things simpler, however, we shall avoid the characteristic functions of the intervals and use approximating trigonometric polynomials instead.

Note that $|\hat{\chi}_I(\ell)| < |I|$ for all $\ell \ne 0$. Since $\hat{\chi}_I(\ell) \to 0$ as $\ell \to \infty$, it follows that if

(8.4) $$\delta = |I| - \sup_{\ell \ne 0} |\hat{\chi}_I(\ell)| ,$$

then $\delta > 0$.

Let A be any arc. Let $\varepsilon > 0$. Since μ is continuous (Zygmund [1 , I, p. 209]), there exists a trigonometric polynomial $P(x)$ satisfying

$$P(x) \ge \chi_A(x) , \int P \, d\mu \le \mu A + \varepsilon .$$

Likewise, there exists a trigonometric polynomial $Q(x)$ such that

$$Q \ge \chi_{I^c} , \quad \int Q \, dm \le |I^c| + \varepsilon .$$

Let N be the maximum of the degrees of P and Q . Then

$$\mu(A \cap E_j) = \int \chi_A(x) \, \chi_{I^c}(m_j x) \, d\mu(x)$$

$$\le \int P(x) \, Q(m_j x) \, d\mu(x)$$

(8.5) $$= \sum_{|r| \le N} \sum_{|\ell| \le N} \hat{P}(-r) \, \hat{Q}(-\ell) \, \hat{\mu}(r + m_j \ell) .$$

We claim that for all sufficiently large j , there is at most one value of $\ell \in [1,N]$ such that for some $r \in [-N,N]$, we have $\hat{\mu}(r + m_j\ell) \ne 0$. For note that if $\hat{\mu}(n) \ne 0$, $n > 0$, then n has the form

$$n = n_{k_1} + \Sigma \pm n_{k_i} , \; k_i < k_1 .$$

Therefore, if $\hat{\mu}(r + m_j\ell) \ne 0$ and $\hat{\mu}(r' + m_j\ell') \ne 0$, we may write

$$r + m_j\ell = n_{k_1} + \Sigma \pm n_{k_i} , \quad k_i < k_1 ,$$

$$r' + m_j\ell' = n_{k_1'} + \Sigma \pm n_{k_i'} , \; k_i' < k_1' ,$$

and say, $k_1' \le k_1$. For large j , $n_{k_1'}$ is large and we have the asymptotic relations

$$\frac{\ell}{\ell'} \sim \frac{r + m_j\ell}{r' + m_j\ell'} = \frac{n_{k_1} + \Sigma \pm n_{k_i}}{n_{k_1'} + \Sigma \pm n_{k_i'}} \sim \frac{n_{k_1}}{n_{k_1'}} \quad .$$

Since $\ell/\ell' \le N$, it follows that $k_1' = k_1$ for sufficiently large j , whence $\ell = \ell'$, as desired.

Let j be so large that there is at most one value of ℓ as above. If it exists, let it be ℓ_o . Then (8.5) becomes

$$\mu(A \cap E_j) = \sum_{|r| \le N} \hat{P}(-r)\, \hat{Q}(0)\, \hat{\mu}(r)$$

$$+ \sum_{|r| \le N} \hat{P}(-r)\, \hat{Q}(-\ell_o) \hat{\mu}(r+m_j\ell_o) + \sum_{|r| \le N} \hat{P}(-r)\, \hat{Q}(\ell_o)\, \hat{\mu}(r-m_j\ell_o)$$

$$= \int Q\, dm \cdot \int P\, d\mu + \sum_{|r| \le N} \hat{P}(-r)\, \hat{Q}(-\ell_o)\, \hat{\mu}(r+m_j\ell_o)$$

$$+ \sum_{|r| \le N} \hat{P}(r)\, \hat{Q}(\ell_o)\, \hat{\mu}(-r-m_j\ell_o) \tag{8.6}$$

$$= \int Q\, dm \cdot \int P\, d\mu + 2\, \mathrm{Re} \sum_{|r| \le N} \hat{P}(-r)\, \hat{Q}(-\ell_o)\, \hat{\mu}(r+m_j\ell_o)$$

since P , Q , μ are real. Since $\ell_o \ne 0$,

$$\hat{Q}(-\ell_o) = -\,(1 - Q)^{\wedge}\,(-\ell_o) \ .$$

But

$$|(1-Q) - \chi_I| = |Q - \chi_{I^c}| \ ,$$

so that

$$|(1-Q)^{\wedge}\,(-\ell_o)| - |\hat{\chi}_I(-\ell_o)| \le \int |Q-\chi_{I^c}|\, dm$$

$$= \int (Q-\chi_{I^c})\, dm \le \varepsilon$$

by choice of Q . By definition of δ ,

$$|\hat{\chi}_I(-\ell_o)| \le |I| - \delta \ ,$$

whence finally

$$|\hat{Q}(-\ell_o)| \le |I| - \delta + \varepsilon \quad .$$

(The idea here was to approximate $|\hat{Q}(-\ell_o)|$ not by the more obvious $|\hat{\chi}_{I^c}(-\ell_o)| < |I^c|$ but by $|\hat{\chi}_I(-\ell_o)| < |I|$.)

Therefore from (8.6), for all large j ,

$$\mu(A \cap E_j) \le (|I^c| + \varepsilon)\,(\mu A + \varepsilon) + \tag{8.7}$$

$$+ 2\,(|I| - \delta + \varepsilon) \Big| \sum_{|r| \le N} \hat{P}(-r) \hat{\mu}(r+m_j\ell_o) \Big| \ ,$$

and it remains to estimate the last sum. The easy estimate of

$$\Big| \int P(x) e(-m_j\ell_o x)\, d\mu(x) \Big| \le \int P\, d\mu \le \mu A + \varepsilon$$

is insufficient by a factor of 2 . But by Lemma 7.8(i), there exists K_o such that for all sufficiently large n ,

$$\sum_{|r|\leq N} \hat{P}(-r)\,\hat{\mu}(r+n) = \widehat{P d\mu}(n) = \hat{\mu}(n_{K_o})\,\widehat{P d\mu}(n - n_{K_o}) .$$

Since $\hat{\mu}(n_{K_o}) = \frac{1}{2}\alpha_{K_o} e(\phi_{K_o})$ and $|\widehat{P d\mu}| \leq \int P d\mu \leq \mu A + \varepsilon$, it follows that for large n ,

$$\Big|\sum_{|r|\leq N} \hat{P}(-r)\,\hat{\mu}(r+n)\Big| \leq \frac{1}{2}(\mu A + \varepsilon) .$$

Hence

$$\mu(A\cap E_j) \leq (mI^c+\varepsilon)(\mu A+\varepsilon) + 2(mI-\delta+\varepsilon)\,\frac{1}{2}\,(\mu A+\varepsilon)$$

$$= (1-\delta+2\varepsilon)(\mu A+\varepsilon)$$

for all large j . (Note that if j is such that ℓ_o does not exist, then this inequality still holds.) Letting $\varepsilon \to 0$, we arrive at (8.1) with $d = 1 - \delta$. □

Now from Theorem 3.1, we know that the converse to Theorem 8.1 becomes true if we weaken the definition of H-sets to a set of the form

$$E = \{x: \overline{\lim_{J\to\infty}}\, \frac{1}{J} \sum_{j=1}^{J} \chi_I(m_j x) < |I|\} .$$

That is, if $\mu \notin R$, then there is a set E of this form such that $\mu E \neq 0$. Suppose, then, that we do not weaken it quite so far.

DEFINITION. A Borel set $E \subset \mathbb{T}$ is called an <u>asymptotic H-set</u> if there is a non-empty open arc I and a sequence $m_j \uparrow \infty$ such that for all $x \in E$

$$\lim_{J\to\infty} \frac{1}{J} \sum_{j=1}^{J} \chi_I(m_j x) = 0 .$$

Asymptotic H-sets are, of course, A-sets , hence U_o-sets . If $\mu \notin R$, is there an asymptotic H-set E such that $\mu E \neq 0$? As suggested in the introduction to this section, the answer is "no." In fact, the Riesz product in Theorem 8.3 is an example of a non-Rajchman measure which annihilates all asymptotic H-sets. This follows from Proposition 8.5 below.

LEMMA 8.4. Let μ be a positive measure on a measurable space X . Suppose that μ has no atoms of infinite measure. Let $\{E_n\}$ be measurable sets and χ_n the characteristic function of E_n . Let

$$E = \{t: \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_n(t) = 1\} .$$

(This is the set of t which are in "almost all" the E_n .) Then

$$\mu E \leq \sup_{\{n_k\}} \mu \bigcap_{k=1}^{\infty} E_{n_k} , \tag{8.8}$$

where the sup runs over all infinite sequences $n_k \to \infty$. In particular, if $\mu \bigcap_1^\infty E_{n_k} = 0$ for all (n_k) , then $\mu E = 0$.

PROPOSITION 8.5. If $\mu E = 0$ for all H-sets E , then $\mu E = 0$ for all asymptotic H-sets E .

PROOF. Let E be an asymptotic H-set

$$E \subset \{x: \lim_{J\to\infty} \frac{1}{J} \sum_1^J \chi_I(m_j x) = 0\} .$$

Let

$$E_j = \{x: m_j x \notin I\} .$$

Since $\bigcap_{\ell=1}^\infty E_{j_\ell}$ is an H-set for every subsequence $\{j_\ell\}_{\ell=1}^\infty$, it has μ-measure 0 . Since

$$E \subset \{x: \lim \frac{1}{J} \sum_1^J \chi_{E_j}(x) = 1\} ,$$

the conclusion follows from the lemma. □

This establishes, as claimed, that asymptotic H-sets do not characterize R , subject to the

PROOR OF LEMMA 8.4. By restricting μ to a subset of E of finite measure, if necessary, it suffices to assume that μ is a finite measure. Let $\varepsilon > 0$. Set $\alpha_i = (1+\varepsilon^{2^{i-1}})^{-1}$ for $i \geq 1$. We construct sets

F_j and integers N_j inductively. Begin with $F_o = E$ and $N_o = 0$. Suppose we have constructed $F_o, F_1, \ldots, F_{j-1}$ and $N_o, \ldots, N_{j-1}$ in such a manner that $F_o \supseteq F_1 \supseteq \cdots \supseteq F_{j-1}$, $N_o < N_1 < \ldots < N_{j-1}$, and, for $1 \leq i \leq j-1$,

$$\mu F_i \geq \alpha_i \, \mu F_{i-1} \tag{8.9}$$

and

$$F_i = \{t \in F_{i-1}: \frac{1}{N_i - N_{i-1}} \sum_{N_{i-1} \leq n \leq N_i} \chi_n(t) \geq \alpha_i\} .$$

Then since $F_{j-1} \subset E$,

$$\mu(F_{j-1} \setminus \{t: \frac{1}{N-N_{j-1}} \sum_{N_{j-1} < n \leq N} \chi_n(t) \geq \alpha_j\})$$

tends to 0 as $N \to \infty$. Thus, there exists $N = N_j$ such that if we set

$$F_j = \{t \in F_{j-1}: \frac{1}{N_j - N_{j-1}} \sum_{N_{j-1} < n \leq N_j} \chi_n(t) \geq \alpha_j\} ,$$

then $F_j \subseteq F_{j-1}$ and $\mu F_j \geq \alpha_j \, \mu F_{j-1}$. This completes the construction of $\{F_j\}$ and $\{N_j\}$.

Let $F = \bigcap_1^\infty F_j$. Then by (8.9) ,

$$\mu F \geq \mu E \prod_{j=1}^\infty \alpha_j = (1-\varepsilon) \, \mu E .$$

For any $j \geq 1$, we have

$$\int_X \prod_{i=1}^{j} \left(\frac{1}{N_i - N_{i-1}} \sum_{N_{i-1} < n \leq N_i} \chi_n(t) \right) d\mu(t)$$

(8.10)
$$\geq \int_F \cdots \geq \mu F \prod_{i=1}^{j} \alpha_i \geq (1 - \varepsilon)^2 \mu E .$$

But also the integral on the left-hand side of (8.10) is equal to

$$\frac{1}{\prod_{i=1}^{j} (N_i - N_{i-1})} \sum_{N_0 < n_1 \leq N_1} \cdots \sum_{N_{j-1} < n_j \leq N_j} \mu \bigcap_{k=1}^{j} E_{n_k} ,$$

which is the average of the terms $\mu \bigcap_{k=1}^{j} E_{n_k}$ over $n_k \in (N_{k-1}, N_k]$. Hence one of the terms is at least the average: for some $n_1(j), \ldots, n_j(j)$ with $n_k(j) \in (N_{k-1}, N_k]$, we have

(8.11)
$$\mu \bigcap_{k=1}^{j} E_{n_k(j)} \geq (1 - \varepsilon)^2 \mu E .$$

Among the sequence $\{n_1(j): j \geq 1\}$, there must be some number $n_1 \in (N_0, N_1]$ which occurs infinitely often. Likewise, among $\{n_2(j): j \geq 2, n_1(j) = n_1\}$ there must be one number $n_2 \in (N_1, N_2]$ occuring infinitely often. Continuing in this way yields a sequence $\{n_k\}_1^\infty$ such that for all K, there is some $j \geq K$ such that $n_k(j) = n_k$ for $1 \leq k \leq K$. Therefore

$$\mu \bigcap_{k=1}^{K} E_{n_k} \geq \mu \bigcap_{k=1}^{j} E_{n_k(j)} \geq (1 - \varepsilon)^2 \mu E$$

by (8.11), whence

$$\mu \bigcap_{k=1}^{\infty} E_{n_k} \geq (1 - \varepsilon)^2 \mu E .$$

This establishes the lemma. □

REMARK. The hypothesis that μ has no atoms of infinite measure is necessary. For example, if μ is the measure on $\mathbb{N}$ which assigns 0 to finite sets and ∞ to infinite sets, and if $E_n = [0, n]$, then $\mu \bigcap_1^\infty E_{n_k} = 0$ for all $\{n_k\}$, while $E = \mathbb{N}$, whence $\mu E = \infty$.

H-sets find their greatest significance as examples of sets of uniqueness, whose definition follows.

DEFINITION. A set $E \subset \mathbb{T}$ is called a <u>set of uniqueness</u>, or <u>U-set</u>, if the only trigonometric series $\sum_{-\infty}^{\infty} c_n e(nt)$ which converges to 0 for all $t \notin E$ is the 0-series: $c_n \equiv 0$.

Rajchman [1,2] was the first to show that H-sets are U-sets and he conjectured that U-sets and H_σ-sets were the same. This question stood unanswered until the paper [1] of Pjateckiĭ-Šapiro (see also Bari [2, II, Chapter XIV, §§15,16]). He showed that Rajchman's conjecture was false by introducing, for each integer

$m \geq 1$, a generalization of H-sets called $H^{(m)}$-sets. For $m = 1$, these are the same as H-sets , but $H^{(m)}$-sets cannot always be written as a countable union of $H^{(m-1)}$-sets . Pjateckiĭ-Šapiro showed that, nevertheless, $H^{(m)}$-sets are U-sets (see also Zygmund [1, I, p. 346]). In a parallel fashion, we shall now show that $H^{(m)}$-sets do not characterize R for any m . We begin with a preliminary definition.

DEFINITION. Let $m \in \mathbb{Z}^+$. A sequence $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^m$ of m-tuples of positive integers is called <u>quasi-independent</u> if for each fixed $\Lambda \in \mathbb{Z}^m$, Λ not the 0-vector , we have

$$|V_k \cdot \Lambda| = |\sum_{i=1}^{m} n_k^{(i)} \ell_i| \to \infty \quad \text{as} \quad k \to \infty ,$$

where $V_k = (n_k^{(1)},\ldots,n_k^{(m)})$ and $\Lambda = (\ell_1,\ldots,\ell_m)$.

Note that if we take $\Lambda = (0,0,\ldots,0,1,0,\ldots,0)$, then by the definition, $n_k^{(i)} \to \infty$ as $k \to \infty$. An example of a quasi-independent sequence is given by any V_k with $n_k^{(1)} \to \infty$ and $n_k^{(i)}/n_k^{(i-1)} \to \infty$ for $1 < i \leq m$, but this is not the only kind of example.

DEFINITION. A Borel set $E \subset \mathbb{T}$ is called an <u>$H^{(m)}$-set</u> if there is a quasi-independent sequence $\{V_k\}_1^\infty$ and a non-empty open set $B \subset \mathbb{T}^m$ such that for all $x \in E$ and all k ,

$$V_k x = (n_k^{(1)} x,\ldots,n_k^{(m)} x) \notin B .$$

We first show that $H^{(m)} \subset U_0$ by using the following generalization of the Rajchman-Milicer-Grużewska Criterion. (That $H^{(m)} \subset U_0$ also follows from the well-known but subtler facts that $H^{(m)} \subset U \subset U_0$; <u>cf</u>. the notes to this chapter.)

NOTATION. For $(n_1,n_2,\ldots,n_m) \in \mathbb{Z}^m$ and $x \in \mathbb{T}$, denote

$$(n_1,\ldots,n_m)x = (n_1 x,\ldots,n_m x) \in \mathbb{T}^m . \tag{8.12}$$

THEOREM 8.6. Let $\mu \in M(\mathbb{T})$ and $m \in \mathbb{Z}^+$. The following are equivalent:

(i) $\mu \in R$.

(ii) For every open set $B \subset \mathbb{T}^m$ and every quasi-independent sequence $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^m$,

$$\lim_{k\to\infty} \int_{\mathbb{T}} \chi_B(V_k x)\, d\mu(x) = mB \cdot \hat{\mu}(0) ,$$

where mB is the m-dimensional normalized Lebesgue measure of B .

(iii) For every function $f \in C(\mathbb{T}^m)$ and every quasi-independent sequence $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^m$,

$$\lim_{k\to\infty} \int_{\mathbb{T}} f(V_k x)\, d\mu(x) = \hat{f}(0,0,\ldots,0)\hat{\mu}(0) . \tag{8.13}$$

PROOF. (i) ⇒ (iii). Given $\{V_k\}$, the set of $f \in C(\mathbb{T}^m)$ for which (8.13) holds is a closed linear subspace of $C(\mathbb{T}^m)$. If $\Lambda \in \mathbb{Z}^m$ and $f(t) = e(\Lambda \cdot t)$ is the corresponding trigonometric polynomial on $\mathbb{T}^m$, then since $|V_k \cdot \Lambda| \to \infty$ for $\Lambda \neq 0$, (i) implies that (8.13) holds for this f. Since the trigonometric polynomials span $C(\mathbb{T}^m)$, it follows that (iii) holds.

(iii) ⇒ (i). Take $V_k = (k, k^2, \ldots, k^m)$ and $f(t_1, \ldots, t_m) = e(-t_1)$.

The proofs that (i) & (iii) ⇒ (ii) and (ii) ⇒ (iii) are parallel to those for Theorem 1.1. □

PROPOSITION 8.7. If $\mu \in R$, then $\mu E = 0$ for all $H^{(m)}$-sets E.

PROOF. Let $E \subset \{x: V_k x \notin B\}$ be an $H^{(m)}$-set. Let $\nu = \mu|E \in R$. Then by the preceding theorem,

$$mB \cdot \mu E = \lim_{k\to\infty} \int \chi_B(V_k x)\, d\nu(x) = 0$$

since for $x \in E$, $\chi_B(V_k x) = 0$. That is, $\mu E = 0$. □

As we mentioned, the converse to Proposition 8.7 fails. This is proved by

THEOREM 8.8. Let m be a positive integer and let μ be the Riesz product (8.2). Assume (8.3) and

$$(8.14) \qquad |\alpha_k| \leq \frac{2}{3^m - 1}.$$

Then μ puts no mass on any $H^{(m)}$-set.

REMARK. This theorem leaves open the (unlikely) possibility that if $\mu E = 0$ for every $H^{(m)}$-set E <u>for all</u> <u>m</u>, then $\mu \in R$; i.e., that the union of the classes $H^{(m)}$ characterizes R.

Our method of proof of Theorem 8.8 is the same as that of Theorem 8.3. The difficult part of the proof is contained in the next two lemmas.

LEMMA 8.9. Let $d > 0$ and let $\{\Lambda^{(j)}\}_{j=1}^{m+1} \subset [-d,d]^m \cap \mathbb{Z}^m$. There is a linear dependence relation

$$(8.15) \qquad \sum_{j=1}^{m+1} c_j \Lambda^{(j)} = 0$$

with $c_j \in \mathbb{Z}$ not all 0 and $|c_j| \leq d^m \cdot m^{m/2}$.

PROOF. Let $\Lambda^{(j)} = (\ell_1^{(j)}, \ldots, \ell_m^{(j)})$. Since we have $m + 1$ vectors $\Lambda^{(j)}$ in an m-dimensional vector space $\mathbb{R}^m$, one of the vectors, say $\Lambda^{(m+1)}$, is linearly dependent on the others:

$$(8.16) \qquad \sum_{j=1}^{m} b_j \Lambda^{(j)} = \Lambda^{(m+1)}.$$

By Cramer's rule, b_j can be written as the quotient of determinants with entries $\ell_i^{(j)}$. Let c_j be the determinant in the numerator of b_j and let $-c_{m+1}$ be the common determinant of the denominators. Hadamard's inequality,

$$|\det(a_{ij})| \le \prod_i (\sum_j |a_{ij}|^2)^{1/2},$$

now gives the result when (8.16) is multiplied through by $-c_{m+1}$, since $|\ell_i^{(j)}| \le d$. □

LEMMA 8.10. Let $\{V_j\}_1^\infty \subset (\mathbb{Z}^+)^m$ be quasi-independent, let $(n_k)_{k=1}^\infty$ be hyperlacunary (i.e., $n_{k+1}/n_k \to \infty$), let $L \in \mathbb{Z}^+$, and let Δ be a finite subset of $\mathbb{Z}$ containing 0. Denote the cardinality of Δ by $|\Delta|$ and let D be any finite subset of $\mathbb{Z}^m$. Then for all sufficiently large j, the number of solutions Λ to

$$(8.17)\quad \begin{cases} |V_j \cdot \Lambda - \sum_{k=1}^\infty \varepsilon_k n_k| \le L, \\ \Lambda \in D,\ \varepsilon_k \in \Delta, \\ \varepsilon_k = 0 \text{ for all but finitely many } k \end{cases}$$

is at most $|\Delta|^m$. As a function of $|\Delta|$ and m, this upper bound is best possible.

Note that this lemma does not bound the number of solutions $(\Lambda, \{\varepsilon_k\})$ to (8.17), but only the number of different Λ among such solutions. In proving Theorem 8.8, we shall use the case $\Delta = \{-1,0,1\}$.

PROOF. We begin by showing that $|\Delta|^m$ is the best possible bound. Let $V_j = (n_{1+j}, n_{2+j}, \ldots, n_{m+j})$ and choose $L \ge \max\{|\varepsilon| : \varepsilon \in \Delta\}$. Then for every j,

there are at least $|\Delta|^m$ solutions, namely: take ε_{i+j} to be arbitrary elements of Δ for $1 \le i \le m$, $\varepsilon_k = 0$ for all other k, and $\Lambda = (\varepsilon_{1+j}, \varepsilon_{2+j}, \ldots, \varepsilon_{m+j})$.

We now prove the rest of the lemma by showing that in some sense the example just given is typical; we show that there exist $k_1, \ldots, k_m$ such that $\varepsilon_{k_1}, \ldots, \varepsilon_{k_m}$ determine the solution $(\varepsilon_k)_1^\infty$ to such an extent that $(\varepsilon_k)_1^\infty$ in turn uniquely determines Λ.

Let $M = \max\{|\varepsilon| : \varepsilon \in \Delta\}$ and fix j. Let d be the maximum absolute value of the coordinates of Λ over all $\Lambda \in D$. Consider any $m + 1$ solutions

$$(\Lambda^{(r)}, \{\varepsilon_k^{(r)}\}_1^\infty), \quad 1 \le r \le m+1,$$

to (8.17). Let $c_1, \ldots, c_{m+1}$ be as in Lemma 8.9. Define

$$h^{(r)} = V_j \cdot \Lambda^{(r)} - \sum_1^\infty \varepsilon_k^{(r)} n_k,$$

so that $|h^{(r)}| \le L$. Then

$$\sum_1^{m+1} c_r h^{(r)} = V_j \cdot \sum_1^{m+1} c_r \Lambda^{(r)} - \sum_1^\infty (n_k \sum_1^{m+1} c_r \varepsilon_k^{(r)})$$

$$= -\sum_1^\infty n_k \delta_k,$$

where $\delta_k = \sum_1^{m+1} c_r \varepsilon_k^{(r)}$. From our bounds on c_r, $h^{(r)}$, and $\varepsilon_k^{(r)}$, we see that

$$(8.18)\begin{cases} |\sum_1^\infty n_k\delta_k| = |\sum_1^{m+1} c_r h^{(r)}| \le (m+1)\, Ld^m m^{m/2}, \\ |\delta_k| = |\sum_1^{m+1} c_r \varepsilon_k^{(r)}| \le (m+1)\, Md^m m^{m/2}. \end{cases}$$

But since $n_{k+1}/n_k \to \infty$, (8.18) implies that there exists some $k_o = k_o(L,M,d,m)$ (k_o does not depend on j) such that $\delta_k = 0$ for all $k \ge k_o$. That is, the vectors

$$(\varepsilon_{k_o}^{(r)}, \varepsilon_{k_o+1}^{(r)}, \ldots),\ 1 \le r \le m+1$$

are linearly dependent.

We have thus demonstrated that for fixed j,

$$\{\{\varepsilon_k\}_{k_o}^\infty : \{\varepsilon_k\}_1^\infty \text{ is a solution of } (8.17)\}$$

belongs to an m-dimensional space. There are therefore m coordinates $\varepsilon_{k_1},\ldots,\varepsilon_{k_m}$ ($k_i \ge k_o$) which determine all ε_k, $k \ge k_o$. Since there are only $|\Delta|$ choices for each ε_{k_i}, there are at most $|\Delta|^m$ solutions $\{\varepsilon_k\}_{k_o}^\infty$ to (8.17). But we claim that for large j, each such solution corresponds to exactly one solution Λ. For let

$$N = \max\{|\sum_{k<k_o} \varepsilon_k n_k| : \varepsilon_k \in \Delta\}.$$

By quasi-independence of $\{V_j\}$, there exists j_o such that for each $j \ge j_o$, we have

$$\inf\{|V_j\cdot\Lambda| : 0 \ne \Lambda \in D - D\} > N + 2L,$$

where $D - D = \{\Lambda_1 - \Lambda_2 : \Lambda_1, \Lambda_2 \in D\}$. Now suppose that $(\Lambda^{(1)}, \{\varepsilon_k\}_{k_o}^\infty)$, $(\Lambda^{(2)}, \{\varepsilon_k\}_{k_o}^\infty)$ are two solutions of (8.17) for some $j \ge j_o$. Then

$$|V_j\cdot(\Lambda^{(1)} - \Lambda^{(2)}) - \sum_{k<k_o} \varepsilon_k n_k| \le 2L.$$

Since $\Lambda^{(1)} - \Lambda^{(2)} \in D - D$, the definition of j_o implies that $\Lambda^{(1)} - \Lambda^{(2)} = 0$. This establishes the claim and finishes the proof. □

PROOF OF THEOREM 8.8. As in the proof of Theorem 8.3, we may assume that $n_{k+1}/n_k \ge 4$. Let $E \subset \bigcap_{j=1}^\infty E_j$ be an arbitrary $H^{(m)}$-set given by $E_j = \{x : V_j x \notin B\}$, where $\{V_j\}$ is quasi-independent and B is a non-empty open set in $\mathbb{T}^m$. To show that $\mu E = 0$, we again establish (8.1).

Define

$$(8.19)\qquad \delta = mB - \sup\{|\hat{\chi}_B(\Lambda)| : 0 \ne \Lambda \in \mathbb{Z}^m\}.$$

Then $\delta > 0$. Let A be any arc, $\varepsilon > 0$, and $P(x)$ be a trigonometric polynomial such that

$$P(x) \ge \chi_A(x),\ \int_{\mathbb{T}} P\,d\mu \le \mu A + \varepsilon.$$

Let L be the degree of P. Let Q be a trigonometric polynomial on $\mathbb{T}^m$ such that

$$Q \le \chi_B , \quad \hat{Q}(0,0,\dots,0) \ge mB - \varepsilon .$$

Let $D = \{\Lambda \in \mathbb{Z}^m : \hat{Q}(\Lambda) \ne 0\}$ be the spectrum of Q. Then

$$\mu(A \cap E_j) = \int_{\mathbb{T}} \chi_A(x)\,(1 - \chi_B(V_j x))\, d\mu(x)$$

$$\le \int P(x)\,(1 - Q(V_j x))\, d\mu(x)$$

$$= \left(\int P d\mu\right)(1 - \hat{Q}(0,\dots,0))$$

(8.20)

$$- \sum_{\substack{|r| \le L \\ 0 \ne \Lambda \in D}} \hat{P}(-r)\,\hat{Q}(-\Lambda)\,\hat{\mu}(r + V_j\Lambda) .$$

If $\hat{\mu}(r + V_j \cdot \Lambda) \ne 0$, then

$$r + V_j \cdot \Lambda = \sum_1^\infty \varepsilon_k n_k$$

for some $\varepsilon_k = -1, 0, 1$, whence

$$\left|V_j \cdot \Lambda - \sum_1^\infty \varepsilon_k n_k\right| \le L .$$

By Lemma 8.10, there are at most 3^m such $\Lambda \in D$ for all sufficiently large j, one of those Λ being $\Lambda = 0$. But for each such $\Lambda \ne 0$, $|V_j \cdot \Lambda| \to \infty$, so that, as in the proof of Theorem 8.3, there exists K_0 such that for large j,

$$\left|\sum_{|r| \le L} \hat{P}(-r)\,\hat{\mu}(r + V_j \cdot \Lambda)\right| \le |\hat{\mu}(n_{K_0})| \int P d\mu .$$

By the hypothesis (8.14) and choice of P, this is

$$\le \frac{1}{3^m - 1}\,(\mu A + \varepsilon) .$$

Furthermore, by (8.19), if $\Lambda \ne 0$, then

$$|\hat{Q}(-\Lambda)| \le |\hat{\chi}_B(-\Lambda)| + \int_{\mathbb{T}^m} |\chi_B - Q|\, dm$$

$$\le mB - \delta + \varepsilon .$$

Therefore for all sufficiently large j, (8.20) yields

$$\mu(A \cap E_j) \le (1 - mB + \varepsilon)(\mu A + \varepsilon)$$

$$+ (3^m - 1)(mB - \delta + \varepsilon)\frac{1}{3^m - 1}(\mu A + \varepsilon)$$

$$= (1 - \delta + 2\varepsilon)(\mu A + \varepsilon) .$$

Letting $\varepsilon \to 0$, we arrive at (8.1) with $d = 1 - \delta$. □

9. Summary

The following notation is convenient.

NOTATION. If C is a class of sets, $C^{\perp}$ denotes the set of $\mu \in M(\mathbb{T})$ for which $|\mu|E = 0$ for all $E \in C$.

We have shown:

$$R = W^{\perp} ,$$

$$R = U_0^{\perp} ,$$

$$R = A^{\perp} ,$$

$$R \supsetneq (W^*)^{\perp} ,$$

$$R \subsetneq \{E\colon s^+(E) \geq s\}^{\perp} \quad \text{for all} \quad s > 0 ,$$

$$R \subsetneq H^{\perp} ,$$

$$R \subsetneq (H^{(m)})^{\perp} \quad \text{for all} \quad m \geq 1 .$$

We have not determined whether

$$R \overset{?}{=} \{E\colon s^+(E) > 0\}^{\perp}$$

or

$$R \overset{?}{=} (\bigcup_1^{\infty} H^{(m)})^{\perp} .$$

The most interesting question is whether

$$R \overset{?}{=} (\text{Borel U-sets})^{\perp} ;$$

its answer is sure to shed more light on the mysteries of U-sets. Another problem, especially interesting to those in the field of diophantine approximation, is to determine $(W^*)^{\perp}$. Related to this is the question of whether Theorem 4.4 is best possible.

These results give us some information on how big sets are of the various given types. Since $H^{\perp} \supsetneq R$, H-sets are rather small. On the other hand, W*-sets are rather large, while W-sets are "just right."

10. Notes

Dunkl and Ramirez [1] give characterizations in the form of Theorem 1.6 of certain subclasses of the measure algebra on a locally compact group.

Lemma 2.6 generalizes, at least in part, to any uniformly convex Banach space:

THEOREM 10.1 (Kakutani [1]). If B is a uniformly convex Banach space and $x_n \in B$ converge weakly to y, then there exists a subsequence $\{x_n'\} \subset \{x_n\}$ such that $\frac{1}{N}\sum_{n=1}^{N} x_n' \to y$ in norm.

Examples of uniformly convex Banach spaces are $L^p(\mu)$ for $p > 1$. The theorem fails for $B = L^1[0,1]$ (Banach and Saks [1]) and for $B = C[0,1]$ (Schreier [1]).

The proof that A-sets belong to U_o is given in Kahane and Salem [1].

Baker [2] demonstrated Theorem 4.4 in the case $\phi(n) = (\log n)^{-1-\varepsilon}$, some $\varepsilon > 0$, and Theorem 4.5 in the case $\phi(n) = (\log\log n)^{-1-\varepsilon}$.

Additional material on infinite convolutions, including those of the form (5.6), and on infinite product measures is collected in Sections 4-7 of Chapter 6 in Graham and McGehee [1].

Corollary 7.7 also follows immediately from the following deep theorem of Körner [2, Theorem 1, p. 278]:

THEOREM 10.2. For $0 \leq s \leq 1$, there exists a Helson-s set E and a measure $\mu \in M^+(E)$ such that for every Borel subset B of E,

$$R(\mu|B) = s\,\mu(B) .$$

Theorem 8.1 has been well-known since Rajchman (see Milicer-Grużewska [3, p. 177]). Several other proofs are known. For example, it follows immediately from the more important fact that H-sets are U-sets (Zygmund [1, I, p. 345]) and that U-sets are U_o-sets (Section IV.2). Of course, it is also a corollary of Theorem 3.1. Our notion of "quasi-independent" is more commonly known as "normal" in English. However, the Russian word used by Pjateckiĭ-Šapiro, who introduced the concept, means "independent."

CHAPTER IV

SUPPLEMENTARY RESULTS

1. The Class J

Given a measure μ , let

$$c = \sup\{|\mu|E : E \in U_o\} .$$

We claim the supremum is attained. For let $E_n \in U_o$ with $|\mu|E_n \to c$. Then evidently $E \equiv \bigcup_1^\infty E_n$ belongs to U_o and $|\mu|E = c$. Thus, if the supremum is attained for the U_o-set E_o , let $\nu = \mu|E_o$ and $\sigma = \mu - \nu$. Then $\sigma E = 0$ for all $E \in U_o$, whence $\sigma \in R$. Since ν is concentrated on a U_o-set , also $\sigma \perp \nu$.

DEFINITION. A Borel measure μ is said to belong to the class J if μ is concentrated on a U_o-set .

This definition is essentially due to Šreĭder [2], who noted that given Corollary III.2.2, we have, as proved above,

THEOREM 1.1. Given any $\mu \in M(\mathbb{T})$, there exist unique measures $\mu_R \in R$ and $\mu_J \in J$ such that $\mu = \mu_R + \mu_J$. For any $\mu_1 \in R$, $\mu_2 \in J$, we have $\mu_1 \perp \mu_2$.

If we combine this theorem with the standard Lebesgue decomposition, we see that any measure μ may be written as

$$\mu = \mu_a + \mu_d + \mu_R + \mu_J ,$$

where $\mu_a << m$, μ_d is discrete, $\mu_R \in R$ and is singular, $\mu_J \in J$ and is continuous, and $\mu_a, \mu_d, \mu_R, \mu_J$ are all mutually singular.

We have already shown (Theorem II.4.1) that R is a <u>band</u>, i.e., if $\nu << \mu \in R$, then $\nu \in R$. It is evident that J is a band as well. Also note that R is a closed ideal of the Banach algebra $M(\mathbb{T})$, while J is a closed subspace. That J is not closed under convolution, hence not a subalgebra, follows from Proposition 4.9 to come.

$M(\mathbb{T})/R$ is a Banach algebra with the quotient norm. The norm is easily evaluated, since as a Banach space M/R is clearly isomorphic to J . Thus

$$\|\mu\|_{M/R} = \|\mu - R\|_{M/R} = \|\mu_J\| = \max\{|\mu|E: E \in U_o\} .$$

From Proposition III.2.3, we have

$$(1.1) \qquad \|\mu\|_{M/R} \geq R(\mu) .$$

While $R(\mu)$ is a norm on M/R , it is not complete. For by the open mapping theorem and (1.1), if $R(\cdot)$ were complete, then it would have to be equivalent to $\|\cdot\|_{M/R}$. But if μ is a Riesz product not in R , then, as we shall see in Theorem 4.4, $\mu \in J$. Since $R(\mu)$ can be made arbitrarily small while $\|\mu\|_{M/R} = \|\mu_J\| = \|\mu\| = 1$, it follows that the norms are not equivalent. (On the other hand, if E is a Helson set, then the norms are equivalent when restricted to $\mu \in M(E)$: .

$$R(\mu) \geq s(E)\cdot\|\mu\| = s(E)\ \|\mu\|_{M/R}\ ,$$

where $s(E)$ is as defined in Section III.7.)

2. Sets of Uniqueness in the Wide Sense

U_o-sets are also known as "sets of uniqueness in the wide sense" because of an alternative definition (Theorem 2.1 below). While various authors have chosen one or the other definition as they preferred, it appears that an explicit theorem fully stating the equivalence of the two definitions has not been set down except in the case of closed sets (Zygmund [1, I, Chap. IX, (6.11), p. 348]). Theorem 2.1 gathers the facts together for this purpose. Our original definition of "U_o-set" is now the standard.

THEOREM 2.1. A Borel set E is a U_o-set if and only if the only Fourier-Stieltjes series $\sum_{-\infty}^{\infty} \hat{\mu}(n)\, e(nt)$ converging to 0 for all $t \notin E$ is the 0-series : $\mu = 0$.

PROOF. We show the contrapositives. Suppose that $E \notin U_o$. Then there exists a non-zero $\mu \in R$ such that $|\mu|E \neq 0$. Let F be a closed subset of E with $|\mu|F \neq 0$ and put $\nu = \mu|F$. Then $\Sigma\hat{\nu}(n)\, e(nt) \to 0$ for $t \notin F$ (Graham and McGehee [1, Theorem 4.2.1 (v) $\Rightarrow$ (iv) , p. 94]), hence for $t \notin E$, whereas $\nu \neq 0$.

On the other hand, suppose $\mu \neq 0$ and $\lim_{N\to\infty} \sum_{-N}^{N} \hat{\mu}(n)\, e(nt) = 0$ for $t \notin E$. If $mE = 1$, then certainly $E \notin U_o$, so assume $mE \neq 1$. Then $\mu \in R$ by the Cantor-Lebesgue theorem (Zygmund [1, I, Chap IX, (1.2), p. 316]). Since $|\mu|E \neq 0$ (Zygmund [1, II, p. 160 (proof of 19i)]), it follows that $E \notin U_o$. □

It follows that Borel U-sets are U_o-sets . Pjateckiĭ-Šapiro [2] (see also Graham and McGehee [1, pp. 104-109]) was the first to show that not all U_o-sets are U-sets . In fact, he gave an example of an A-set which is not a U-set :

THEOREM 2.2. If $0 < \gamma < 1/2$, the set

$$\{x: (\forall K)\ \frac{1}{K}\sum_{k=1}^{K} \chi_{[\frac{1}{2},1)}(2^{k-1}x) \leq \gamma\}$$

is not a U-set .

3. Purity Theorems and Infinite Convolutions

If μ is a probability measure which is the weak* limit of an infinite convolution of discrete probability measures, then μ is "of pure type": either μ is discrete, absolutely continuous, or purely singular. This is the classical Jessen-Wintner purity law, Corollary 3.3 below (Jessen and Wintner [1, p. 86], Stout [1, pp. 98-99]; Brown and Moran [1] have a strengthening of this theorem). As noted by van Kampen [1, Theorem VIII, p. 444], this is part of a more general purity law, Theorem 3.2. As a consequence of this and Corollary III.2.2, we shall show that μ is (purely) in R or in J . In certain cases, we shall give criteria for deciding which of these two alternatives holds. The general purity law is proved by introducing independent random variables on a probability space. The following assumptions will be shared by several theorems:

(3.1) Let X_n be independent discrete random variables on a probability space (Ω,P) with values in $\mathbb{T}$. Assume that $X = \sum_{n=1}^{\infty} X_n$ exists a.s. [P]. Let μ be the distribution of X : for Borel sets $E \subset \mathbb{T}$,

$$\mu(E) \equiv P(X \in E) .$$

Here and in all that follows, $\mathbb{R}$ could replace $\mathbb{T}$ without other change.

NOTATION. If $E, F \subset \mathbb{T}$, $E + F$ denotes the set $\{e + f: e \in E, f \in F\}$.

The key to the proof of the purity law is the following theorem of Jessen and Wintner.

THEOREM 3.1. Assume (3.1). Let H be the group

$$(3.2)\quad \{\sum_{n=1}^{N} m_n t_n : N \in \mathbb{N}, m_n \in \mathbb{Z}, (\exists i)\ P(X_i = t_n) > 0\} .$$

If $E \subset \mathbb{T}$ is any Borel set, then either $\mu(H + E) = 0$ or $\mu(H + E) = 1$.

PROOF. We shall merely give a sketch; details are in Stout [1, p. 98], for example. Let D_n be the set of t such that $P(X_n = t) > 0$. Then $P(X_n \in D_n) = 1$, D_n is countable, and H is the subgroup of $\mathbb{T}$ generated by $\bigcup_{n=1}^{\infty} D_n$. By restricting to a subset of Ω of probability 1, we may assume that $\forall n\ X_n \in D_n$ and $X = \sum_1^\infty X_n$ everywhere. Then $\{X \in H + E\}$ is a tail event with respect to $\{X_n\}$. By the Kolmogorov 0 - 1 law (Stout [1, p. 95]), $P(X \in H + E) = 0$ or 1, as desired. □

THEOREM 3.2 (Purity Law). Assume (3.1). Let C be any class of Borel sets in $\mathbb{T}$ which is closed under countable unions and under translation. Then either μ puts no mass on any set in C or μ is concentrated on some set in C.

PROOF. Suppose $\mu E > 0$ for some $E \in C$. Let H be as in (3.2). Then by Theorem 3.1, $\mu(H + E) = 1$. Since H is countable, $H + E$ is a countable union of translates of E : $H = \bigcup_{h \in H} (h+E)$. Hence $H + E \in C$ and so μ is concentrated on a set in C. □

This theorem is called a purity law for the following reason. If C is a class of sets closed under countable unions and

$$A_C = \{\mu \in M(\mathbb{T}) : (\exists E \in C)\ |\mu|E = \|\mu\|\} ,$$

$$I_C = \{\mu \in M(\mathbb{T}) : (\forall E \in C)\ |\mu|E = 0\} ,$$

then $M(\mathbb{T}) = A_C \oplus I_C$ and $A_C \perp I_C$. (The proof is the same as that of Theorem 1.1.) If C is also closed under translation, then the purity law says that for μ of the form (3.1), μ belongs purely to A_C or to I_C.

For the classical Jessen-Wintner purity law, we consider the classes of countable sets or of Lebesgue-measure-zero sets. Theorem 3.2 gives immediately

COROLLARY 3.3 (Jessen-Wintner). If (3.1) holds, then μ is discrete or continuous, and is absolutely continuous or singular.

This is usually reformulated in the way we had originally stated it to give three possibilities: $\mu \in M_d$, $\mu \in L^1$, or $\mu \in M_s$, where M_d, L^1, M_s are the discrete, absolutely continuous, and purely singular measures, respectively. We now use Corollary III.2.2 and the class U_o for C to obtain

COROLLARY 3.4. If (3.1) holds, then μ is either in R or in J.

Note that combining Corollaries 3.3 and 3.4 shows that either $\mu \in M_d$, $\mu \in L^1$, $\mu \in M_s \cap R$, or $\mu \in M_c \cap J$, where M_c are the continuous measures.

PROOF. For any $\nu \in R$, the translate $\nu_t \in R$, since $\hat{\nu}_t(n) = e(-nt)\,\hat{\nu}(n)$. Thus, every translate of R is equal to R. It follows that U_o is closed under translation. Also, U_o is clearly closed under countable unions. Therefore, Theorem 3.2 yields $\mu E = 0$ for all $E \in U_o$ or else μ is concentrated on a U_o-set. By Corollary III.2.2, this is equivalent to the desired conclusion. □

REMARK. Even if $\sum_1^\infty X_n$ does not converge a.s., every a.s. limit point is of pure type. For if $\sum_1^{N_k} X_n \to X$ a.s. as $k \to \infty$, set $Y_k = \sum_{N_{k-1}<n\le N_k} X_n$. Then $\{Y_k\}$ are discrete and independent, so the purity law applies to their sum, X.

A standard argument shows that an infinite convolution of discrete probability measures can be represented in the form (3.1), and hence that the previous results hold for such measures. We summarize this as follows.

THEOREM 3.5. Let $\mu_n \in M(\mathbb{T})$ be discrete probability measures with $\mathop{\ast}_1^N \mu_n \to \mu$ weak*. If

$$H = \{\sum_{n=1}^{N} m_n t_n : N \in \mathbb{N},\ m_n \in \mathbb{Z},\ (\exists i)\ \mu_i(\{t_n\}) > 0\}$$

and $E \subset \mathbb{T}$ is Borel, then $\mu(H + E) = 0$ or 1. If C is as in Theorem 3.2, then either μ puts no mass on every set of C or μ is concentrated on some set of C. Either $\mu \in M_d$, $\mu \in L^1$, $\mu \in M_s \cap R$, or $\mu \in M_c \cap J$.

NOTE. For the corresponding theorem with $\mathbb{R}$ in place of $\mathbb{T}$, our proof works only under the additional assumption that μ is a probability measure. (This is automatic for weak* convergence in $M(\mathbb{T})$ since $1 \in C(\mathbb{T})$.) However, the theorem is valid even if $\|\mu\| < 1$, since then $\mu = 0$ (Proposition 3.6). For the necessary theorems concerning R and J in $M(\mathbb{R})$, see Section V.1.

PROOF. Let (Ω, P) be the probability space $(\mathop{\times}_1^\infty \mathbb{T}, \mathop{\times}_1^\infty \mu_n)$. Denote the n-th coordinate projection on Ω by X_n. Then X_n are independent random variables with respect to P and have distributions μ_n, hence are discrete. Since $\mathop{\ast}_1^\infty \mu_n$ converges, it follows that

$\sum_1^\infty X_n$ converges a.s. [P] (see Brown and Moran [1, Proposition 1 and Theorem 1]; or Jessen and Wintner [1, p. 84, Theorem 32] combined with Billingsley [1, Theorem 2.1, pp. 11-12]). Also, by considering the Fourier-Stieltjes transform, we see that μ is the distribution of X (Jessen and Wintner [1, p. 84, Theorem 32]). Hence, all the previous theorems apply to μ . □

PROPOSITION 3.6. If $\mu_n \in M(\mathbb{R})$ are positive measures with norm at most 1 and if $\mu = \lim_{N\to\infty} \mathop{\ast}_1^N \mu_n$ weak* , then $\|\mu\| = \prod_1^\infty \|\mu_n\|$ or $\mu = 0$.

PROOF. We claim that it suffices to consider the case where $\|\mu_n\| = 1$. Note that

$$C_N \equiv \prod_1^N \|\mu_n\| = \| \mathop{\ast}_1^N \mu_n \| .$$

If $C_N \to 0$, then certainly $\mu = 0$. Otherwise, if $C_N \to C \neq 0$, then

$$\lim \mathop{\ast}_1^N \mu_n = \lim C_N \mathop{\ast}_1^N (\mu_n/\|\mu_n\|)$$

$$= C \lim \mathop{\ast}_1^N (\mu_n/\|\mu_n\|)$$

and $\mu_n/\|\mu_n\|$ has norm 1 . Thus, the proposition is reduced to the case of probability measures μ_n ; we wish to show that either $\|\mu\| = 1$ or $\mu = 0$.

Suppose that $\mu \neq 0$. Let X_n be independent random variables on a probability space (Ω, P) with distributions μ_n . Since $|\hat{\mu}_n(t)| \le 1$, $\lim_{N\to\infty} \prod_1^N |\hat{\mu}_n(t)|$ exists for all $t \in \mathbb{R}$. If the limit were zero a.e. [m], then $\lim_N \prod_1^N \hat{\mu}_n(t)$ would also be zero a.e. , whence μ would be zero (Loève [1, p. 190, Corollary 2]). Hence the limit is positive on a set of positive measure. By Loève [1, p. 251, Corollary 2], there exist $a_n \in \mathbb{R}$ so that $\Sigma(X_n - a_n)$ converges a.s. [P] . Let ν_n be the distribution of $X_n - a_n$,

$$\tau_N = \mathop{\ast}_1^N \nu_n , \quad \sigma_N = \mathop{\ast}_1^N \mu_n .$$

Then there exists a probability measure ν such that $\tau_N \to \nu$ weak* (Brown and Moran [1, Proposition 1 and Theorem 1] or Loève [1, p. 168 (c) and p. 181(A)]). Since there are only a denumerable number of mass-points of μ and ν , let $M_j \uparrow \infty$ be a sequence such that both $\pm M_j$ are continuity points for both μ and ν . Denote $(-M_j, M_j)$ by I_j . Since χ_{I_j} may be approximated from above and from below arbitrarily closely in $L^1(\mu)$ and in $L^1(\nu)$ by functions in $C_o(\mathbb{R})$, it follows that

(3.3) $$\lim_{N\to\infty} \sigma_N(I_j) = \mu I_j ,$$

$$\lim_{N\to\infty} \tau_N(I_j) = \nu I_j .$$

The proof will be complete if we show that $\lim_{j\to\infty} \mu I_j = 1$.

Note that for all Borel E , $\tau_N(E) = P(\sum_1^N (X_n - a_n) \in E)$ $= P(\sum_1^N X_n \in E + \sum_1^N a_n) = \sigma_N(E + \sum_1^N a_n)$. We claim that there exists a sequence $N_k \uparrow \infty$ such that $\sum_{n=1}^{N_k} a_n$ converges to some (finite) number a . For if not, then $|\sum_1^N a_n| \to \infty$. To establish a contradiction, we shall show that $\mu I_j = 0$ for all j (which implies $\mu = 0$) . Fix j and pick $\varepsilon > 0$. Let K be such that $\nu E < \varepsilon$ for $E = \{t \in \mathbb{R} : |t| > K\}$. Let N_o be such that for $N \geq N_o$,

$$|\tau_N - \nu|(E) < \varepsilon \quad \text{and} \quad I_j - \sum_1^N a_n \subset E .$$

Then if $N \geq N_o$, we have

$$\sigma_N(I_j) = \tau_N(I_j - \sum_1^N a_n) \leq \tau_N(E)$$

$$\leq \nu(E) + \varepsilon < 2\varepsilon .$$

Therefore $\sigma_N(I_j) \to 0$ as $N \to \infty$, which shows that $\mu I_j = 0$ (by 3.3), as desired.

Thus the existence of N_k such that $\sum_1^{N_k} a_n \to a$ is established. We claim now that for each j ,

$$(3.4) \qquad \lim_{k\to\infty} \tau_{N_k}(I_j - \sum_1^{N_k} a_n) = \lim_{k\to\infty} \tau_{N_k}(I_j - a) .$$

Fix j and choose $\delta_k > \sum_{n>N_k} a_n$ so that $\delta_k \to 0$ as $k \to \infty$ and that for each k , the four points $\pm M_j \pm \delta_k$ are continuity points of ν . Let $E_k = (M_j - \delta_k, M_j + \delta_k)$. Then

$$|\tau_{N_k}(I_j - \sum_1^{N_k} a_n) - \tau_{N_k}(I_j - a)| \leq \tau_{N_k}(E_k \cup - E_k) .$$

Since $\nu(\pm E_k) \to 0$ as $k \to \infty$ and for each ℓ , $\tau_{N_k}(\pm E_\ell) \to \nu(\pm E_\ell)$, it follows that $\tau_{N_k}(E_k \cup - E_k) \to 0$. This proves (3.4). Hence

$$\mu I_j = \lim_k \sigma_{N_k}(I_j) = \lim_k \tau_{N_k}(I_j - \sum_1^{N_k} a_n)$$

$$= \lim_k \tau_{N_k}(I_j - a) \geq \lim_{\varepsilon\to 0} \nu(-M_j - a + \varepsilon, M_j - a - \varepsilon) ;$$

note that since $\pm M_j - a$ may not be continuity points of ν , we cannot assert that $\lim_k \tau_{N_k}(I_j - a) = \nu(I_j - a)$. However, since $M_{j-1} < M_j$, the last limit above is at least $\nu(I_{j-1} - a)$, whence $\mu I_j \geq \nu(I_{j-1} - a)$. As $j \to \infty$, the latter quantity $\to 1$. Therefore, $\mu I_j \to 1$, as desired. □

We denote by μ^m the m-fold convolution $\mu * \mu * \ldots * \mu$. It is evident that if $\mu \in R[\mu \notin R]$, then $\mu^m \in R[\mu^m \notin R]$ for all $m \geq 1$. Now if μ is an infinite convolution of discrete probability measures, then so is μ^m . Since μ , μ^m are pure, we deduce

COROLLARY 3.7. If μ is an infinite convolution of discrete probability measures on $\mathbb{T}$ or on $\mathbb{R}$, then either $\mu^m \in R$ for all $m \geq 1$ or $\mu^m \in J$ for all $m \geq 1$.

As noted for random variables, weak* limit points of infinite convolutions of discrete probability measures are pure.

We now turn to a more detailed examination of certain infinite convolutions and, in particular, we shall be able to tell whether they belong to R or to J .

The measure (II.5.5),

$$(3.5)\qquad \mu_\theta = \mathop{\Large *}_{n=1}^{\infty} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta((\theta-1)\theta^{-n})] \quad , \theta > 2 ,$$

is well-known to be a Rajchman measure if θ is not a P-V number and to be supported on a U-set if θ is a P-V number (Zygmund [1, II, pp. 147-152]). Hence:

PROPOSITION 3.8. If θ is not a P-V number, $\mu_\theta \in R$. If θ is a P-V number, $\mu_\theta \in J$.

With regard to the measure (II.5.6),

$$(3.6)\quad \begin{cases} \mu = \mathop{\Large *}_{n=1}^{\infty} [p_n\delta(0) + q_n\delta(2^{-n})] , \\ p_n + q_n = 1 , \Sigma p_n = \infty = \Sigma q_n , \end{cases}$$

we have a similar result.

THEOREM 3.9. For μ as in (3.6), let $b = \overline{\lim_{n\to\infty}} |p_n - q_n|$. Then

$$(3.7)\qquad \frac{2b}{\pi} \le R(\mu) \le b .$$

If $b > 0$, then μ is concentrated on a W-set . Hence $\mu \in R$ if $b = 0$ and $\mu \in J$ if $b > 0$.

PROOF. See Graham and McGehee [1, pp. 183, 187-8] for the proof of (3.7).

Suppose $b > 0$. Let

$$(3.8)\qquad s_n(x) = \chi_{[0,\frac{1}{2})}(2^{n-1}x) .$$

Then $\{s_n\}$ are independent random variables with respect to μ and have expectations

$$\int s_n \, d\mu = p_n .$$

Let $\{n_k\}$ be such that $p_{n_k} \to p$ for some $p \ne \frac{1}{2}$. By Lemma III.2.5, there exists a subsequence $\{n_k'\} \subset \{n_k\}$ for which $\{2^{n_k'-1}x\}$ has an asymptotic distribution for almost every x $[\mu]$. By the strong law of large numbers,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_{[0,\frac{1}{2})}(2^{n_k'-1}x) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k'}(x)$$

$$= \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} p_{n_k'} = p \ne m\,[0,\tfrac{1}{2})$$

for a.e. x $[\mu]$. It follows that $\{2^{n_k'-1}x\}$ is Weyl-distributed for a.e. x $[\mu]$. □

Actually, in most cases, there is no need to appeal to Lemma III.2.5 because we can calculate the asymptotic distribution of $\{2^{n_k'-1}x\}$ explicitly for a certain sequence $\{n_k'\}$:

THEOREM 3.10. Let μ be as in (3.6). Assume that

(3.9) $$\underline{\lim}\ p_n > 0 \ , \quad \underline{\lim}\ q_n > 0 \ .$$

Then there exist sequences $\{n_k\}$, $\{\alpha_k\}$ such that

(3.10) $$\lim_{k\to\infty} p_{n_k+m} = \alpha_m \quad \text{for all} \quad m \geq 1 \ .$$

For any such sequences, if $\beta_k = 1 - \alpha_k$ and

(3.11) $$\nu = \mathop{\ast}_{m=1}^{\infty} [\alpha_m \delta(0) + \beta_m \delta(2^{-m})] \ ,$$

then

(3.12) $$\{2^{n_k} x\} \sim \nu \quad \text{a.e. } [\mu] \ .$$

EXAMPLE. If $p_n \to p$, we can take $n_k = k$ and $\alpha_m = p$, whence

$$\{2^n x\} \sim \mathop{\ast}_{1}^{\infty} [p\delta(0) + q\delta(2^{-m})] \quad \text{a.e. } [\mu] \ .$$

In fact, n_k can be chosen arbitrarily, so that $\{2^{n_k} x\}$ also has the same asymptotic distribution for a.e. x $[\mu]$. If $p = \frac{1}{2}$, then the distribution is uniform: μ puts no mass on the set of non-normal numbers base 2 . This shows why the more complicated construction of Theorem III.6.1 was needed.

PROOF. The existence of $\{n_k\}$, $\{\alpha_k\}$ satisfying (3.10) follows from an easy diagonal argument. Also note that by (3.9), $\underline{\lim}\ \alpha_m > 0$, $\underline{\lim}\ \beta_m > 0$, so that $\Sigma\alpha_m = \infty = \Sigma\beta_m$.

Let s_n be as in (3.8). By the strong law of large numbers,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_{[0,\frac{1}{2})} (2^{n_k} x) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k+1}(x)$$

$$= \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} p_{n_k+1} = \alpha_1 = \nu[0,\tfrac{1}{2}) \quad \text{a.e. } [\mu] \ .$$

Likewise, since $s_{n_k+1} s_{n_k+2}$ and $s_{n_{k+2}+1} s_{n_{k+2}+2}$ are independent,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_{[0,\frac{1}{4})} (2^{n_k} x) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k+1}(x)\, s_{n_k+2}(x)$$

$$= \lim_{K\to\infty} \frac{1}{K} \sum_{\substack{k=1 \\ k \text{ odd}}}^{K} s_{n_k+1}(x)\, s_{n_k+2}(x)$$

$$+ \lim_{K\to\infty} \frac{1}{K} \sum_{\substack{k=1 \\ k \text{ even}}}^{K} s_{n_k+1}(x)\, s_{n_k+2}(x)$$

$$= \tfrac{1}{2}\alpha_1\alpha_2 + \tfrac{1}{2}\alpha_1\alpha_2 = \alpha_1\alpha_2 = \nu[0,\tfrac{1}{4}) \quad \text{a.e. } [\mu]$$

and therefore

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_{[\frac{1}{4},\frac{1}{2})} (2^{n_k} x) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k+1}(x)\, [1 - s_{n_k+2}(x)]$$

$$= \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k+1}(x) - \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} s_{n_k+1}(x)\, s_{n_k+2}(x)$$

$$= \alpha_1 - \alpha_1\alpha_2 = \alpha_1\beta_2 = \nu[\tfrac{1}{4},\tfrac{1}{2}) \quad \text{a.e. } [\mu] \ .$$

In general, a similar argument shows that for any dyadic interval $I = \left[\frac{a}{2^N}, \frac{a+1}{2^N}\right)$,

$$(3.13) \qquad \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_I (2^{n_k} x) = \nu I \quad \text{a.e. } [\mu].$$

It follows that (3.13) holds for every interval I whose endpoints are points of continuity of ν (see the argument in Section II.2), which shows (3.12). □

In the same way, if $\theta > 2$ is an <u>integer</u> and

$$\mu_\theta = \mathop{\ast}_{n=1}^{\infty} [p_n \delta(0) + q_n \delta((\theta-1)\,\theta^{-n})],$$

$$\underline{\lim}\; p_n > 0 , \quad \underline{\lim}\; q_n > 0 ,$$

we can find a

$$\sigma = \mathop{\ast}_{m=1}^{\infty} [\alpha_m \delta(0) + \beta_m \delta((\theta-1)\,\theta^{-m})]$$

such that μ_θ is concentrated on the set

$$\mathbb{S} = \{x : (\theta^{n-1} x) \sim \sigma\} .$$

If $\theta = 3$ and $p_n \to \frac{1}{2}$, then σ is the Cantor-Lebesgue measure. This should be compared with our remarks about the Cantor-Lebesgue measure in Section II.3.

Even though for different sequences $\{p_n\}$ tending to $\frac{1}{2}$, the different measures μ_3 are concentrated on E, it is still possible for them to be mutually singular.

Indeed, if $\{p_n\}$, $\{p'_n\}$ are two sequences tending to $\frac{1}{2}$, the corresponding measures μ_3, μ'_3 will be mutually singular if

$$\sum_{n=1}^{\infty} (p_n - p'_n)^2 = \infty .$$

This follows from

THEOREM 3.11. Let

$$(3.14) \quad \begin{cases} \mu = \mathop{\ast}\limits_{n=1}^{\infty} [p_{n,1}\,\delta(0) + p_{n,2}\,\delta(3^{-n}) + p_{n,3}\,\delta(2\cdot 3^{-n})] , \\ \mu' = \mathop{\ast}\limits_{n=1}^{\infty} [p'_{n,1}\,\delta(0) + p'_{n,2}\,\delta(3^{-n}) + p'_{n,3}\,\delta(2\cdot 3^{-n})] , \\ p_{n,1} + p_{n,2} + p_{n,3} = p'_{n,1} + p'_{n,2} + p'_{n,3} = 1 , \quad p_{n,i} \geq 0 , \end{cases}$$

$$(3.15) \quad \begin{cases} \infty = \sum\limits_{n=1}^{\infty} (1 - p_{n,1}) = \sum\limits_{n=1}^{\infty} (1 - p_{n,3}) \\ \quad = \sum\limits_{n=1}^{\infty} (1 - p'_{n,1}) = \sum\limits_{n=1}^{\infty} (1 - p'_{n,3}) . \end{cases}$$

Then $\mu \perp \mu'$ if for some $i = 1, 2, 3$,

$$(3.16) \qquad \sum_{n=1}^{\infty} (p_{n,i} - p'_{n,i})^2 = \infty .$$

REMARK. Of course, a similar theorem holds for measures of the form (3.6) or of the form

$$(3.17)\quad \begin{cases} \mu = \mathop{\times}\limits_{n=1}^{\infty} [\sum_{i=1}^{q} p_{n,i}\ \delta((i-1)\ q^{-n})] , \\ \sum_{i=1}^{q} p_{n,i} = 1 ,\quad p_{n,i} \geq 0 . \end{cases}$$

PROOF. The maps ϕ of Section III.5 which define the infinite convolutions μ , μ' from infinite product measures are measure-copreserving because of (3.15). Thus

$$s_{n,i} \equiv \chi_{[(i-1)\ 3^{-n}, i3^{-n})}(3^{n-1}\ x)$$

are, for fixed i , independent random variables with respect to both μ and μ' . Choosing the i for which (3.16) holds, we find that the conclusion follows immediately from Theorem 1 of Brown [1] . □

In the next section, very similar results will be described for Riesz products.

4. Riesz Products

We will use the following more compact notation for Riesz products:

$$(4.1)\quad \begin{cases} d\mu = \prod_{k=1}^{\infty} [1 + \mathrm{Re}(\gamma_k e(n_k x))]\ dm , \\ |\gamma_k| \leq 1 . \end{cases}$$

Note that γ_k may be complex.

All the results in this section follow from a principle which is almost the same as Theorem III.4.1, namely

THEOREM 4.1. If μ is a probability measure and $\{m_k\}$ is any sequence with

$$(4.2)\quad \sum_{K=1}^{\infty} \frac{1}{K^3} \mathrm{Re}\{ \sum_{1\leq \ell < k \leq K} [\hat{\mu}(m_k - m_\ell) - \hat{\mu}(m_k)\ \hat{\mu}(-m_\ell)] \} < \infty ,$$

then

$$(4.3)\quad \lim_{K\to\infty}\{\frac{1}{K}\sum_{k=1}^{K} e(m_k x) - \frac{1}{K}\sum_{k=1}^{K} \hat{\mu}(-m_k)\} = 0 \quad \text{a. e. } [\mu] .$$

PROOF. Let

$$f_K(x) = \frac{1}{K}\sum_{k=1}^{K} [e(m_k x) - \hat{\mu}(-m_k)] .$$

Then it is easily calculated that

$$\|f_K\|^2_{L^2(\mu)} = \frac{1}{K} - \frac{1}{K^2}\sum_{k=1}^{K} |\hat{\mu}(m_k)|^2$$

$$+ \frac{2}{K^2}\ \mathrm{Re} \sum_{1\leq \ell<k\leq K} [\hat{\mu}(m_k-m_\ell) - \hat{\mu}(m_k)\ \hat{\mu}(-m_\ell)] .$$

Hence (4.2) is equivalent to

$$\sum_{K=1}^{\infty} \frac{1}{K} \|f_K\|^2_{L^2(\mu)} < \infty .$$

The proof of Theorem III.4.1 shows that

$$\lim_{K\to\infty} f_K(x) = 0 \quad \text{a.e. } [\mu],$$

which is (4.3). □

Our needs will be more than met by the following simple

COROLLARY 4.2. Let

(4.4) $\Delta_K = \#\{(k,\ell) \mid 1\leq \ell<k\leq K ,\ \hat{\mu}(m_k-m_\ell) \neq \hat{\mu}(m_k)\ \hat{\mu}(-m_\ell)\}$.

If μ is a probability measure, if

$$(4.5) \qquad \sum_{K=1}^{\infty} \frac{\Delta_K}{K^3} < \infty ,$$

and if $\hat{\mu}(m_k) \to \alpha$ as $k \to \infty$, then

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} e(m_k x) = \bar{\alpha} \quad \text{a.e. } [\mu] .$$

COROLLARY 4.3. Let μ be a probability measure with $\hat{\mu}(m_k) \to \alpha \neq 0$. Let Δ_K be as in (4.4) and assume $\Delta_K = O(1)$. Then there exists a subsequence $\{m_k'\} \subset \{m_k\}$ such that for all $n \geq 1$, μ^n is concentrated on $W(\{m_k'\})$, the maximal W-set corresponding to $\{m_k'\}$.

PROOF. If we denote the dependence of Δ_K on the measure and sequence by $\Delta_K = \Delta_K(\mu,\{m_k\})$, then note that $\Delta_K(\mu,\{m_k'\}) = O(1)$ for any $\{m_k'\} \subset \{m_k\}$. Also, since $\widehat{\mu^n} = \hat{\mu}^n$, $\Delta_K(\mu^n,\{m_k'\}) = \Delta_K(\mu,\{m_k'\})$ and $\widehat{\mu^n}(m_k') \to \alpha^n$.

By Lemma III.2.5 and a diagonal argument, there exists $\{m_k'\} \subset \{m_k\}$ such that for all $n \geq 1$ and for μ^n-almost all x, $\{m_k' x\}$ has an asymptotic distribution $\nu = \nu(n,x)$. Since $\Delta_K(\mu^n,\{m_k'\}) = O(1)$, (4.5) holds, whence Corollary 4.2 gives $\hat{\nu}(1) = \alpha^n$. Since $\alpha \neq 0$, ν is not Lebesgue measure for any n or any x . This completes the proof. □

We now show that Riesz products are purely in R or J .

THEOREM 4.4. Let μ be the Riesz product (4.1) with

$$(4.6) \qquad n_{k+1}/n_k \geq \frac{3+\sqrt{5}}{2} = 2.618^{+} .$$

Then $\mu \in R$ if $\gamma_k \to 0$ and $\mu \in J$ if $\gamma_k \not\to 0$. In the latter case, there is a W-set E such that for all $m \geq 1$, μ^m is concentrated on E .

PROOF. Since we are not assuming $n_{k+1}/n_k \geq 3$, we must first show that (4.1) is well-defined, i.e., that there is a unique weak* limit of the partial products of (4.1). This will follow if we establish that any $n \in \mathbb{Z}$ can be written in the form

(4.7) $$n = \Sigma \, \epsilon_k n_k \; , \quad \epsilon_k = -1 \; , \; 0 \; , \; 1 \; ,$$

in only a finite number of ways.

Let $q = \frac{3+\sqrt{5}}{2}$. Since $q + \frac{1}{q} = 3$,

$$q + \frac{1}{q} + \frac{1}{q^{k-1}} > 3 \; ,$$

or

$$q^k - 2q^{k-1} + q^{k-2} > q^{k-1} - 1 \; ,$$

or

$$q - 1 > \frac{q^{k-1}-1}{q-1} \cdot \frac{1}{q^{k-2}} = \frac{1-q^{-k+1}}{1-q^{-1}}$$

$$= 1 + \frac{1}{q} + \frac{1}{q^2} + \ldots + \frac{1}{q^{k-2}} \; .$$

Since $n_{k-1} \geq q^{j-1} n_{k-j}$ $(j \geq 0)$, it follows that

$$\frac{n_k}{n_{k-1}} - 1 > 1 + \frac{n_{k-2}}{n_{k-1}} + \frac{n_{k-3}}{n_{k-1}} + \ldots + \frac{n_1}{n_{k-1}} \; ,$$

or

(4.8) $$n_k - n_{k-1} > n_{k-1} + n_{k-2} + \ldots + n_1 \; .$$

If $k \geq \frac{\log n}{\log q} + 2$, then

$$n_k - n_{k-1} - n_{k-2} - \ldots - n_1 > n_{k-1} \geq n \; ,$$

whence if (4.7) holds, $\epsilon_k = 0$ for $k \geq \frac{\log n}{\log q} + 2$. This establishes the claim.

Furthermore, if $k \neq \ell$, then (4.8) also shows that $n_k - n_\ell$ may be written in the form (4.7) in exactly one way. This means that for $k \neq \ell$

(4.9) $$\hat{\mu}(n_k - n_\ell) = \hat{\mu}(n_k)\, \hat{\mu}(-n_\ell) \; .$$

Note that $R(\mu) = \overline{\lim_{k\to\infty}} |\gamma_k|$. Therefore $\mu \in R$ if and only if $\gamma_k \to 0$. Suppose $\gamma_k \not\to 0$. Let $\gamma_{k_j} \to \alpha \neq 0$. If Δ_K is as in (4.4) for the sequence $\{n_{k_j}\}$, then $\Delta_K = 0$ by (4.9). Thus, the theorem follows from Corollary 4.3. □

REMARK 1. If $\{\gamma_k\}$ has more than one limit point, then μ is concentrated on an intersection of W-sets in the obvious way. Other W-sets on which μ is concentrated may be obtained as follows: Let $\{m_k\}$ be a sequence of numbers of the form

$$m_k = \Sigma \, \epsilon_\ell(k)\, n_\ell \quad , \quad \epsilon_\ell = -1 \; , \; 0 \; , \; 1$$

such that $\hat{\mu}(m_k) \to \beta \neq 0$ and such that $k_1 \neq k_2 \Rightarrow \forall \ell \;\, \epsilon_\ell(k_1) \cdot \epsilon_\ell(k_2) = 0$. (If $\gamma_{k_j} \to \beta \neq 0$, then $\{n_{k_j}\}$ is such a sequence.) Then there exists a subsequence $\{m_k'\} \subset \{m_k\}$ such that μ is concentrated on $\{x \,|\, \exists \, \nu\colon \{m_k' x\} \sim \nu \text{ and } \hat{\nu}(1) = \beta\}$.

REMARK 2. The measure μ^m is a Riesz product:

$$d\mu^m = \prod_{k=1}^{\infty} [1 + \mathrm{Re}\{\gamma_k^m\, e(n_k x)\}]\ dm .$$

In the previous section, we showed that general infinite convolutions of discrete probability measures belong either to R or to J . For certain special cases, we determined which alternative held and we showed that in the latter case, the measure is concentrated on a specified W-set . We have now shown that the general Riesz product is either in R or in J . Also, we have determined which alternative holds in general and shown that in the latter case, the measure is concentrated on a W-set . Thus, general Riesz products are more tractable than general infinite convolutions. Nevertheless, in order to specify on which W-set a Riesz product in J is concentrated, we are again forced to consider special cases: While we are able to specify $\hat{\nu}(1)$ for the limiting distribution ν in the general case, the higher coefficients remain unknown. This is because for $m > 1$, the representations of mn_ℓ in the form (4.7) are unknown. Therefore, we do not know if (4.2) holds.

As mentioned, however, for certain sequences $\{n_k\}$, we can resolve these problems. The easiest case is when $\{n_k\}$ is hyperlacunary, i.e. $n_{k+1}/n_k \to \infty$. The other case we will deal with is $n_k = q^{k-1}$, $q \geq 3$ an integer.

THEOREM 4.5. Let μ be the Riesz product (4.1) with $\{n_k\}$ hyperlacunary. Let $\gamma_{k_\ell} \to \gamma$ and set

$$(4.10)\qquad d\nu(x) = (1 + \mathrm{Re}\{\gamma\, e(x)\})\ dm(x) .$$

Then

$$(4.11)\qquad \{n_{k_\ell} x\}_{\ell=1}^{\infty} \sim \nu \quad \text{a.e. } [\mu] .$$

PROOF. Certainly for $m = -1 , 0 , 1$,

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} e(mn_{k_\ell} x) = \hat{\nu}(-m) \quad \text{a.e. } [\mu] .$$

For $|m| \geq 2$, there exists $k_o = k_o(m)$ such that $m\,n_k$ and $m\,n_k - m\,n_\ell$ have <u>no</u> representations of the form (4.7) for $k \geq k_o$, $k > \ell$. Therefore for fixed m , $\Delta_K = O(1)$ for the sequence $\{m\,n_k\}$ and Corollary 4.2 gives

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} e(m\,n_{k_\ell} x) = 0 = \hat{\nu}(-m) \quad \text{a.e. } [\mu]$$

for $|m| \geq 2$. □

THEOREM 4.6. Let μ be the Riesz product

$$(4.12)\qquad d\mu = \prod_{k=0}^{\infty} [1 + \mathrm{Re}\{\gamma_k e(q^k x)\}]\ dm$$

with $q \geq 3$ an integer. Let $k_\ell \uparrow \infty$ be such that $\gamma_{k_\ell + j} \to \alpha_j$ for $j \geq 1$ and set

$$(4.13) \qquad d\nu = \prod_{j=0}^{\infty} [1 + \mathrm{Re}\{\alpha_j e(q^j x)\}] \, dm .$$

Then

$$(4.14) \qquad \{q^{k_\ell} x\}_{\ell=1}^{\infty} \sim \nu \quad \text{a.e. } [\mu] .$$

PROOF. The number of representations

$$(4.15) \qquad n = \sum_{o}^{\infty} \varepsilon_j q^j \quad , \quad \varepsilon_j = -1 , 0 , 1$$

is at most one and is the same as the number of representations of nq^k . Define

$$(4.16) \qquad j_o(n) = \begin{cases} \min\{j: \varepsilon_j \neq 0\} & \text{if } n \text{ has the form (4.15)}, \\ 0 & \text{otherwise} , \end{cases}$$

$$(4.17) \qquad j_1(n) = \min\{j: q^{j-1} \geq |n|\} .$$

Fix n of the form (4.15). Note that $\varepsilon_j = 0$ unless $j_o \leq j < j_1$. Let $m_\ell = n \cdot q^{k_\ell}$. For any complex number z , we denote

$$z^{(\varepsilon)} = \begin{cases} z & \text{if } \varepsilon = 1 , \\ 1 & \text{if } \varepsilon = 0 , \\ \bar{z} & \text{if } \varepsilon = -1 . \end{cases}$$

Then

$$m_\ell = \sum_{j=o}^{\infty} \varepsilon_j q^{k_\ell + j} \quad ,$$

whence

$$(4.18) \qquad \begin{aligned} \hat{\mu}(-m_\ell) &= \prod_{j=o}^{\infty} \left(\tfrac{1}{2} \overline{\gamma}_{k_\ell + j}\right)^{(\varepsilon_j)} \\ &\to \prod_{j=o}^{\infty} \left(\tfrac{1}{2} \overline{\alpha}_j\right)^{(\varepsilon_j)} = \hat{\nu}(n) \end{aligned}$$

as $\ell \to \infty$. (Since $\varepsilon_j \neq 0$ for only finitely many j , these products are really finite.)

Now if $\ell - p > j_1(n)$, then $k_\ell - k_p > j_1(n)$. Therefore if $\ell - p > j_1(n)$, then the powers of q are distinct in the two sums

$$m_\ell - m_p = \sum_{j \geq j_o(n)} \varepsilon_j q^{k_\ell + j} + \sum_{j \leq j_1(n)} (-\varepsilon_j) \, q^{k_p + j} ,$$

so that

$$\hat{\mu}(m_\ell - m_p) = \hat{\mu}(m_\ell) \, \hat{\mu}(-m_p) .$$

Thus for Δ_K as in (4.4),

$$\Delta_K \leq j_1(n) K ,$$

so (4.5) certainly holds. By Corollary 4.2 and (4.18), it follows that

$$(4.19) \qquad \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} e(-n \, q^{k_\ell} x) \to \hat{\nu}(n) \quad \text{a.e. } [\mu] .$$

Now if n does not have a representation of the form (4.15), we claim that neither do m_ℓ , m_p , or $m_\ell - m_p$ for $\ell - p > j_1(n)$, where again $m_\ell = n \, q^{k_\ell}$. For suppose

$$nq^{k_\ell} - nq^{k_p} = \sum_{j=o}^{\infty} \varepsilon_j q^j .$$

Then

$$(4.20) \quad nq^{k_\ell} - \sum_{j=k_\ell}^{\infty} \varepsilon_j q^j = nq^{k_p} + \sum_{j=o}^{k_\ell - 1} \varepsilon_j q^j .$$

The left-hand side is divisible by q^{k_ℓ}, while the right is at most (in absolute value)

$$q^{j_1(n)-1} q^{k_p} + \sum_{j=o}^{k_\ell - 1} q^j < q^{k_\ell}$$

since $k_\ell - k_p > j_1(n)$. Therefore, both sides of (4.20) must be equal to 0 :

$$nq^{k_\ell} = \sum_{j=k_\ell}^{\infty} \varepsilon_j q^j ,$$

$$n = \sum_{j=o}^{\infty} \varepsilon_{j+k_\ell} q^j ,$$

contradicting the assumption that n has no such representation. Our claim is established.

Therefore $\Delta_K \leq j_1(n) \cdot K$ and (4.19) holds for such n as well. This shows (4.14). □

An especially interesting corollary is

COROLLARY 4.7. Let μ be as in (4.12) with $\gamma_k \to 0$. Then the maximal W*-set of any sequence $\{q^{k_\ell}\}_{\ell=1}^{\infty}$ has μ-measure 0 .

Note that this is true no matter how slowly $\gamma_k \to 0$. This should be compared with the example following Theorem 3.10 and with Theorem III.6.1.

Let us also remark that since $\log(1+r \cos 2\pi t) \in C(\mathbb{T})$ for $-1 < r < 1$, if

$$d\mu = \prod_{k=o}^{\infty} (1 + r \cos 2\pi 3^k x) \; dm ,$$

then from Theorems 4.5 and II.1.2,

$$\frac{1}{K+1} \sum_{k=o}^{K} \log(1+r \cos 2\pi 3^k x) \to \int_{\mathbb{T}} \log(1+r \cos 2\pi t) \; d\mu(t)$$

for almost every $x \; [\mu]$. The fact that these limits exist and are equal for μ - a.e. x is a theorem of Y. Meyer and B. Weiss (see Peyrière [1]).

As in the preceding section, our discussion of the sets on which Riesz products are concentrated would be incomplete without mentioning the following orthogonality result of Brown and Moran (see Graham and McGehee [1, p. 203]):

THEOREM 4.8. Let

$$d\mu = \prod_{k=1}^{\infty} [1 + \mathrm{Re}\{\gamma_k e(n_k x)\}] \; dm ,$$

$$d\mu' = \prod_{k=1}^{\infty} [1 + \mathrm{Re}\{\gamma_k' \, e(n_k x)\}] \; dm .$$

Then $\mu \perp \mu'$ if

$$\sum_{k=1}^{\infty} |\gamma_k - \gamma_k'|^2 = \infty .$$

The proof is very similar to that of Theorem 3.11.

An interesting consequence of Theorem 4.4 is

PROPOSITION 4.9. There exist $\mu, \nu \in J$ such that $\mu * \nu = m$ and $\mu \perp \nu$. There exist W-sets E_1, E_2 such that $E_1 + E_2 = \{s + t \mid s \in E_1, t \in E_2\}$ has full Lebesgue measure.

PROOF. Choose sequences $\{n_k\}$, $\{m_k\}$ so that if μ, ν are any corresponding Riesz products not in R, then the sets of frequencies on which $\hat{\mu}, \hat{\nu}$ are supported are disjoint except for $\{0\}$. Then $\mu, \nu \in J$ and $\hat{\mu} \cdot \hat{\nu} = \hat{m}$, i.e. $\mu * \nu = m$. Furthermore, use of the random variables $e(n_k x)$ in Theorem 1 of Brown [1] yields $\mu \perp \nu$. Finally, if μ, ν are concentrated on the W-sets E_1, E_2, then m is concentrated on $E_1 + E_2$. □

REMARK. We may also prove Proposition 4.9 by using infinite convolutions. Let $\mu = \mathop{\ast}\limits_{n=1}^{\infty} [\frac{1}{2}\delta(0) + \frac{1}{2}\delta(2^{-2n})]$ and

$\nu = \mathop{\ast}\limits_{n=1}^{\infty} [\frac{1}{2}\delta(0) + \frac{1}{2}\delta(2^{-2n+1})]$. Then μ is supported on the H-set $F_1 = \{x \in \mathbb{T} : \text{every odd binary digit is } 0\}$ and ν is supported on the H-set $F_2 = \{x \in \mathbb{T} : \text{every even binary digit is } 0\}$. Hence $\mu, \nu \in J$, $\mu \perp \nu$, and $\mu * \nu = m$. This also exhibits $\mathbb{T} = F_1 + F_2$ as the sum of two H-sets. From this and Proposition 5.14 (or Theorem 3.9), we may find W-sets $E_i \subset F_i$ such that $m(E_1 + E_2) = 1$.

COROLLARY 4.10. There exists $\mu \in J$ such that $\mu * \mu \notin J$.

PROOF. Let $\nu_1, \nu_2 \in J$ be such that $\nu_1 * \nu_2 \notin J$. Let $\sigma = \nu_1 + \nu_2$. Then

$$\nu_1 * \nu_2 = \frac{1}{2}(\sigma^2 - \nu_1^2 - \nu_2^2),$$

whence one of the three squares on the right is not in J. (For ν_1, ν_2 as in the previous proof, $\nu_1^2 \in J$ and $\nu_2^2 \in J$, so that $\sigma^2 \notin J$.) □

5. Containment Relations

We have seen that the class of W-sets is contained in the class of W*-sets . Here, we consider other such relations among the classes of sets we have been discussing. It will be useful to begin with the following result.

THEOREM 5.1. A finite or countable union of W*-sets is a W*-set . Briefly, $W^*_\sigma = W^*$.

Dress [1] demonstrated this theorem for finite unions. The extension to countable unions requires a slight modification of his concept of "mixing" two sequences. We shall also simplify the proofs of the corresponding lemmas by employing Weyl's criterion.

NOTATION. If N is a sequence and $k \in \mathbb{Z}^+$, $N(k)$ denotes the k-th element of N .

DEFINITION. Let N and M be two sequences. The sequence $P = N \operatorname{mix} M$ is defined by

$$P(1) = N(1) , P(2) = M(1) ,$$

$$P(2^k + r) = N(2^{k-1} + r) \text{ for } 0 < r \le 2^{k-1} , k \ge 1 ,$$

$$P(2^k + r) = M(2^{k-1} + r) \text{ for } 2^{k-1} < r \le 2^k , k \ge 1 .$$

Thus

$$P(3) = N(2) ,$$

$$P(4) = M(2) ,$$

$$P(5) = N(3) , P(6) = N(4) ,$$

$$P(7) = M(3) , P(8) = M(4) ,$$

$$P(9) = N(5) , P(10) = N(6) , P(11) = N(7) , P(12) = N(8) ,$$

$$\ldots .$$

DEFINITION. If $N \subset \mathbb{Z}$ is any sequence, $K \in \mathbb{Z}^+$, $x \in \mathbb{T}$, and $\ell \in \mathbb{Z}$, let

$$S_N(K, x, \ell) = \sum_{k=1}^{K} e(-\ell N(k)x) .$$

Let

$$W^*(N) = \{x \in \mathbb{T}: (\exists \ell \neq 0)\ S_N(K,x,\ell) \neq o(K)\} .$$

Note that in this definition, N is not required to be a strictly increasing sequence of positive integers. If N is such a sequence, however, then $W^*(N)$ is the maximal W*-set corresponding to N as in Section II.2. Note also that even if N and M are increasing sequences, $N \operatorname{mix} M$ need not be.

LEMMA 5.2. Fix $N \subset \mathbb{Z}$, $x \in \mathbb{T}$, $\ell \in \mathbb{Z}$. Then

$$S_N(2^k, x, \ell) = o(2^k)$$

if and only if

$$S_N(2^{k+1},x,\ell) - S_N(2^k,x,\ell) = o(2^k) .$$

PROOF. Assuming the first condition, we have

$$S(2^{k+1}) - S(2^k) = o(2^{k+1}) + o(2^k) = o(2^k) ,$$

which is the second condition. Conversely, if the second condition holds, the first follows from

$$S(2^k) = \sum_{j=1}^{k} (S(2^j) - S(2^{j-1})) + S(1) . \square$$

LEMMA 5.3. Fix $N \subset \mathbb{Z}$, $x \in \mathbb{T}$, $\ell \in \mathbb{Z}$. Let

$$M_k = \max\{|S_N(K,x,\ell) - S_N(2^k,x,\ell)| : 2^k < K \leq 2^{k+1}\} .$$

Then

$$S_N(K,x,\ell) = o(K) \Leftrightarrow M_k = o(2^k) .$$

PROOF. If $S_N(K,x,\ell) = o(K)$, then

$$S_N(K,x,\ell) - S_N(2^k,x,\ell) = o(K) + o(2^k) .$$
$$= o(2^k)$$

uniformly for $2^k < K \leq 2^{k+1}$, i.e., $\forall \varepsilon > 0 \, \exists k_o \, \forall k > k_o$ if $2^k < K \leq 2^{k+1}$, then $|S_N(K) - S_N(2^k)|/2^k < \varepsilon$. That is, $M_k = o(2^k)$.

Conversely, suppose $M_k = o(2^k)$. By the previous lemma, $S_N(2^k,x,\ell) = o(2^k)$. Hence, if $2^k < K \leq 2^{k+1}$,

$$S(K) = (S(K) - S(2^k)) + S(2^k)$$
$$= o(2^k) + o(2^k) = o(K) . \square$$

PROPOSITION 5.4. Let $N, M \subset \mathbb{Z}$. Then

$$W^*(N \text{ mix } M) = W^*(N) \cup W^*(M) .$$

PROOF. Let $P = N$ mix M. Suppose $x \notin W^*(N) \cup W^*(M)$. Then for $0 \neq \ell \in \mathbb{Z}$ and $K \in \mathbb{Z}^+$, there exist K_1, $K_2 \geq 0$ such that $K = K_1 + K_2$ and

$$S_P(K,x,\ell) = S_N(K_1,x,\ell) + S_M(K_2,x,\ell)$$
$$= o(K_1) + o(K_2) = o(K) .$$

Thus $x \notin W^*(P)$.

Conversely, suppose $x \notin W^*(P)$. Fix $\ell \neq 0$. Since $S_P(K,x,\ell) = o(K)$, we have

$$S_P(2^j + K) - S_P(2^{j+1}) = o(2^j)$$

uniformly for $K \in (2^j, 2^{j+1}]$. But

$$S_P(2^j+K) - S_P(2^{j+1}) = S_N(K) - S_N(2^j) .$$

Thus Lemma 5.3 gives $S_N(K,x,\ell) = o(K)$, i.e., $x \notin W^*(N)$. Similarly, the relation

$$S_P(2^{j+1} + K) - S_P(2^{j+1} + 2^j) = S_M(K) - S_M(2^j)$$

for $K \in (2^j, 2^{j+1}]$ gives $x \notin W^*(M)$. $\square$

DEFINITION. Given N, $A \subset \mathbb{Z}$, define $N' = N$ add A by

$$N'(1) = N(1) + A(1) ,$$
$$N'(2^k + r) = N(2^k + r) + A(k + 2)$$

for $0 < r \leq 2^k$, $k \geq 0$. Thus

$N'(1) = N(1) + A(1)$,

$N'(2) = N(2) + A(2)$,

$N'(3) = N(3) + A(3)$, $N'(4) = N(4) + A(3)$,

$N'(5) = N(5) + A(4)$, $N'(6) = N(6) + A(4)$,

$N'(7) = N(7) + A(4)$, $N'(8) = N(8) + A(4)$,

... .

PROPOSITION 5.5. For any $N, A \subset \mathbb{Z}$, we have

$$W^*(N \text{ add } A) = W^*(N) .$$

PROOF. Let $N' = N$ add A . It is sufficient to show that $W^*(N') \subset W^*(N)$ since $N = N'$ add $(-A)$. Let $x \notin W^*(N)$, $0 \neq \ell \in \mathbb{Z}$. Then $S_N(K,x,\ell) = o(K)$. For $K \in (2^j, 2^{j+1}]$, $|S_N(K,x,\ell) - S_N(2^j,x,\ell)| = |S_{N'}(K,x,\ell) - S_{N'}(2^j,x,\ell)|$. Therefore, by two uses of Lemma 5.3, it follows that $S_{N'}(K,x,\ell) = o(K)$. Hence $x \notin W^*(N')$. □

Given strictly increasing sequences N_1 , $N_2 \subset \mathbb{Z}^+$, it is clear that A_1 , $A_2 \subset \mathbb{N}$ can be chosen inductively so that N_1' mix N_2' is strictly increasing, where $N_i' = N_i$ add A_i $(i = 1,2)$. If A_1 , A_2 are chosen with least possible elements $A_i(k)$, then we define

$$N_1 \text{ Mix } N_2 = (N_1 \text{ add } A_1) \text{ mix}(N_2 \text{ add } A_2) .$$

From Propositions 5.4 and 5.5, we immediately deduce

PROPOSITION 5.6. If N_1 , $N_2 \subset \mathbb{Z}^+$ are strictly increasing, then there exists $N \subset \mathbb{Z}^+$ strictly increasing such that $W^*(N_1) \cup W^*(N_2) = W^*(N)$.

Therefore finite unions of W^*-sets are W^*-sets . To show the same for countable unions, we define the countable mixture of sequences as follows. Given N_1 , N_2 , ..., let

$$P_o = N_1 \text{ mix } \mathbb{N} ,$$

$$P_1 = N_1 \text{ mix}(N_2 \text{ mix } \mathbb{N}) ,$$

$$P_2 = N_1 \text{ mix}(N_2 \text{ mix}(N_3 \text{ mix } \mathbb{N})) ,$$

... .

($\mathbb{N}$ plays the role of blanks or holding places which get filled by successive N_i's .) It is clear that for all k , there exists $j_o(k)$ such that for $j \geq j_o(k)$, $P_j(k) = P_{j_o(k)}(k)$. Define $M_o = \text{mix}(N_1, N_2, \ldots)$ by

$$M_o(k) = P_{j_o(k)}(k) .$$

(Another notation could be $M_o = N_1 \text{ mix}(N_2 \text{ mix}(N_3 \text{ mix } \ldots))$.) Note that if we set

$$M_i = \text{mix}(N_{i+1}, N_{i+2}, \ldots) ,$$

then for all $i \geq 0$,

(5.1) $\quad M_i = N_{i+1} \text{ mix } M_{i+1}$.

Thus, M_o is the sequence

$\{N_1(1), N_2(1), N_1(2), N_3(1), N_1(3), N_1(4), N_2(2), N_4(1), N_1(5), N_1(6), N_1(7), N_1(8), N_2(3), N_2(4), N_3(2), N_5(1), N_1(9), N_1(10), \ldots, N_1(16), N_2(5), N_2(6), N_2(7), N_2(8), N_3(3), N_3(4), N_4(2), N_6(1), N_1(17), \ldots\}$.

The subsequence of M_o consisting of the terms from N_j for $j > i$ is M_i. Similarly, if $N_i \subset \mathbb{Z}^+$ are strictly increasing, then there exists $M_o = \mathrm{Mix}(N_1, N_2, \ldots)$ defined analogously, which could be denoted

$$M_o = N_1 \text{ Mix } (N_2 \text{ Mix}(N_3 \text{ Mix } \ldots)) .$$

The sequence M_o is strictly increasing. Note that

(5.2) $\quad \mathrm{Mix}(N_1, N_2, \ldots) = \mathrm{mix}(N_1 \text{ add } A_1, N_2 \text{ add } A_2, \ldots)$

for the least possible $A_i \subset \mathbb{N}$; $\{A_i\}_1^\infty$ can also be chosen inductively as in the definition of the Mix of two sequences.

LEMMA 5.7. Let $P = N \text{ mix } M$. Let $K \in \mathbb{Z}^+$ and let L be the number of terms of M among the first K terms of P. Then

$$\frac{1}{3} K \leq L \leq \frac{1}{2} K$$

for $K \geq 2$.

PROOF. Let K_1 be the number of terms of N among the first K terms of P. Then the required inequalities follow from

$$K_1 + L = K ,$$

$$L \leq K_1 \leq 2L \quad \text{for } K \geq 2 . \ \square$$

COROLLARY 5.8. Let $M_o = \mathrm{mix}(N_1, N_2, \ldots)$. Let $K \in \mathbb{Z}^+$ and let K_i be the number of terms of N_i among the first K terms of M_o. Then

(5.3) $$\frac{1}{2} 3^{-i+1} K \leq K_i \leq 2^{-i+1} K$$

for $K \geq 2 \cdot 3^{i-1}$ and

(5.4) $$\sum_{j>i} K_j \leq K_i .$$

PROOF. Let $M_i = \mathrm{mix}(N_{i+1}, N_{i+2}, \ldots)$, so that by (5.1), $M_{i-1} = N_i \text{ mix } M_i$, $i \geq 1$. Let L_i be the number of terms of M_i among the first K of M_o. Since L_i is the number of terms of M_i among the first L_{i-1} of M_{i-1}, the lemma gives

(5.5) $$\frac{1}{3} L_{i-1} \leq L_i \leq \frac{1}{2} L_{i-1}$$

if $L_{i-1} \geq 2$. Since $L_o = K$, it follows by induction that for $i \geq 1$, (5.6) $$3^{-i}K \le L_i \le 2^{-i} K$$

if $K \ge 2\cdot 3^{i-1}$. Now

$$K_i = L_{i-1} - L_i .$$

By (5.5),

$$\frac{1}{2} L_{i-1} \le K_i \le L_{i-1} .$$

Combining this with (5.6), we obtain (5.3).

Also,

$$K_i \ge \frac{1}{2} L_{i-1} \ge L_i = \sum_{j>i} (L_{j-1} - L_j) = \sum_{j>i} K_j .$$

This is (5.4). □

THEOREM 5.9. Given $N_1, N_2, \ldots \subset \mathbb{Z}$, let $M_o = \mathrm{mix}(N_1, N_2, \ldots)$. Then

(5.7) $$W^*(M_o) = \bigcup_{i=1}^{\infty} W^*(N_i) .$$

If $N_i \subset \mathbb{Z}^+$ are strictly increasing, let $M_o = \mathrm{Mix}(N_1, N_2, \ldots)$. Then (5.7) again holds.

PROOF. Let $N_i \subset \mathbb{Z}$, $M_i = \mathrm{mix}(N_{i+1}, N_{i+2}, \ldots)$. By (5.1) and Proposition 5.4,

$$\begin{aligned} W^*(M_o) &\supset W^*(N_1) \cup W^*(M_1) \\ &\supset W^*(N_1) \cup W^*(N_2) \cup W^*(M_2) \\ &\supset \ldots , \end{aligned}$$

whence

$$W^*(M_o) \supset \bigcup_{i=1}^{\infty} W^*(N_i) .$$

Conversely, let $x \notin \bigcup_{i=1}^{\infty} W^*(N_i)$:

$$\forall i \ \forall \ell \ne 0 \quad S_{N_i}(K,x,\ell) = o(K) .$$

Fix $\ell \ne 0$. Let $\varepsilon > 0$. For $K \in \mathbb{Z}^+$, let K_i be the number of terms of N_i among the first K of M_o . Then

(5.8) $$K = \sum_{i=1}^{\infty} K_i$$

and

(5.9) $$S_{M_o}(K,x,\ell) = \sum_{i=1}^{\infty} S_{N_i}(K_i,x,\ell) .$$

Let i_o be such that $2^{-i_o+1} \le \varepsilon$. Let K' be large enough that

(5.10) $$|S_{N_i}(K,x,\ell)| \le \varepsilon K$$

for $1 \le i \le i_o$ and $K \ge K'$.

Let $K \ge 2\cdot 3^{i_o - 1} K'$. Then by (5.3),

$$K_i \ge K'$$

for $1 \le i \le i_o$. By (5.9),

$$|S_{M_o}(K,x,\ell)| \leq \sum_{i\leq i_o} |S_{N_i}(K_i,x,\ell)| + \sum_{i>i_o} K_i \,.$$

By (5.10) and (5.4), this is

$$\leq \varepsilon \sum_{i\leq i_o} K_i + K_{i_o} \,,$$

which, by (5.8), (5.3), and choice of i_o, is

$$\leq \varepsilon K + 2^{-i_o+1} K$$

$$\leq 2\varepsilon K \,.$$

This shows that $S_{M_o}(K,x,\ell) = o(K)$. Therefore $x \notin W^*(M_o)$, which completes the proof of the first half of the theorem.

The second half now follows from (5.2) and Proposition 5.5. □

We have thus proved a result stronger than Theorem 5.1: a finite or countable union of maximal W^*-sets is a maximal W^*-set for some sequence.

We use this first to prove

THEOREM 5.10. Every $H^{(m)}$-set is a W^*-set.

PROOF. Let $E \subseteq \{x\colon (\forall k)\ V_k x \notin B\}$ be an $H^{(m)}$-set, where $\{V_k\} \subset \mathbb{Z}^m$ is quasi-independent and B is a non-empty open set in $\mathbb{T}^m$. Since for every $0 \neq \Lambda \in \mathbb{Z}^m$, $|V_k \cdot \Lambda| \to \infty$ as $k \to \infty$ and since there are only countably many Λ, a diagonal argument provides a subsequence $\{k_\ell\}$ such that for every $\Lambda \neq 0$, $V_{k_\ell} \cdot \Lambda$ is eventually

strictly monotonic (increasing or decreasing). By re-labeling, we assume this is the whole sequence.

Let P be a trigonometric polynomial in m variables such that

$$\hat{P}(0) > 0\,, \quad P \leq \chi_B \,.$$

Then for $x \in E$, $\chi_B(V_k x) = 0$, whence $P(V_k x) \leq 0$. Therefore

$$0 \geq \frac{1}{K}\sum_{k=1}^{K} P(V_k x) = \hat{P}(0) + \sum_{\Lambda\neq 0} (\hat{P}(\Lambda)\cdot \frac{1}{K}\sum_{k=1}^{K} e(\Lambda\cdot V_k x)) \,.$$

Letting $K \to \infty$, it follows that for some $\Lambda \neq 0$,

$$\frac{1}{K}\sum_{k=1}^{K} e(V_k\cdot\Lambda x) \not\to 0 \,.$$

But by assumption, for k larger than some k_o, $V_k\cdot\Lambda \uparrow \infty$ or $-V_k\cdot\Lambda \uparrow \infty$ as $k \to \infty$. Therefore, $x \in W^*(\{|V_k\cdot\Lambda|\}_{k>k_o})$.

There are only a finite number of Λ for which $\hat{P}(\Lambda) \neq 0$, whence there are a finite number of W^*-sets whose union contains E. □

COROLLARY 5.11. Symmetric perfect sets (Cantor sets) of constant ratio of dissection θ^{-1} (Zygmund [1, I, Chap. V, §3, pp. 194-195]) with θ a P-V number are W^*-sets.

PROOF. In fact, they are finite unions of $H^{(m)}$-sets (Zygmund [1, II, Chap. XII, §11, pp. 152-156]). □

A relation of a different kind is

PROPOSITION 5.12. Every uncountable U_o-set contains an uncountable W-set.

PROOF. Let E be an uncountable U_o-set. Let F be a non-empty perfect subset. Let μ be a continuous probability measure supported on F. Since $F \in U_o$, $\mu \in J$. Hence there is a W-set $E' \subset F$ which has positive μ-measure. Since μ is continuous, E' is uncountable. □

We can say more:

PROPOSITION 5.13. If E is a U_o-set, for any positive measure μ concentrated on E, there exists a W_σ-set $F \subset E$ such that $\mu(E \setminus F) = 0$.

PROOF. Given such E, μ, let

$$\alpha = \sup\{\mu F : F \in W_\sigma\} .$$

Then just as in Section 1, the sup is attained. Since $\mu \in J$, $\alpha = \|\mu\|$. □

DEFINITION. If C is a class of sets, a set E is said to be almost in C if for every positive measure μ concentrated on E, there exists $F \subset E$ such that $F \in C$ and $\mu(E \setminus F) = 0$.

Using this definition, we restate Proposition 5.13 as

PROPOSITION 5.13'. The class U_o coincides with the class of almost W_σ-sets.

PROPOSITION 5.14. Every H-set is almost a W-set.

REMARK. This shows, as promised in Section II.3, that every measure supported on an H-set is concentrated on a W-set. Also, a proof combining the proof of Theorem 5.10 with the one following shows that $H^{(m)}$-sets are almost finite unions of W-sets.

PROOF. Let $E \subseteq \{x : \forall k\ n_k x \notin I\}$ be an H-set. Let μ be a positive measure concentrated on E. Let $\{n_k'\} \subset \{n_k\}$ be such that $\{n_k' x\}$ has an asymptotic distribution for almost all $x[\mu]$ (Lemma III.2.5). Since $n_k' x \notin I$, this distribution is not uniform for $x \in E$. That is, μ is concentrated on a W-set. □

On the other hand, by Theorems III.2.1 and III.8.3, we have

PROPOSITION 5.15. Not every W-set is almost an H_σ-set.

Similarly, we have

PROPOSITION 5.16. Not every W*-set is almost a W_σ-set. Every U_o-set is almost a W*-set.

The second statement follows from Proposition 5.13' and Theorem 5.1.

Proposition III.8.5 is equivalent to

PROPOSITION 5.17. Every asymptotic H-set is almost an H_σ-set.

PROPOSITION 5.18. Dirichlet sets are H-sets. Weak Dirichlet sets are almost H_σ-sets.

PROOF. Let E be a Dirichlet set. Then there exist $n_k \uparrow \infty$ and $\delta_k \to 0$ such that

$$\| e(n_k x) - 1 \|_{L^\infty(E)} < \delta_k .$$

Therefore E is contained in the H-set

$$\bigcap_{k=1}^{\infty} \{x: |e(n_k x) - 1| < \delta_k\} .$$

It immediately follows that weak Dirichlet sets are almost H_σ-sets. □

Note that not every H-set is even a weak Dirichlet set. For example, the standard Cantor-Lebesgue measure

$$\mu = \mathop{\times}_{n=1}^{\infty} [\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(2\cdot 3^{-n})]$$

is supported on the Cantor middle-thirds set E. Now

$$|\hat{\mu}(m)| = \prod_{n=1}^{\infty} |\tfrac{1}{2} + \tfrac{1}{2} e(-2m3^{-n})|$$

$$\leq |\tfrac{1}{2} + \tfrac{1}{2} e(-2m3^{-n})|$$

$$= |\cos(2\pi m 3^{-n})| .$$

Choose n so that $3^{n-1}|m$ but $3^n \nmid m$. Then $|\cos(2\pi m3^{-n})| = \frac{1}{2}$, so that $R(\mu) \leq \frac{1}{2}$, while $\|\mu\| = 1$. Thus $s^+(E) \leq \frac{1}{2}$. Since weak Dirichlet sets are those for which $s^+ = 1$, our claim follows.

We now aim to identify the class of weak Dirichlet sets with several other classes of sets. We shall use the following

DEFINITION. A Borel set $E \subset \mathbb{T}$ is called

(i) an $\underline{N_o\text{-set}}$ if there exist $n_k \uparrow \infty$ such that for $x \in E$, $\sum_{k=1}^{\infty} |\sin \pi n_k x| < \infty$ (Bari [2, II, Chap. XII, §7, p. 293]);

(ii) an N-set if there exist $a_k, b_k \in \mathbb{R}$ such that for $x \in E$, $\sum_{k=1}^{\infty} |a_k \cos 2\pi kx + b_k \sin 2\pi kx| < \infty$ but $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)^{1/2} = \infty$ (Zygmund [1, I, Chap. VI, §2, p. 236]);

(iii) an R-set if there exist $a_k, b_k \in \mathbb{R}$ such that for $x \in E$, $\sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx)$ converges but $a_k^2 + b_k^2 \not\to 0$ (Bari [2, II, Chap. XII, §4, p. 287]).

Using the fact that $|\sin 2\pi n_k x| \leq 2|\sin \pi n_k x|$, we see that every N_o-set is both an N-set and an R-set.

DEFINITION. Given a Borel set $E \subset \mathbb{T}$, let

$$s_\infty(E) = \inf\{\frac{R(\mu)}{\|\hat{\mu}\|_\infty} : 0 \neq \mu \in M(E)\} .$$

Note that

(5.11) $s(E) \leq s_\infty(E) \leq s^+(E)$.

LEMMA 5.19. For any Borel set E ,

$$s_\infty(E) = \sup\{s: R(\mu) \geq s|\hat{\mu}(0)| \text{ for all } \mu \in M(E)\} .$$

PROOF. It is clear that

$$R(\mu) \geq s_\infty(E)|\hat{\mu}(0)|$$

for $\mu \in M(E)$. Conversely, suppose

$$R(\mu) \geq s|\hat{\mu}(0)|$$

for all $\mu \in M(E)$. For $\mu \in M(E)$, set $d\mu_n(t) = e(-nt)\, d\mu(t)$.
Then $\mu_n \in M(E)$, so that

$$R(\mu) = R(\mu_n) \geq s|\hat{\mu}_n(0)| = s|\hat{\mu}(n)| .$$

Since n is arbitrary, it follows that

$$R(\mu) \geq s\|\hat{\mu}\|_\infty .$$

The lemma now follows. □

We shall prove

THEOREM 5.20. The following conditions are equivalent:

(i) $s^+(E) = 1$.

(ii) E is a weak Dirichlet set.

(iii) E is almost a countable union of increasing Dirichlet sets.

(iv) E is almost an N-set .

(v) E is almost an N_o-set .

(vi) E is almost an R-set .

(vii) $s_\infty(E) = 1$.

(viii) For all $\mu \in M^+(E)$, $\overline{\lim_n} \operatorname{Re} \hat{\mu}(2n) = \|\mu\|$.

(ix) For all $\mu \in M^+(E)$, $\underline{\lim_n} \int |\sin 2\pi nx|\, d\mu(x) = 0$.

Salem [1,2] showed the equivalence of (iv), (v), (vi) and (viii). The proof also appears in Bari [2, II, Chap. XII, §10 and Chap. XIII, §6]. The equivalence of (ii), (iv) and (ix) is also known already (Lindahl and Poulsen [1, pp. 148-149]). Körner [1, p. 259, Lemma 5.2] showed that (ii) ⇒ (vii). By (5.11), (vii) ⇒ (i) and by Theorem III.7.4, (i) ⇒ (ii). Although it only remains to show the equivalence with (iii), we prefer to give a full proof of the theorem here, assuming certain facts about N-sets and R-sets . While our method of proof is not fundamentally different from those just cited, some parts of our proof do use a different approach. Also, we shall isolate or extend some lemmas of independent interest. We begin with the establishment of these lemmas.

PROPOSITION 5.21. If $\mu \geq 0$, $R(\mu) = \|\mu\|$, and $m \in \mathbb{Z}^+$, then there exist $n_k \to \infty$ such that $\hat{\mu}(mn_k) \to \|\mu\|$ as $k \to \infty$.

EXAMPLE. Let $\mu = \delta(0) + \delta(\frac{1}{2})$. Then

$$\hat{\mu}(n) = \begin{cases} 2 , & n \text{ even} \\ 0 , & n \text{ odd} . \end{cases}$$

PROOF. Assume that $\|\mu\| = 1$. For some ℓ , there exists $\{n_k\}$ such that $|\hat{\mu}(n_k)| \to 1$, $n_k \equiv \ell (\text{mod } m)$, and $n_{k+1} - n_k \to \infty$. If $|\hat{\mu}(n_k)| = e(\theta_k)\, \hat{\mu}(n_k)$, then we may assume $\{\theta_k\}$ converges. By (III.7.1),

$$\text{Re}\{e(\theta_{k+1} - \theta_k)\, \hat{\mu}(n_{k+1} - n_k)\} \geq \frac{1}{2}[|\hat{\mu}(n_{k+1})| + |\hat{\mu}(n_k)|]^2 - 1,$$

whence both sides converge to 1 as $k \to \infty$. Therefore $\hat{\mu}(n_{k+1} - n_k) \to 1$. But $n_{k+1} - n_k$ is a multiple of m . □

We have the immediate

COROLLARY 5.22. If $\mu \geq 0$, then $R(\mu) = \|\mu\|$ if and only if $\overline{\lim}\ \text{Re}\ \hat{\mu}(2n) = \|\mu\|$.

PROPOSITION 5.23. For $\mu \geq 0$, the following are equivalent:

(i) $\overline{\lim}\ \text{Re}\ \hat{\mu}(2n) = \|\mu\|$;

(ii) $\underline{\lim} \int \sin^2 2\pi nx\, d\mu(x) = 0$;

(iii) $\underline{\lim} \int |\sin 2\pi nx|\ d\mu(x) = 0$.

PROOF. The equivalence of (i) and (ii) follows from

$$\text{Re}\ \hat{\mu}(2n) = \int \cos 4\pi nx\, d\mu(x) = \int (1 - 2\sin^2 2\pi nx)\, d\mu(x) .$$

The equivalence of (ii) and (iii) follows from

$$\int \sin^2 2\pi nx\, d\mu(x) \leq \int |\sin 2\pi nx|\ d\mu(x)$$
$$\leq \|\mu\|^{\frac{1}{2}} \left(\int \sin^2 2\pi nx\, d\mu(x)\right)^{\frac{1}{2}} . \ \square$$

PROPOSITION 5.24. Let μ be a probability measure on $\mathbb{T}$ and let ν be a measure such that $|\nu| \leq \mu$. Then for all $n, m \in \mathbb{Z}$ and all $\theta \in \mathbb{R}$,

$$(5.12)\quad |e(\theta)\, \hat{\nu}(n+m) - \hat{\nu}(m)| \leq [2(1 - \text{Re}\{e(\theta)\, \hat{\mu}(n)\})]^{1/2} .$$

In particular,

$$(5.13)\quad ||\hat{\nu}(n+m)| - |\hat{\nu}(m)|| \leq [2(1 - |\hat{\mu}(n)|)]^{1/2} .$$

PROOF. The proof is quite similar to that of Lemma III.7.1. The arithmetic-quadratic mean inequality gives

$$|e(\theta)\, \hat{\nu}(n+m) - \hat{\nu}(m)| = \left|\int (e(\theta)\, e(-(n+m)t) - e(-mt))\, d\nu(t)\right|$$
$$\leq \int |e(\theta)\, e(-nt) - 1|\ d\mu(t)$$
$$\leq \left(\int |\ldots|^2\, d\mu\right)^{1/2}$$
$$= [2(1 - \text{Re}\{e(\theta)\, \hat{\mu}(n)\})]^{1/2} .$$

If we choose θ so that $e(\theta)\, \hat{\mu}(n) = |\hat{\mu}(n)|$ and use the fact that

$$||\hat{\nu}(n+m)| - |\hat{\nu}(m)|| \leq |e(\theta)\, \hat{\nu}(n+m) - \hat{\nu}(m)| ,$$

then (5.13) follows. □

Proposition 5.24 in a somewhat more specialized form is called "the increments inequality" by Loève [1, p. 195].

COROLLARY 5.25. For any measure $\mu \in M(\mathbb{T})$, if $R(|\mu|) = \|\mu\|$, then $R(\mu) = \|\hat{\mu}\|_\infty$.

PROOF. We may take $\|\mu\|$ to be 1. It is evident that $R(\mu) \leq \|\hat{\mu}\|_\infty$. Now $|\mu|$ is a probability measure dominating μ in absolute value. Hence, if $R(|\mu|) = 1$, then the preceding proposition implies that for all $\varepsilon > 0$ and all m, there are infinitely many n such that $|\hat{\mu}(n+m)|$ differs from $|\hat{\mu}(m)|$ by less than ε. Hence $R(\mu) \geq |\hat{\mu}(m)|$ for all m, from which we deduce $R(\mu) \geq \|\hat{\mu}\|_\infty$. □

PROPOSITION 5.26. If $\mu \in M^+(\mathbb{T})$ and $R(\mu) = \|\mu\|$, then μ is concentrated on a countable union of increasing Dirichlet sets.

REMARK. This proposition holds for complex μ as well by Corollary 7.2 below.

PROOF. We saw in the first part of the proof of Theorem III.7.4 that for each $\varepsilon > 0$, there exists a Dirichlet set E such that $\mu(E^c) < \varepsilon$. Choose Dirichlet sets E_n such that $\mu(E_n^c) < 2^{-n}$ and set $F_N = \bigcap_{n \geq N} E_n$. We have

$$\mu(F_N^c) \leq \sum_{n \geq N} \mu(E_n^c) \leq 2^{-N+1},$$

so that $\mu(F_N) \geq \|\mu\| - 2^{-N+1}$. Since $F_N \subset E_N$, F_N is a Dirichlet set. It is clear that $F_N \subset F_{N+1}$ and that μ is concentrated on $\bigcup_{N \geq 1} F_N$. □

PROOF OF THEOREM 5.20. That (i) ⇔ (ii) is Theorem III.7.4. That (i) ⇔ (viii) ⇔ (ix) follows from Corollary 5.22 and Proposition 5.23. By (5.11), (vii) ⇒ (i) and by Corollary 5.25, (i) ⇒ (vii).

Suppose that E is almost an N-set. Let $\mu \in M^+(E)$. Then (Zygmund [1, I, Chap. VI, §2, pp. 236-237]) there exists $(\rho_n)_1^\infty$ such that

$$\sum_{n=1}^\infty \rho_n |\sin 2\pi nx| < \infty \quad \text{for } \mu\text{-almost all } x \in E$$

and $\sum_1^\infty \rho_n = \infty$.

Hence

$$\lim_{N\to\infty} \frac{\sum_1^N \rho_n |\sin 2\pi nx|}{\sum_1^N \rho_n} = 0 \quad \text{for } \mu\text{-a.a. } x \in E,$$

whence

$$0 = \lim_{N\to\infty} \int_E \frac{\sum_1^N \rho_n |\sin 2\pi nx|}{\sum_1^N \rho_n} \, d\mu(x)$$

$$= \lim_{N\to\infty} \frac{\sum_1^N \rho_n \int_{\mathbb{T}} |\sin 2\pi nx| d\mu(x)}{\sum_1^N \rho_n} .$$

Therefore (ix) follows.

Conversely, suppose (ix) holds. Then let n_k be such that

$$\int |\sin 2\pi n_k x| \, d\mu(x) \le 2^{-k} .$$

We have

$$\int \sum_1^\infty |\sin 2\pi n_k x| \, d\mu(x) < \infty .$$

Let F be the subset of E where the integrand is finite. Then $\mu(E \setminus F) = 0$ and F is an N_o-set (with sequence $\{2n_k\}$).

We have shown (iv) ⇒ (ix) ⇒ (v). Since (v) ⇒ (iv) and (v) ⇒ (vi), it remains to show that (vi) ⇒ (i) ⇒ (iii) ⇒ (ii).

Assume (vi). Let $\mu \in M^+(E)$. Since any translate of an R-set is an R-set (Bari [2, II, Chap. XII, §4, pp. 287-288]), there is a translate $E-t$ of E which is almost an R-set and which contains 0. It is easy to show (Bari [2, II, Chap. XII, §4, p. 288]), that there exists $\{n_k\}$ such that $\lim \sin 2\pi n_k x = 0$ for μ_t-a.a. $x \in E-t$, where $\mu_t(F) \equiv \mu(F+t)$ is the translate

of μ. Hence (ix) holds for $E-t$. Therefore (i) holds for $E-t$ and hence for E.

Now Proposition 5.26 immediately implies (i) ⇒ (iii). Finally, it is evident from the definitions that (iii) ⇒ (ii). □

REMARK 1. It is not hard to show that R-sets are H_σ-sets (Bari [2, II, Chap. XII, §6, p. 293]); this fact is due to Rajchman.

REMARK 2. Closed weak Dirichlet sets and closed N-sets are identical (Lindahl and Poulsen [1, pp. 148-150]).

We now show that if E is an H-set, then $s^+(E) > 0$. In fact, we shall prove the following stronger assertion: if E is an $H^{(r)}$-set for some $r \ge 1$, then $s_\infty(E) > 0$. For this purpose, we generalize Theorem III.1.4 as follows.

THEOREM 5.27. Let $\mu \in M(\mathbb{T})$, $r \in \mathbb{Z}^+$, $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^r$ be quasi-independent, and let $f \in L^2(\mathbb{T}^r, m)$ be Borel-measurable, where m is (normalized) Lebesgue measure on $\mathbb{T}^r$. For $y, t \in \mathbb{T}^r$, denote $f_t(y) = f(y-t)$. Using the notation (III.8.12), set

$$g_k(t) = \int_{\mathbb{T}} f_t(V_k x) \, d\mu(x) - \hat{f}(0)\, \hat{\mu}(0) .$$

Then $g_k(t)$ exists for almost all $t\,[m]$ and

$$\limsup_{k\to\infty} \|g_k\|_{L^2(\mathbb{T}^r, m)} \le \|f - \hat{f}(0)\|_{L^2(\mathbb{T}^r, m)} R(\mu) .$$

PROOF. From Fubini's theorem, we deduce the existence of $g_k(t)$ a.e. [m] and also that

$$\hat{g}_k(0) = 0 ,$$

$$\hat{g}_k(\Lambda) = \hat{f}(-\Lambda)\, \hat{\mu}(V_k \cdot \Lambda)$$

for $0 \neq \Lambda \in \mathbb{Z}^m$. Therefore

$$\|g_k\|_{L^2}^2 = \|\hat{g}_k\|_{\ell^2}^2 = \sum_{\Lambda \neq 0} |\hat{f}(-\Lambda)\, \hat{\mu}(V_k \cdot \Lambda)|^2$$

Since $|V_k \cdot \Lambda| \to \infty$ for each $\Lambda \neq 0$, we have

$$\limsup_{k\to\infty} \|g_k\|_{L^2} \leq R(\mu) \left(\sum_{\Lambda \neq 0} |\hat{f}(-\Lambda)|^2 \right)^{1/2}$$

$$= \|f - \hat{f}(0)\|_{L^2}\, R(\mu) . \square$$

COROLLARY 5.28. Let $\mu \in M(\mathbb{T})$, $r \in \mathbb{Z}^+$, $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^r$ quasi-independent, $I_1,\dots,I_r$ arcs of $\mathbb{T}$, $I = I_1 \times \dots \times I_r$, m Lebesgue measure on $\mathbb{T}^r$. For $t = (t_1,\dots,t_r)$, denote $I + t = (I_1 + t_1) \times \dots \times (I_r + t_r)$. Let

$$(5.14) \quad g_k(t) = \int_{\mathbb{T}} \chi_{I+t}(V_k x)\, d\mu(x) - mI \cdot \hat{\mu}(0) .$$

Then

$$(5.15) \quad \limsup_{k\to\infty} \|g_k\|_{L^2(\mathbb{T}^r, m)} \leq (mI \cdot mI^c)^{1/2}\, R(\mu) .$$

THEOREM 5.29. Let $I_1,\dots,I_r$ be non-empty open arcs of $\mathbb{T}$, $I = I_1 \times \dots \times I_r$, $\{V_k\}_1^\infty \subset (\mathbb{Z}^+)^r$ be quasi-independent, and let E be an $H^{(r)}$-set contained in

$$\{x \in \mathbb{T} : (\forall k)\ V_k x \notin I\} .$$

Then

$$s_\infty(E) \geq \left(\tfrac{2}{9}\right)^{r/2} mI .$$

PROOF. Let $\mu \in M(E)$. Let J_ℓ be the subarc of I_ℓ having common left endpoint and having length $\frac{2}{3}|I_\ell|$. Let $J = J_1 \times \dots \times J_r$ and define g_k as in (5.14) but for J, not I. For $t = (t_1,\dots,t_r)$ satisfying $0 \leq t_\ell \leq \frac{1}{3}|I_\ell|$, we have $J + t \subset I$, whence

$$g_k(t) = -mJ \cdot \hat{\mu}(0) .$$

Considering the integral of $|g_k|^2$ only over this set of t, we see that

$$\|g_k\|_{L^2}^2 \geq (mJ)^2\, |\hat{\mu}(0)|^2 \prod_{\ell=1}^{r} \left(\tfrac{1}{3}|I_\ell|\right)$$

$$= \left(\tfrac{4}{27}\right)^r (mI)^3\, |\hat{\mu}(0)|^2 .$$

By (5.15), it follows that

$$R(\mu) \geq (mJ)^{-1/2} (\tfrac{4}{27})^{r/2} (mI)^{3/2} \quad |\hat{\mu}(0)|$$

$$= (\tfrac{2}{9})^{r/2} \quad mI \cdot |\hat{\mu}(0)| \; .$$

The theorem now follows by Lemma 5.19. □

REMARK. Let B be a ball in $\mathbb{T}^r$ of radius at most $\frac{1}{2}$, where we use the (induced) Euclidean metric. An argument similar to the above shows that for $H^{(r)}$-sets E contained in $\{x \in \mathbb{T} : (\forall k)\ V_k x \notin B\}$, we have the inequality

$$s_\infty(E) \geq (\tfrac{2}{9})^{r/2} \quad mB \; .$$

Körner proved the following deep

THEOREM 5.30. There exists a Helson-1 set which is not a U-set.

See Graham and McGehee [1, pp. 114-117] for a proof. We deduce immediately

COROLLARY 5.31. There exists a weak Dirichlet set which is not a U-set.

Since H_σ-sets are U-sets, we obtain the following complement to Proposition 5.18:

COROLLARY 5.32. There is a weak Dirichlet set which is not an H_σ- set.

On the other hand, by using Proposition 5.18, we see

COROLLARY 5.33. There is a set which is almost an H_σ-set, hence is almost a U-set, but which is not a U-set.

At least sometimes, N-sets are translates of W*-sets:

THEOREM 5.34 (Salem). If $\rho_n \geq 0$, $\Sigma\, \rho_n = \infty$, and one of the following hold:

(i) ρ_n is decreasing;

(ii) ρ_{n+k}/ρ_n is bounded over all n, k;

(iii) $\sum_{n=1}^{N} \rho_n^{-1} = O(N^2)$;

then the N-set consisting of the points of absolute convergence of $\sum_{n=1}^{N} \rho_n \cos(m_n x - \alpha_n)$ is a translate of a W*-set corresponding to $\{m_n\}$.

See Salem [1] for the proof. Another theorem we mention without proof is due to Arbault [1]:

THEOREM 5.35. Not every H-set is an N_σ-set. In fact, the Cantor middle-thirds set is such an example.

6. Baire Category

The difference in size between W-sets and W*-sets, illustrated by Theorems III.2.1 and III.6.1, shows up also when we consider their Baire category. All W-sets are meager (i.e. are of first category), while many W*-sets are co-meager (i.e. have meager complement, and hence are of second category). Not all maximal W*-sets are co-meager; some sequences, in fact, have $\mathbb{Q}$ as their maximal W*-set. The example where the sequence $\{n_k\}$ is just $\{k\}$ was mentioned in Section II.1. Another example, due to I.M. Vinogradov [1], is that where n_k is the k-th prime; again $W^*(\{n_k\}) = \mathbb{Q}$.

THEOREM 6.1. Every W-set is meager. In fact, the set

$$\{x: (\exists m \neq 0) \varlimsup_{K\to\infty} |\tfrac{1}{K} \sum_{k=1}^{K} e(mn_k x)| > 0\}$$

is meager.

PROOF. It suffices to show that

$$E = \{x: \varliminf_{K\to\infty} |\tfrac{1}{K} \sum_{k=1}^{K} e(n_k x)| > \varepsilon\}$$

is meager for any $\varepsilon > 0$. But

$$E \subseteq \bigcup_{K=1}^{\infty} F_K ,$$

where

$$F_K = \bigcap_{k=K}^{\infty} \{x: |\tfrac{1}{K} \sum_{j=1}^{k} e(n_j x)| \geq \varepsilon\} .$$

Since F_K is a W*-set, it has Lebesgue measure zero. Being closed, F_K is nowhere dense. Therefore E is meager. □

An exactly parallel proof with "$e(n_k x)$" replaced by "$\chi_I(n_k x)$" and "$>\varepsilon$" replaced by "$< mI - \varepsilon$" shows that A-sets are meager.

The easiest, though rather special, example of co-meager W*-sets is given by

THEOREM 6.2. If $n_{k+1}/n_k \geq 2$ is an integer for all k, then the maximal W*-set of $\{n_k\}$ is co-meager. In fact, the set

$$E = \{x: \varlimsup_{K\to\infty} |\tfrac{1}{K} \sum_{k=1}^{K} e(n_k x)| = 1\}$$

is co-meager.

PROOF. If $t \notin E$, then for some rational $r < 1$ and some $N \in \mathbb{N}$, t belongs to the set

$$F_{N,r} = \{x: K \geq N \Rightarrow |\tfrac{1}{K} \sum_{k=1}^{K} e(n_k x)| \leq r\} .$$

That is, E^c is a countable union of sets $F_{N,r}$, so it suffices to show that $F_{N,r}$ is nowhere dense. Since $F_{N,r}$ is closed, this amounts to showing that $F_{N,r}$ contains no arc J. But for large enough K, J contains some point $x = m/n_K$, $m \in \mathbb{N}$. For all $k \geq K$, n_k/n_K is an integer, so $e(n_k x) = 1$. Therefore $x \in F_{N,r}\setminus J$. □

Somewhat more generally, we may show that

THEOREM 6.3. If $n_{k+1}/n_k \geq q > 4$, then $W^*((n_k))$ is co-meager. In fact,

$$E = \{x: \overline{\lim_{K\to\infty}} |\frac{1}{K} \sum_{k=1}^{K} e(n_k x)| \geq \cos \frac{2\pi}{q}\}$$

is co-meager.

PROOF. As above, it suffices to show that

$$F = \{x: K \geq N \Rightarrow |\frac{1}{K} \sum_{k=1}^{K} e(n_k x)| \leq \cos \pi d\}$$

contains no (non-trivial) arc J for $\frac{2}{q} < d < \frac{1}{2}$.

Let I be the closed arc $[-\frac{1}{q}, \frac{1}{q}]$ and let J be any arc. Choose $n_{K_o} \geq 2/|J|$. We claim that J contains a point x such that for $k \geq K_o$, $n_k x \in I$, i.e. $\operatorname{Re} e(n_k x) \geq \cos \frac{2\pi}{q}$. Given this, the result follows, since

$$\overline{\lim_{K\to\infty}} |\frac{1}{K} \sum_{k=1}^{K} e(n_k x)| \geq \overline{\lim_{K\to\infty}} \operatorname{Re} \frac{1}{K} \sum_{k=1}^{K} e(n_k x)$$

$$\geq \cos \frac{2\pi}{q} > \cos \pi d,$$

whence $x \notin F$. The object of the following lemma is to prove our claim (apply it to the sequence $(1, n_{K_o}, n_{K_o+1}, n_{K_o+2}, \ldots)$ and $I_1 = J$, $I_k = I$ for $k \geq 2$). □

LEMMA 6.4. Let $n_{k+1}/n_k \geq 2$. Let I_k be closed arcs of $\mathbb{T}$ with $|I_k| \geq 2n_k/n_{k+1}$. Then there exists $x \in \mathbb{T}$ such that for all k, $n_k x \in I_k$.

PROOF. This is a nested-intervals argument. For the duration of the proof, we use "arc" to mean "closed arc."

Let J_1 be any arc of length $1/n_1$. Then $x \mapsto n_1 x$ maps J_1 bijectively to $\mathbb{T}$, so there is an arc $J_1' \subset J_1$ of length $|I_1|/2n_1$ which is mapped into I_1. (Note that we cannot assert that such a J_1' exists of length $|I_1|/n_1$ since the portion of J_1 which is mapped onto I_1 may consist of two subarcs whose endpoints contain the endpoints of J_1.) Since

$$\frac{1}{n_2} \leq \frac{|I_1|}{2n_1},$$

there is an arc $J_2 \subset J_1'$ which is mapped bijectively to $\mathbb{T}$ by $x \mapsto n_2 x$. The arc J_2 contains, as before, a subarc J_2' of length $|I_2|/2n_2$ which is mapped into I_2 by $x \mapsto n_2 x$. If we continue in this manner, we obtain a sequence $J_1' \supset J_2' \supset \ldots$ of arcs such that $x \mapsto n_k x$ maps J_k' into I_k. Since J_k' are closed, the point $x \in \bigcap_{k=1}^{\infty} J_k'$ satisfies the desired conclusion. □

In the hyperlacunary case, we may prove more than Theorem 6.3, namely

THEOREM 6.5. If $n_{k+1}/n_k \to \infty$, then the set

$$E = \{x: D_x = \Delta\}$$

is co-meager, where D_x is the set of limit points of $\{\frac{1}{K}\sum_{k=1}^{K} e(n_k x)\}_{K=1}^{\infty}$ and $\Delta = \{z \in \mathbb{C}: |z| \leq 1\}$.

PROOF. If $D_x \neq \Delta$, then there exists a rational $z \in \Delta$ which does not belong to D_x. There is also a rational $\varepsilon > 0$ and an $N \in \mathbb{N}$ such that

$$(6.1) \qquad K \geq N \Rightarrow \left|\frac{1}{K}\sum_{k=1}^{K} e(n_k x) - z\right| \geq \varepsilon .$$

Therefore, if F is the closed set of x for which (6.1) holds, it suffices to show that F contains no arc J. We do this by finding, for any J, an $x \in J$ such that

$$(6.2) \qquad \lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} e(n_k x) = z .$$

Let the line through z and 0 hit the circle $\{\zeta: |\zeta| = 1\}$ at ω and $-\omega$. Then z is a convex combination of ω and $-\omega$;

$$z = \alpha\omega - \beta\omega , \alpha + \beta = 1 , 0 \leq \alpha \leq 1 .$$

Choose K_o so that $n_{K_o} \geq 2/|J|$ and $k \geq K_o \Rightarrow n_{k+1}/n_k \geq 2$. Let $\omega = e(t)$, $t \in \mathbb{T}$. Choose any sequence $N \subset [K_o, \infty)$ of density α (i.e., $\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \chi_N(k) = \alpha$). For $k \geq K_o$, put

$$t_k = \begin{cases} t & \text{if } k \in N , \\ t + \pi & \text{if } k \notin N \end{cases}$$

and let I_k be any arc about t_k of length $2n_k/n_{k+1}$. Thus, $e[I_k]$ is an arc about either ω or $-\omega$. Apply Lemma 6.4 to the sequence $\{1, n_{K_o}, n_{K_o+1}, \ldots\}$ and the arcs $J, I_{K_o}, I_{K_o+1}, \ldots$ to obtain a point x. Then $x \in J$ and since $|I_k| \to 0$, it is clear that (6.2) holds. □

Turning briefly to some other classes of sets, we note that all N-sets are meager (Zygmund [1, I, p. 233]). It seems to be unknown whether all Borel U-sets are meager, though it is known that not all Lebesgue-measurable U-sets are meager (Holščevnikova [1]). For that matter, it is apparently unknown whether U_o-sets are meager, although this seems unlikely.

Finally, let us remark that using regularity, one may show that $\mathbb{T}$ is almost meager. For if $\mu \in M(\mathbb{T})$, let $E = \mathbb{T}\setminus\mathbb{Q}$. Let $E' = (\bigcup_{n=1}^{\infty} F_n) \cup \mathbb{Q}$ have full μ-measure with F_n being closed subsets of E. Since F_n is nowhere dense, E' is meager. This remark shows the existence of meager sets having full μ-measure for $\mu = m$ or any other measure.

7. Miscellaneous

Lemma III.7.1 can be improved in several ways. For example, by introducing a second measure and a weight, we obtain

THEOREM 7.1. Let μ be a probability measure and ν a complex measure with $|\nu| \leq \mu$. Let $|\hat{\nu}(n)| = e(\theta_n)\,\hat{\nu}(n)$. Then for any m , $n \in \mathbb{Z}$,

$$\text{(7.1)}\qquad \begin{aligned} &\operatorname{Re}\{e(\theta_n - \theta_m)\,\hat{\mu}(n-m)\} \\ &\geq |\hat{\nu}(n)|\cdot|\hat{\nu}(m)| - \sqrt{(1-|\hat{\nu}(n)|^2)(1-|\hat{\nu}(m)|^2)}\ . \end{aligned}$$

PROOF. Let $a = |\hat{\nu}(n)|$, $b = |\hat{\nu}(m)|$ and $c = \operatorname{Re}\{e(\theta_n - \theta_m)\,\hat{\mu}(n-m)\}$. Let w be a positive number to be chosen later. By the arithmetic-quadratic mean inequality, we have

$$\begin{aligned} (a+wb)^2 &= \left|\int (e(\theta_n)\,e(-nt) + w\,e(\theta_m)\,e(-mt))\,d\nu(t)\right|^2 \\ &\leq \left(\int|\ldots|\,d|\nu|\right)^2 \leq \left(\int|\ldots|\,d\mu\right)^2 \\ &\leq \int |e(\theta_n)\,e(-nt) + w\,e(\theta_m)\,e(-mt)|^2\,d\mu(t) \\ &= 1 + w^2 + 2wc\ . \end{aligned}$$

Therefore

$$c \geq ab - \tfrac{1}{2}\left(\frac{1-a^2}{w} + (1-b^2)w\right)\ .$$

If $b \neq 1$, choose $w = \sqrt{(1-a^2)/(1-b^2)}$. Then (7.1) results. If $b = 1$, then we let $w \to \infty$ to obtain (7.1). □

Note that Lemma III.7.1 is obtained from putting $\nu = \mu$ and using $w = 1$ in the proof.

COROLLARY 7.2. If $\mu \in M(\mathbb{T})$ and $R(\mu) = \|\mu\|$, then $R(|\mu|) = \|\mu\|$.

PROOF. Let $\nu = \mu/\|\mu\|$. Choose n_k so that $|\hat{\nu}(n_k)| \to 1$ and $|n_{k+1} - n_k| \to \infty$. Then by Theorem 7.1 applied to the probability measure $|\nu|$ and the measure ν ,

$$\overline{\lim_{k\to\infty}}\ |\widehat{|\nu|}\,(n_{k+1} - n_k)| \geq 1\ ,$$

whence $R(|\nu|) = 1$. □

Another improvement of Lemma III.7.1 is obtained by considering more Fourier-Stieltjes coefficients.

PROPOSITION 7.3. Let μ be a probability measure and $|\nu| \leq \mu$. Let n_1 , n_2 , ..., $n_K \in \mathbb{Z}$ and set $|\hat{\nu}(n_k)| = e(\psi_k)\,\hat{\nu}(n_k)$ for $1 \leq k \leq K$. Then

$$\text{(7.2)}\qquad \begin{aligned} &\frac{1}{K} + \frac{2}{K^2}\sum_{1\leq k<\ell\leq K} \operatorname{Re}\{e(\psi_k - \psi_\ell)\,\hat{\mu}(n_k - n_\ell)\} \\ &\geq \left(\frac{1}{K}\sum_{k=1}^{K}|\hat{\nu}(n_k)|\right)^2\ . \end{aligned}$$

PROOF. The arithmetic-quadratic mean inequality yields

$$\left(\frac{1}{K}\sum_{1}^{K}|\hat{\nu}(n_k)|\right)^2 = \left|\int \frac{1}{K}\sum_{1}^{K} e(\psi_k)\, e(-n_k t)\, d\nu(t)\right|^2$$

$$\leq \int \left|\frac{1}{K}\sum_{1}^{K} e(\psi_k)\, e(-n_k t)\right|^2 d\mu(t)$$

$$= \frac{1}{K^2}\sum_{k=1}^{K}\sum_{\ell=1}^{K} e(\psi_k - \psi_\ell)\,\hat{\mu}(n_k - n_\ell)$$

$$= \frac{1}{K} + \frac{2}{K^2}\sum_{1\leq k<\ell\leq K} \operatorname{Re}\{e(\psi_k - \psi_\ell)\,\hat{\mu}(n_k - n_\ell)\} \quad . \square$$

COROLLARY 7.4. If μ is a probability measure and $|\nu| \leq \mu$, then $R(\mu) \geq R(\nu)^2$.

COROLLARY 7.5. If $\mu \in M(\mathbb{T})$ and $\|\mu\| = 1$, then $R(|\mu|) \geq R(\mu)^2$.

We may now generalize Proposition 5.21.

PROPOSITION 7.6. Let μ be a probability measure and $|\nu| \leq \mu$. For every $m \in \mathbb{Z}^+$, we have

$$\overline{\lim_{n\to\infty}} \operatorname{Re}\hat{\mu}(m\,n) \geq R(\nu)^2 .$$

PROOF. Let $|\hat{\nu}(n_k)| \to R(\nu)$. We may assume that $n_{k+1} - n_k \to \infty$ and that $e(\psi_k)$ converges, using the terminology of Proposition 7.3. Since there are infinitely many n_k congruent to some $p \pmod m$, we may also assume that $n_k \equiv p \pmod m$ for all k. Then by taking the lim sup of (7.2), we obtain

$$\overline{\lim_{K\to\infty}} \frac{1}{\binom{K}{2}} \sum_{1\leq k<\ell\leq K} \operatorname{Re}\hat{\mu}(n_k - n_\ell) \geq R(\nu)^2 .$$

Hence $\overline{\lim}_{\substack{k,\ell\to\infty \\ k<\ell}} \operatorname{Re}\hat{\mu}(n_k - n_\ell) \geq R(\nu)^2$. Since $n_k - n_\ell$ is divisible by m, the result follows. $\square$

While by Corollary 7.5, $R(|\mu|)$ cannot be much smaller than $R(\mu)$, $R(\mu)$ can be arbitrarily smaller than $R(|\mu|)$.

THEOREM 7.7. For every $\varepsilon > 0$, there exists a measure μ such that $R(|\mu|) = \|\mu\| = 1$ and $R(\mu) < \varepsilon$.

PROOF. Let E be any countable set which is not a Helson set; then E is a weak Dirichlet set. (See the remarks preceding Proposition III.7.5.) Thus $s^+(E) = 1$ while $s(E) = 0$. By definition, then, for any $\varepsilon > 0$, there exists $\mu \in M(E)$ with $R(\mu) < \varepsilon$ and $\|\mu\| = 1$. Since $|\mu| \in M^+(E)$, also $R(|\mu|) = 1$. $\square$

As noted in the above proof, countable sets have $s^+ = 1$. In other words, if μ is a positive discrete measure, then $R(\mu) = \|\mu\|$. If μ is discrete but not necessarily positive, then by Corollary 5.25, $R(\mu) = \|\hat{\mu}\|_\infty$. This also follows from the fact that $\hat{\mu}$ is an almost periodic sequence. Moreover, combining this latter fact with Wiener's theorem (Katznelson [1, p. 42]), it may be shown that for any measure μ, $R(\mu) \geq R(\mu_d)$, where μ_d is the discrete part of μ. See Graham and McGehee [1, p. 110] for the details.

We now study those measures μ for which $R(\mu) = \|\mu\|$. Recall that by Proposition III.2.3, such a measure μ is concentrated on a W-set. By Corollary 7.2, Proposition 5.26, and Theorem 5.20, μ is also concentrated on a weak Dirichlet set, hence (by Proposition 5.18) on an H_σ-set. The next theorem describes the measures for which $|\hat{\mu}|$ attains its maximum, $\|\mu\|$.

THEOREM 7.8. Let $\mu \in M(\mathbb{T})$ and $m \in \mathbb{Z}$. Then $|\hat{\mu}(m)| = \|\mu\|$ if and only if μ has the form

$$d\mu(t) = e(mt + \theta) \ d|\mu|(t)$$

for some θ. Let $n \neq 0$. Then $|\hat{\mu}(m)| = |\hat{\mu}(m+n)| = \|\mu\|$ if and only if μ has the form

$$\mu = \sum_{k=0}^{n-1} e(\frac{km}{n} + \theta) \ a_k \ \delta(\frac{k}{n} + \psi)$$

for some $a_k \geq 0$ and some θ, ψ. If μ is positive, then $|\hat{\mu}(n)| = \|\mu\|$ for some $n \neq 0$ if and only if the support of μ is finite and contained in a translate of $\mathbb{Q}$.

PROOF. Set $|\hat{\mu}(k)| = e(\theta_k) \ \hat{\mu}(k)$. Suppose that $|\hat{\mu}(m)| = \|\mu\|$. Let $d\mu = f \ d|\mu|$. Then

$$\|\mu\| = e(\theta_m) \ \hat{\mu}(m) = \int e(\theta_m) \ e(-mt) \ f(t) \ d|\mu|(t) ,$$

whence

$$0 = \int (1 - e(\theta_m) \ e(-mt) \ f(t)) \ d|\mu|(t)$$

$$= \int (1 - \operatorname{Re}\{e(\theta_m - mt) \ f(t)\}) \ d|\mu|(t) .$$

Since the integrand is non-negative a.e. $[|\mu|]$, it must in fact be zero a.e.:

$$\operatorname{Re}\{e(\theta_m - mt) \ f(t)\} = 1 \quad \text{a.e. } [|\mu|].$$

Since $|e(\theta_m - mt) \ f(t)| \leq 1$ a.e., it follows that

$$e(\theta_m - mt) \ f(t) = 1 \quad \text{a.e. } [|\mu|] ,$$

or

$$f(t) = e(mt - \theta_m) \quad \text{a.e. } [|\mu|].$$

Therefore

(7.3) $$d\mu(t) = e(mt - \theta_m) \ d|\mu|(t) .$$

Conversely, if $d\mu = e(mt + \theta) \ d|\mu|$, then clearly $\hat{\mu}(m) = e(\theta)\|\mu\|$, whence $|\hat{\mu}(m)| = \|\mu\|$.

Now suppose that $|\hat{\mu}(m)| = |\hat{\mu}(m+n)| = \|\mu\|$ for some $n \neq 0$. From the first part of the theorem, we have

$$d\mu(t) = e(mt - \theta_m) \ d|\mu|(t)$$

$$= e((m+n)t - \theta_{m+n}) d|\mu|(t) .$$

Therefore

$$e((m+n)t - \theta_{m+n}) = e(mt - \theta_m) \quad \text{a.e. } [|\mu|],$$

or

$$e(nt + \theta_m - \theta_{m+n}) = 1 \quad \text{a.e. } [|\mu|].$$

Putting $\psi = (\theta_{m+n} - \theta_m)/n$, we see that $|\mu|$ is supported in $\{\frac{k}{n} + \psi: 0 \leq k \leq n-1\}$. For some $a_k \geq 0$, then,

$$|\mu| = \sum_{k=o}^{n-1} a_k\ \delta(\frac{k}{n} + \psi) ,$$

whence by (7.3),

$$\mu = \sum_{k=o}^{n-1} e(\frac{km}{n} + \theta)\ a_k\ \delta(\frac{k}{n} + \psi) ,$$

where $\theta = m\psi - \theta_m$.

Conversely, if μ has this form, then μ has the form (7.3), so that $|\hat{\mu}(m)| = \|\mu\|$. Furthermore,

$$\hat{\mu}(m+n) = \sum_{k=o}^{n-1} e(\theta - k - m\psi - n\psi)\ a_k$$

$$= e(\theta - m\psi - n\psi)\|\mu\| .$$

Finally, if μ is positive, then $\hat{\mu}(0) = \|\mu\|$. Hence the second part of the theorem shows that $|\hat{\mu}(n)| = \|\mu\|$ for some $n \neq 0$ if and only if μ is supported on a set of n equally spaced points of $\mathbb{T}$ for some $n \neq 0$. This is clearly equivalent to the assertion of the theorem. □

The Fourier-Stieltjes coefficients of μ exhibit markedly different behavior, depending on whether $|\hat{\mu}(m)| = \|\mu\|$ for two values of m or not.

THEOREM 7.9. For $\mu \in M(\mathbb{T})$, either

(i) for some m and some $n \neq 0$, $|\hat{\mu}(m)| = |\hat{\mu}(m+n)| = \|\mu\|$, in which case $|\hat{\mu}(m+kn)| = \|\mu\|$ for all $k \in \mathbb{Z}$;

or

(ii) for any sequence $(n_k)_{k=1}^{\infty}$ of distinct integers, if $|\hat{\mu}(n_k)| \to \|\mu\|$, then $|n_{k+1} - n_k| \to \infty$.

PROOF. We assume $\|\mu\| = 1$ for convenience. If $|\hat{\mu}(m)| = |\hat{\mu}(m+n)| = 1$ for some $n \neq 0$, then the conclusion of (i) follows easily from Theorem 7.8.

To prove the remainder of the theorem, we must show that if $|\hat{\mu}(n_k)| \to 1$, (n_k) are distinct, and $|n_{k+1} - n_k| \not\to \infty$, then $|\hat{\mu}(m)| = 1$ for two values of m . Since $|n_{k+1} - n_k| \not\to \infty$, there exists $N \neq 0$ such that $n_{k+1} - n_k = N$ for infinitely many k . For such k , (7.1) yields

$$|\widehat{|\mu|}(N)| \geq |\hat{\mu}(n_{k+1})|\cdot|\hat{\mu}(n_k)| - \sqrt{(1-|\hat{\mu}(n_{k+1})|^2)(1-|\hat{\mu}(n_k)|^2)} .$$

Since the right side tends to 1 as $k \to \infty$ and since $|\widehat{|\mu|}(N)| \leq \|\mu\| = 1$, we have $|\widehat{|\mu|}(N)| = 1$. Proposition 5.24 now implies that $\hat{\mu}$ is periodic with period N . From the assumption $|\hat{\mu}(n_k)| \to 1$, it follows that $|\hat{\mu}(m)| = 1$ for infinitely many values of m . □

An immediate consequence of this result is

COROLLARY 7.10. For all $\mu \in M(\mathbb{T})$, the set $\{n \in \mathbb{Z}: |\hat{\mu}(n)| = \|\mu\|\}$ is an arithmetic progression or is empty.

While in general $\overline{\lim}_{n\to\infty} |\hat{\mu}(n)|$ is not necessarily equal to $\overline{\lim}_{n\to-\infty} |\hat{\mu}(n)|$ (Graham and McGehee [1, pp. 29-30]), it is interesting that if $\overline{\lim}_{n\to\infty} |\hat{\mu}(n)| = \|\mu\|$, then also $\overline{\lim}_{n\to-\infty} |\hat{\mu}(n)| = \|\mu\|$. This follows from the more general Theorem 7.11 below. We shall employ the following

NOTATION. For $\mu \in M(\mathbb{T})$, write

$$R_+(\mu) = \overline{\lim_{n\to\infty}} |\hat{\mu}(n)| ,$$

$$R_-(\mu) = \overline{\lim_{n\to-\infty}} |\hat{\mu}(n)| .$$

THEOREM 7.11. For any $\mu \in M(\mathbb{T})$ with $\|\mu\| = 1$,

$$|R_+(\mu) - R_-(\mu)| \leq [2(1-R(|\mu|))]^{1/2} \leq [2(1-R(\mu)^2)]^{1/2} .$$

PROOF. Without loss of generality, assume that $R_+(\mu) \geq R_-(\mu)$. Set $\nu = |\mu|$. By Proposition 5.24,

$$|\hat{\mu}(n)| - |\hat{\mu}(m)| \leq [2(1 - |\hat{\nu}(n-m)|)]^{1/2}$$

for all $n, m \in \mathbb{Z}$. If we take the lim inf of both sides as $m \to -\infty$, we obtain

$$|\hat{\mu}(n)| - R_-(\mu) \leq [2(1 - R(|\mu|))]^{1/2} .$$

Taking the lim sup as $n \to +\infty$ now yields the first inequality of the theorem. The second inequality follows from Corollary 7.5. □

By modifying the above argument, we may obtain many similar inequalities (Theorem 7.12).

NOTATION. For $\mu \in M(\mathbb{T})$, denote

$$r_+(\mu) = \underline{\lim}_{n\to\infty} |\hat{\mu}(n)| , \quad r_-(\mu) = \underline{\lim}_{n\to-\infty} |\hat{\mu}(n)| ,$$

$$r(\mu) = \underline{\lim}_{|n|\to\infty} |\hat{\mu}(n)| = \min\{r_+(\mu), r_-(\mu)\} ,$$

$$R_o(\mu) = \min\{R_+(\mu), R_-(\mu)\}, \quad r_o(\mu) = \max\{r_+(\mu), r_-(\mu)\} ,$$

$$D(\mu) = [2\|\mu\|(\|\mu\| - R(|\mu|))]^{1/2}, \quad d(\mu) = [2\|\mu\|(\|\mu\| - r(|\mu|))]^{1/2} .$$

THEOREM 7.12. For any $\mu \in M(\mathbb{T})$,

$$|R_+(\mu) - R_-(\mu)| \leq D(\mu) , \quad |r_+(\mu) - r_-(\mu)| \leq D(\mu) ,$$

$$R(\mu) - r(\mu) \leq d(\mu) ,$$

$$\|\hat{\mu}\|_\infty \leq R_o(\mu) + D(\mu) , \quad r_o(\mu) \leq \inf_m |\hat{\mu}(m)| + D(\mu) ,$$

$$\|\hat{\mu}\|_\infty \leq r(\mu) + d(\mu) , \quad R(\mu) \leq \inf_m |\hat{\mu}(m)| + d(\mu) .$$

We omit the easy proofs. In order to illustrate these inequalities, we prove

COROLLARY 7.13. If $\mu \in M(\mathbb{T})$ is concentrated on a weak Dirichlet set, then

$$R_+(\mu) = R_-(\mu) = \|\hat{\mu}\|_\infty$$

and

$$r_+(\mu) = r_-(\mu) = \inf_m |\hat{\mu}(m)| .$$

PROOF. Since weak Dirichlet sets have $s^+ = 1$ (Theorem III.7.4), we have $R(|\mu|) = \|\mu\|$. Therefore $D(\mu) = 0$, whence by Theorem 7.12,

$$R_+(\mu) = R_-(\mu) , \quad r_+(\mu) = r_-(\mu) ,$$

$$\|\hat{\mu}\|_\infty \leq R_o(\mu) , \quad r_o(\mu) \leq \inf_m |\hat{\mu}(m)| .$$

Also, it is evident from the definitions that

$$R_o(\mu) \leq \|\hat{\mu}\|_\infty , \inf_m |\hat{\mu}(m)| \leq r_o(\mu) .$$

Thus the result follows. □

Finally, we give a short proof of a theorem of Milicer-Grużewska [1].

PROPOSITION 7.14. Let $\mu \in M(\mathbb{T})$ and E be a set of positive Lebesgue measure. Set

$$\nu_{(t)} = \mu|(E - t) .$$

If $\nu_{(t)} \in R$ for almost all t [m], then $\mu \in R$.

PROOF. Using Fubini's theorem, we have

$$\int_{\mathbb{T}} \hat{\nu}_{(t)}(n)\, dm(t) = \int_{\mathbb{T}}\int_{\mathbb{T}} e(-n\tau)\, \chi_{E-t}(\tau)\, d\mu(\tau)\, dm(t)$$

$$= \iint_{\mathbb{T}\mathbb{T}} e(-n\tau)\, \chi_{E-\tau}(t)\, dm(t)\, d\mu(\tau)$$

$$= \int_{\mathbb{T}} e(-n\tau)\, mE\, d\mu(\tau)$$

$$= mE \cdot \hat{\mu}(n) .$$

Since $|\hat{\nu}_{(t)}(n)| \leq \|\nu_{(t)}\| \leq \|\mu\|$, if $\hat{\nu}_{(t)}(n) \to 0$ for almost all t [m], then the bounded convergence theorem gives $\hat{\mu}(n) \to 0$. □

8. Notes

Van Kampen and Wintner [1 , Theorem 5, p. 652] generalize Proposition 3.6.

Kakutani's criterion for mutual singularity of infinite product measures (Graham and McGehee [1 , p. 170]) , if used instead of Theorem 1 of Brown [1] in the proof of Theorem 3.11,would give Theorem 3.11 a somewhat different form.

Peyrière [1] has additional information on the sets which may receive positive measure from a Riesz product.

Corollary 5.11 should be compared to Theorem II.2.2 in Mendes France [1] .

Šalát [1] proves that maximal W*-sets are co-meager for sequences $\{q^k\}$, q an integer, by establishing a result similar to Theorem 6.5.

A set $E \subset \mathbb{Z}$ is called a <u>Rajchman set</u> if whenever $\hat{\mu}$ is supported on E , then $\mu \in R$. Host and Parreau [1] characterize such sets. From their characterization and Theorem 4.4, it follows immediately that if E is not a Rajchman set, then there exists $\mu \in J$ such that $\hat{\mu}$ is supported on E .

Some more inequalities between values of $\hat{\mu}$ for probability measures μ are collected by Kawata [1 , pp. 95-101] .

CHAPTER V

LOCALLY COMPACT ABELIAN GROUPS

1. Characterizations of $M_o(G)$

Most of the main results that we have described for the circle carry over with little difficulty to the general LCA case. It is usually only a matter of making the proper definitions. Thus, most of our proofs will cite earlier theorems and proofs and be rather concise. This chapter also provides a summary of most of the important theorems of earlier chapters.

The dual of an LCA group G will be denoted $\hat{G}$. The (finite) complex regular Borel measures on G are denoted by $M(G)$, the positive ones by $M^+(G)$, and those μ for which $\hat{\mu}(\gamma) \to 0$ as $\gamma \to \infty$ by $M_o(G)$; recall that $\lim_{\gamma\to\infty} \hat{\mu}(\gamma) = 0$ means that for each $\varepsilon > 0$, there exists a compact set $K \subset \hat{G}$ such that if $\gamma \notin K$, then $|\hat{\mu}(\gamma)| < \varepsilon$. When the group G is understood, $M_o(G)$ will also be denoted by R and be called the <u>Rajchman measures</u>. If $\hat{G}$ is compact (i.e. , G is discrete), then the condition $\gamma \to \infty$ is impossible to fulfil. In this case, $M_o(G) = M(G)$ and the problem of characterizing $M_o(G)$ disappears. Nevertheless, we will not need to specially exclude this case from our theorems, as they will be vacuously fulfilled.

The all-important property of being a band still holds for $M_o(G)$. First, note

THEOREM 1.1. For any LCA group G and any $\mu \in M^+(G)$, the set of trigonometric polynomials

$\{\sum_{k=1}^{K} a_k \gamma_k \mid a_k \in \mathbb{C}, \gamma_k \in \hat{G}, K \in \mathbb{N}\}$ is dense in $L^1(G,\mu)$.

For the proof, see Hewitt and Ross [1, II, pp. 211-212]. (This, by the way, is the only place where regularity of $\mu \in M(G)$ is needed.) It follows as in Section II.4 that $M_o(G)$ is a band:

THEOREM 1.2. If $\nu \ll \mu \in M_o(G)$, then $\nu \in M_o(G)$. $\mu \in M_o(G)$ if and only if $|\mu| \in M_o(G)$.

DEFINITION. A Borel set $E \subset G$ is a <u>U_o-set</u> if $\mu E = 0$ for all $\mu \in M_o(G)$.

How do we define W-sets? Once we observe that the elements of $\hat{G}$ map G into $\mathbb{T}$, this becomes easy:

DEFINITION. A Borel set $E \subset G$ is a <u>W-set</u> if there is a sequence $\{\gamma_k\}_{k=1}^{\infty} \subset \hat{G}$ tending to ∞ such that for all $x \in E$, $\{\gamma_k(x)\}_{k=1}^{\infty}$ is Weyl-distributed.

Recall that when $G = \mathbb{T}$, the characters of $\mathbb{T}$ are $\{e(nx)\}_{n=-\infty}^{\infty}$. Identifying $\mathbb{T}$ with $\{|z|=1\}$ or with $\mathbb{R}/\mathbb{Z}$ as necessary, we see that the new definition of W-sets agrees with the old. Actually, there is one minor difference. In the old definition, n_k was required to be strictly increasing to infinity, while here we only require $\gamma_k \to \infty$ (we do not even require γ_k to be distinct). This does make a difference in the definition of W-sets on $\mathbb{T}$ since we cannot merely rewrite a sequence n_k in increasing order, as the following result shows.

PROPOSITION 1.3. If $n_k \uparrow \infty$ and E is the corresponding maximal W-set on $\mathbb{T}$, there exists a sequence $|m_k| \to \infty$ such that $\{m_k\}_1^{\infty} = \mathbb{Z}$ as sets and yet the maximal W-set of $\{m_k\}$ is E.

PROOF. It is easily checked that

$$\{m_k\}_1^{\infty} = \{n_1, 0, n_2, n_3, 1, n_4, n_5, n_6, n_7, -1, n_8, n_9, n_{10}, n_{11}, n_{12}, n_{13}, n_{14}, n_{15}, 2, \ldots\}$$

works. □

REMARK. Likewise, any given W-set in G corresponds to a sequence $\{\gamma_k\}$ containing any given countable subset of $\hat{G}$.

It was convenient when dealing with $G = \mathbb{T}$ to require $n_k \uparrow \infty$ as it led to a slightly stronger theorem. In the general case, however, we give up any such requirement with no great loss.

If $\hat{G}$ is compact, then clearly there are no W-sets. Hence, it is vacuously true that a measure μ lies in $M_o(G)$ if and only if $\mu E = 0$ for every W-set E.

When we identify $\mathbb{T}$ with $\{|z|=1\}$, Weyl's criterion takes this form:

THEOREM 1.4. $\{z_k\}$ has an asymptotic distribution if and only if

(1.1) $$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} z_k^m \text{ exists}$$

for every $m \in \mathbb{Z}$, in which case the limits are $\hat{\nu}(-m)$, where ν is the limiting distribution.

Suppose that for some sequence $\{\gamma_k\}_1^\infty \subset \hat{G}$ and some $m \geq 2$, all γ_k have order m : $\gamma_k^m = \text{id}$, the identity of $\hat{G}$. Then every $x \in G$ satisfies (1.1) for $z_n = \gamma_n(x)$ and this m. It follows from Lemma 1.5 below that G is almost a W-set, whence $W \not\subset U_o$. This unsatisfactory situation has two possible solutions: we can work only with those groups for which this cannot happen, or we can modify the definition of W-sets and work with a new class of sets. The first solution involves the following groups.

DEFINITION. G is a <u>Weyl group</u> if for all $m \neq 0$ and for all sequences $\gamma_k \to \infty$ in $\hat{G}$, we have $\gamma_k^m \to \infty$ as $k \to \infty$.

The structure of Weyl groups is given in Section 2. The second solution involves the

DEFINITION. A Borel set $E \subset G$ is a <u>W_1-set</u> if there is a sequence $\gamma_k \to \infty$ in $\hat{G}$ such that for all $x \in E$, there exists $\nu \in M(\mathbb{T})$ such that $\{\gamma_k(x)\} \sim \nu$ and $\hat{\nu}(1) \neq 0$.

Since the definition of W-set only requires $\nu \neq m$, this is a stricter definition: $W_1 \subset W$. By using W_1-sets,

we may show that for <u>all</u> LCA groups, U_o-sets characterize $M_o(G)$.

To do this, we shall need the fact that if $\mu \notin R$, then there exists a sequence (not merely a net) $\gamma_k \to \infty$ such that $\hat{\mu}(\gamma_k)$ converges to a non-zero value. This follows from Theorem 2.4 in the following section. For if $\mu \notin R$, then there exists $\varepsilon > 0$ and a net $\{\gamma_\alpha\}_{\alpha \in A}$ tending to ∞ in $\hat{G}$ such that for all $\alpha \in A$, $|\hat{\mu}(\gamma_\alpha)| > \varepsilon$ (for example, the directed set A may be taken as the collection of compact subsets of $\hat{G}$ directed by inclusion and γ_α is a point outside of α such that $|\hat{\mu}(\gamma_\alpha)| > \varepsilon$). Theorem 2.4 provides a subsequence $\{\gamma_{\alpha_k}\}_{k=1}^\infty$ tending to ∞. Since $\varepsilon < |\hat{\mu}(\gamma_{\alpha_k})| \leq \|\mu\|$, there is a further subsequence $\{\gamma_{\alpha'_k}\}$ such that $z = \lim_{k\to\infty} \hat{\mu}(\gamma_{\alpha'_k})$ exists. Since $z \neq 0$ and $\gamma_{\alpha'_k} \to \infty$, this is the required sequence.

The fundamental lemma needed for characterizing R has the same proof as has Lemma III.2.5:

LEMMA 1.5. Given $\mu \in M(G)$ and $\gamma_k \to \infty$, there exists a subsequence $\gamma'_k \to \infty$ such that $\{\gamma'_k(t)\}$ has an asymptotic distribution for almost every t $[|\mu|]$.

Note that the weak <u>sequential</u> compactness of the unit ball in Hilbert space is very important to the proof.

We now state the basic theorem. Recall that for a class of Borel sets C, $C^{\perp}$ denotes $\{\mu \in M(G): |\mu|(E) = 0 \ \forall E \in C\}$.

THEOREM 1.6. For any LCA group G, $M_o(G) = W_1^{\perp}$. For Weyl groups G, $M_o(G) = W^{\perp}$.

PROOF. Suppose $\mu \in R$. We desire to show that if $\gamma_k \to \infty$, then $\mu\, W_1(\{\gamma_k\}) = 0$ and, if G is a Weyl group, then $\mu\, W(\{\gamma_k\}) = 0$. As in Section III.2, $\gamma_k \to 0$ weakly in $L^2(|\mu|)$ and, if G is a Weyl group, also $\gamma_k^m \to 0$ weakly in $L^2(|\mu|)$ for $m \neq 0$. Therefore

$$\frac{1}{K}\sum_{k=1}^{K} \gamma_k^m \to 0 \text{ weakly in } L^2(|\mu|)$$

for $m = 1$ in general and for all $m \neq 0$ if G is a Weyl group. From Lemma III.2.4, it follows that for almost all t $[|\mu|]$, if $\frac{1}{K}\sum_{k=1}^{K}\gamma_k^m(t)$ has a limit as $K \to \infty$, then that limit is 0. This establishes the two parts of the theorem in one direction.

Conversely, let $\mu \notin R$ and let $\hat{\mu}(\gamma_k) \to \alpha \neq 0$, $\gamma_k \to \infty$. Let γ_k' be as in Lemma 1.5, $f_K = \frac{1}{K}\sum_{k=1}^{K}(\gamma_k')^{-1}$, and

$$E = \{t\colon (\exists \nu)\ \{\gamma_k'(t)\} \sim \nu \text{ and } \hat{\nu}(1) \neq 0\}.$$

Since $\lim f_K$ exists a.e. $[|\mu|]$ and is 0 off E, we have

$$\int_E \lim f_K \, d\mu = \lim \int_{\mathbb{T}} f_K \, d\mu = \lim \frac{1}{K}\sum_{1}^{K} \hat{\mu}(\gamma_k') = \alpha \neq 0 .$$

Therefore the W_1-set E has positive $|\mu|$-measure. □

We have a similar situation for abnormal sets.

DEFINITION. A Borel set $E \subset G$ is an <u>A-set</u> if there is a sequence $\gamma_k \to \infty$ in $\hat{G}$ such that for all $x \in E$, $\{\gamma_k(x)\}$ is badly distributed.

In order to show that $A \subset U_o$ for Weyl groups, we need to generalize the Rajchman-Milicer-Grużewska criterion:

THEOREM 1.7. Let G be an LCA group and $\mu \in M(G)$. Consider the following three conditions:

(i) $\mu \in R$.

(ii) For every arc $I \subset \{|z| = 1\}$,

$$\lim_{\gamma\to\infty} \int_G (\chi_I \circ \gamma)(t)\, d\mu(t) = |I| \cdot \hat{\mu}(\mathrm{id}) .$$

(iii) For every $f \in C(\{|z| = 1\})$,

$$\lim_{\gamma\to\infty} \int_G (f\circ\gamma)(t)\, d\mu(t) = \hat{f}(0)\cdot\hat{\mu}(\mathrm{id}) . \tag{1.2}$$

Then (ii) ⇔ (iii) ⇒ (i). If G is a Weyl group, then (i) ⇒ (iii) also, so that all three conditions are equivalent.

PROOF. The proof is exactly parallel to that of Theorem III.1.1. We only note here that in showing (i) ⇒ (iii), the assumption that G is a Weyl group is needed in order to assert that (1.2) holds for $f(z) = z^k$, $k \in \mathbb{Z}$. □

The proof of Theorem III.3.1 now extends to yield

THEOREM 1.8. For Weyl groups G, $\mu \in M_o(G)$ if and only if $\mu E = 0$ for all A-sets E. For any LCA group G, if $\mu E = 0$ for all A-sets E, then $\mu \in M_o(G)$.

As before, we define J to be the class of measures concentrated on a U_o-set.

THEOREM 1.9. For any LCA group G, $M(G) = R \oplus J$ and $R \perp J$.

2. The Structure of Weyl Groups

Baker [1,2] has defined the term "Weyl group" in a very different way than we have. The object here is to show that the two notions in fact coincide. This is the content of the next theorem.

DEFINITION (Baker [1]). A group is <u>almost torsion-free</u> if for every $m \in \mathbb{Z}^+$, it contains at most finitely many elements of order m.

THEOREM 2.1. An LCA group G is a Weyl group if and only if it possesses an open subgroup of the form $\mathbb{R}^n \times G_1$, where $n \in \mathbb{N}$ and G_1 is a compact group whose dual is almost torsion-free. Furthermore, if $\mathbb{R}^n \times G_1$ is any open subgroup of a Weyl group with G_1 compact, then $\hat{G}_1$ is almost torsion-free.

The proof is a series of short lemmas. The idea is to prove it for compact G and build up from there via the structure theorem for LCA groups. Recall that if H is a closed subgroup of G and Λ is the annihilator of H in $\hat{G}$, then $\hat{H}$ is $\hat{G}/\Lambda$. If $\phi: \hat{G} \to \hat{H}$ is the natural map ensuing from this, then ϕ is a continuous open homomorphism (see Rudin [1, pp. 35-36]).

LEMMA 2.2. With notation as above, if H is open in G and $K \subset \hat{H}$ is compact, then $\phi^{-1}(K)$ is compact in $\hat{G}$.

PROOF. We first claim that there exists compact $K_1 \subset \hat{G}$ such that $\phi(K_1) = K$. For let N_x be a neighborhood of x with compact closure N_x^- for each $x \in \phi^{-1}(K)$. Then $\{\phi(N_x)\}_{x \in \phi^{-1}(K)}$ is an open cover of K, so let $(\phi(N_{x_i}))_{i=1}^n$ be a finite subcover. Now let $K_1 = \phi^{-1}(K) \cap \left[\bigcup_{i=1}^n N_{x_i}^-\right]$; note $\phi^{-1}(K)$ is closed while $\bigcup_1^n N_{x_i}^-$ is compact.

Now $\phi^{-1}(K) = K_1 + \Lambda$. Since Λ is isomorphically homeomorphic to $\widehat{G/H}$ (Rudin [1, p. 35] and G/H is discrete (Rudin [1, p. 40]), it follows that Λ is compact. Therefore so is $\phi^{-1}(K)$. □

LEMMA 2.3. Let H be an open subgroup of G and $\phi: \hat{G} \to \hat{H}$ the natural map. Let $\{\gamma_\alpha\}_{\alpha \in A} \subset \hat{G}$, $\{\beta_\alpha\} \subset \hat{H}$ be nets such that $\phi(\gamma_\alpha) = \beta_\alpha$. Then $\gamma_\alpha \to \infty$ if and only if $\beta_\alpha \to \infty$.

PROOF. Let $\gamma_\alpha \to \infty$. Let $K \subset \hat{H}$ be compact. Let $\alpha_o \in A$ be such that $\alpha \geq \alpha_o \Rightarrow \gamma_\alpha \notin \phi^{-1}(K)$. Then $\beta_\alpha \notin K$ for $\alpha \geq \alpha_o$. That is, $\beta_\alpha \to \infty$.

Conversely, let $\beta_\alpha \to \infty$. Let $K_1 \subset \hat{G}$ be compact. Then $\phi(K_1)$ is compact, so for some α_o, $\alpha \geq \alpha_o \Rightarrow \beta_\alpha \notin \phi(K_1)$. Then $\alpha \geq \alpha_o \Rightarrow \gamma_\alpha \notin K_1$. Thus $\gamma_\alpha \to \infty$. □

THEOREM 2.4. Let Γ be an LCA group and $\{\gamma_\alpha\}$ be a net in Γ tending to ∞. Then there exists a subsequence γ_k' tending to ∞.

PROOF. Let $\Gamma = \hat{G}$. By the structure theorem, there exists an open subgroup $H \subset G$ of the form $\mathbb{R}^n \times G_1$, where $n \in \mathbb{N}$ and G_1 is compact. Let $\phi: \hat{G} \to \hat{H}$ be the natural map.

Then $\phi(\gamma_\alpha) \to \infty$. But $\hat{H} = \hat{\mathbb{R}}^n \times \hat{G}_1$ and $\hat{G}_1$ is discrete. Let $\phi(\gamma_\alpha) = (\beta_\alpha, \delta_\alpha)$, $\beta_\alpha \in \hat{\mathbb{R}}^n$, $\delta_\alpha \in \hat{G}_1$. Note $\phi(\gamma_\alpha) \to \infty \Leftrightarrow \beta_\alpha \to \infty$ or $\delta_\alpha \to \infty$. If $\beta_\alpha \to \infty$, let β_{α_k} be a subsequence tending to ∞. If $\delta_\alpha \to \infty$, let δ_{α_k} be any subsequence of distinct terms. Then $\delta_{\alpha_k} \to \infty$. In either case, let $\gamma_k' = \gamma_{\alpha_k}$. Then $\phi(\gamma_k') \to \infty$. Now Lemma 2.3 applies again to give $\gamma_k' \to \infty$. □

This theorem was used in the last section, but it is not needed for the proof of Theorem 2.1.

LEMMA 2.5. Let H be an open subgroup of an LCA group G. Then G is a Weyl group if and only if H is a Weyl group.

PROOF. This is an immediate corollary of Lemma 2.3. □

LEMMA 2.6. If G_o, G_1 are Weyl groups, so is $G_o \times G_1$ and conversely.

PROOF. This is obvious. □

LEMMA 2.7. A compact abelian group G is a Weyl group if and only if $\hat{G}$ is almost torsion-free.

PROOF. Since $\hat{G}$ is discrete, the compact subsets of $\hat{G}$ are precisely the finite subsets.

Suppose $\hat{G}$ is not almost torsion-free. Let $\gamma_k^m = \mathrm{id}$ for some $m \geq 2$ and every $k \in \mathbb{Z}^+$, where $\{\gamma_k\}$ are distinct. Then $\gamma_k \to \infty$, but $\gamma_k^m \not\to \infty$, so G is not a Weyl group.

Conversely, suppose G is not a Weyl group. Let $\gamma_k \to \infty$ with $\gamma_k^m \to \infty$ for some $m \neq 0$. Then there exists a finite set K such that $\gamma_{k_i}^m \in K$ for infinitely many γ_{k_i}. Some element of K, say γ, is equal to infinitely many $\gamma_{k_{i_j}}^m$. Since

$$(\gamma_{k_{i_j}} \gamma_{k_{i_1}}^{-1})^m = \gamma\gamma^{-1} = \text{id} ,$$

there are infinitely many elements, $\gamma_{k_{i_j}} \gamma_{k_{i_1}}^{-1}$, of order dividing $|m|$. □

PROOF OF THEOREM 2.1. Suppose G is a Weyl group. By the structure theorem, there exists an open subgroup $\mathbb{R}^n \times G_1$, G_1 compact. By the lemmas, $\mathbb{R}^n \times G_1$, hence G_1, is a Weyl group, hence $\hat{G}_1$ is almost torsion-free. Note also that this holds for <u>all</u> open subgroups $\mathbb{R}^n \times G_1$, G_1 compact.

Conversely, if $\mathbb{R}^n \times G_1$ is an open subgroup of G with G_1 compact and $\hat{G}_1$ almost torsion-free, then by the lemmas and the obvious fact that $\mathbb{R}^n$ is a Weyl group, it follows that G_1 and hence G are Weyl groups. □

REMARK. It follows from Theorem 2.4 that G is a Weyl group if and only if for all $m \in \mathbb{Z}\backslash\{0\}$, $\lim_{\gamma\to\infty} \gamma^m = \infty$ in $\hat{G}$.

3. Riesz Products

Let us briefly recall how Riesz products are defined in <u>compact</u> abelian groups (see Graham and McGehee [1, pp. 196-198]). A subset $\Phi \subset \hat{G}$ is said to be <u>dissociate</u> if every $\omega \in \hat{G}$ can be expressed in at most one way as

$$(3.1) \qquad \omega = \prod_{\theta\in\Phi} \theta^{\varepsilon_\theta} ,$$

where

$$(3.2) \qquad \varepsilon_\theta = \begin{cases} 0, \pm 1 & \text{if } \theta^2 \neq 1 , \\ 0, + 1 & \text{if } \theta^2 = 1 , \end{cases}$$

and $\varepsilon_\theta \neq 0$ for only finitely many θ. (The usual example on the circle is a set of characters $e(n_k t)$ with $n_{k+1}/n_k \geq 3$.) Let $a: \Phi \to \mathbb{C}$ be any function satisfying

$$(3.3) \qquad |a(\theta)| \leq \tfrac{1}{2} \text{ if } \theta^2 \neq 1 , \ -1 \leq a(\theta) \leq 1 \text{ if } \theta^2 = 1 .$$

Define the polynomial

$$(3.4) \qquad q_\theta = \begin{cases} 1 + a(\theta)\theta + \overline{a(\theta)\theta} & \text{if } \theta^2 \neq 1 , \\ 1 + a(\theta)\theta & \text{if } \theta^2 = 1 . \end{cases}$$

Then the Riesz product μ based on Φ and a is the weak* limit in $M(G)$ of $(\prod_{\theta\in\phi} q_\theta)\lambda$, where the limit is taken over the net of finite $\phi \subset \Phi$ ordered by inclusion and λ is Haar measure on G. The Fourier-Stieltjes transform of μ is given by

$$(3.5)\qquad \hat{\mu}(\omega) = \begin{cases} \prod_{\theta\in\Theta} a(\theta)^{(\varepsilon_\theta)} & \text{if } \omega \text{ has the form (3.1),} \\ 0 & \text{otherwise,} \end{cases}$$

where for $z \in \mathbb{C}$, we write

$$z^{(\varepsilon)} = \begin{cases} z, & \varepsilon = 1 \\ 1, & \varepsilon = 0 \\ \bar{z}, & \varepsilon = -1 . \end{cases}$$

The elements of the form (3.1) are called <u>words in</u> Θ. The set of all words in Θ is denoted $\Omega(\Theta)$. We have $R(\mu) = \overline{\lim_{\theta\to\infty}} |a(\theta)|$ if Θ is infinite, where $R(\mu)$ denotes $\overline{\lim_{\gamma\to\infty}}|\hat{\mu}(\gamma)|$. Note that since $\hat{G}$ is discrete, compact sets are finite and ∞ is a limit point of Θ if Θ is infinite.

The proofs of the next three results are the same as those in Section IV.4. From now on, we assume Θ is infinite.

THEOREM 3.1. If $\mu \in M(G)$ is a probability measure and $\{\gamma_k\} \subset \hat{G}$ is any sequence with

$$\sum_{K=1}^{\infty} \frac{1}{K^3} \operatorname{Re}\Big(\sum_{1\le \ell<k\le K} [\hat{\mu}(\gamma_k\bar{\gamma}_\ell) - \hat{\mu}(\gamma_k)\,\hat{\mu}(\bar{\gamma}_\ell)]\Big) < \infty ,$$

then

$$\lim_{K\to\infty}\Big(\frac{1}{K}\sum_{k=1}^{K} \gamma_k(x) - \frac{1}{K}\sum_{k=1}^{K} \hat{\mu}(\bar{\gamma}_k)\Big) = 0 \quad \text{a.e. } [\mu] .$$

COROLLARY 3.2. Let

$$\Delta_K = \#\{(k,\ell) \mid 1 \le \ell < k \le K ,\ \hat{\mu}(\gamma_k\bar{\gamma}_\ell) \ne \hat{\mu}(\gamma_k)\hat{\mu}(\bar{\gamma}_\ell)\} .$$

If $\mu \in M(G)$ is a probability measure, if

$$\sum_{K=1}^{\infty} \frac{\Delta_K}{K^3} < \infty ,$$

and if $\hat{\mu}(\gamma_k) \to \alpha$ as $k \to \infty$, then

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \gamma_k(x) = \bar{\alpha} \quad \text{a.e. } [\mu] .$$

THEOREM 3.3. Let Θ be dissociate and let a satisfy (3.3). Let μ be the Riesz product based on Θ and a. If $\lim_{\theta\to\infty} a(\theta) = 0$, then $\mu \in R$. Otherwise, μ^m is concentrated on a W_1-set and $\mu^m \in J$ for every $m \ge 1$.

In order to formulate and prove the analogue of Theorem III.7.6, we must define the analogue to (III.7.3): $n_{k+1}/n_k \ge q > 3$. (Actually, we will define the analogue to the weaker hypothesis (III.7.15).) For $\theta \in \Theta \cup \Theta^{-1} = \Theta \cup \bar{\Theta}$, let us use the notation

$$\Theta_\theta = \begin{cases} \Theta\setminus\{\theta\} & \text{if } \theta \in \Theta , \\ \Theta\setminus\{\bar{\theta}\} & \text{if } \theta \in \bar{\Theta} . \end{cases}$$

We call Θ <u>superdissociate</u> if Θ is dissociate and if for every finite set $K \subset \hat{G}$, there exists a finite $K_1 \subset \hat{G}$ such that for every $\gamma_1 \in \hat{G}\setminus K_1$, there exists $\theta_0 \in \Theta \cup \bar{\Theta}$

such that for all $\gamma \in K$, if $\gamma\gamma_1 \in \Omega(\Phi)$, then $\gamma\gamma_1$ has the form

$$\gamma\gamma_1 = \theta_o \omega$$

for some $\omega \in \Omega(\Phi_{\theta_o})$ and such that for all $\gamma \in K$, $\gamma\gamma_1\,\theta_o^{-2} \notin \Omega(\Phi_{\theta_o})$. ("Super" refers not to the complexity of the definition, but to the fact that more than $n_{k+1}/n_k \geq 3$ is required in the case of the circle. It is recommended that the reader verify that most of the proof of Lemma III.7.8 was devoted to verifying that $\{e(n_k x)\}$ is superdissociate; K_o there corresponds to θ_o here.) It is not hard to see that Φ is superdissociate if and only if Φ is dissociate and there exist functions $\theta_o(K,\gamma)$ and $L(K)$ for $K \subset \hat{G}$ finite and $\gamma \in \hat{G}$ such that

$$(3.6)\quad \begin{cases} \theta_o(K,\gamma) \in \Phi \cup \bar{\Phi} \\ L(K) \text{ is a finite subset of } \hat{G}, \end{cases}$$

and such that for all finite $K \subset \hat{G}$,

$$(3.7)\quad \begin{cases} \text{for all } \gamma \in K \text{ and all } \gamma_1 \in \hat{G}\setminus L(K), \\ \text{(a) if } \gamma\gamma_1 \in \Omega(\Phi), \text{ then } \gamma\gamma_1 = \theta_o(K,\gamma_1)\omega \\ \text{for some } \omega \in \Omega(\Phi_{\theta_o(K,\gamma_1)}), \text{ and} \\ \text{(b) } \gamma\gamma_1\,\theta_o(K,\gamma_1)^{-2} \notin \Omega(\Phi_{\theta_o(K,\gamma_1)}). \end{cases}$$

Such functions $\theta_o(K,\gamma)$, $L(K)$ will be called <u>SD functions</u>. Given Φ, we shall denote

$$\Psi(K) = \{\gamma_1 \in \hat{G}: \forall \gamma \in K \quad \gamma\gamma_1 \notin \Omega(\Phi)\}.$$

As we have mentioned, $\theta_o(K,\gamma)$ corresponds to $K_o(N,m)$ of Lemma III.7.8. We shall need analogues (Lemmas 3.5 and 3.6) of (III.7.7) and (III.7.8), for which we require

LEMMA 3.4. Let Φ be superdissociate and $\theta_o(K,\gamma)$, $L(K)$ corresponding SD functions. Suppose that for some K and some $\gamma, \gamma_1, \theta \in \hat{G}$, we have $\gamma_1 \notin L(K)$, $\gamma \in K$, $\gamma\gamma_1 \in \Omega(\Phi)$, and $\gamma\theta^{-1} \in K$. Then $\theta_o(K,\gamma_1) \neq \theta$.

PROOF. Suppose to the contrary that $\theta = \theta_o(K,\gamma_1)$. Then for some $\omega \in \Omega(\Phi_\theta)$, $\gamma\gamma_1 = \theta_o(K,\gamma_1)\omega = \theta\omega$. Also, $(\gamma\theta^{-1})\gamma_1 = \omega \in \Omega(\Phi)$, so for some $\omega' \in \Omega(\Phi_\theta)$, $(\gamma\theta^{-1})\gamma_1 = \theta\omega'$. But then $\theta\omega' = \omega$, which contradicts dissociativity of Φ. □

LEMMA 3.5. Let Φ be superdissociate. Given a finite set $K_o \subset \hat{G}$ and an infinite set $B \subset \hat{G}\setminus\Psi(K_o)$, there exists SD functions $\theta_o(K,\gamma)$, $L(K)$ and $(\beta_n)_{n=1}^{\infty} \subset B$ such that $\beta_n \to \infty$ and $\theta_o(K_o,\beta_n) \to \infty$ as $n \to \infty$.

PROOF. For each $\beta \in B$, there exists $\gamma_o \in K_o$ such that $\gamma_o\beta \in \Omega(\Phi)$ since $\beta \notin \Psi(K_o)$. Without loss of generality, we may assume that γ_o is the same for all $\beta \in B$. Fix this γ_o.

Let $\theta_1(K,\gamma)$, $L(K)$ be SD functions for Φ. It is clear that $\beta_n \in B$, $K_n \subset \hat{G}$ $(n \geq 1)$ may be chosen inductively so as to satisfy

$$(3.8)\begin{cases} \beta_n \in B\setminus(\{\beta_1,\beta_2,\ldots,\beta_{n-1}\}\cup L(K_{n-1})) \\ K_n = K_o\cup\{\gamma_o\theta_1(K_o,\beta_1)^{-1},\gamma_o\theta_1(K_1,\beta_2)^{-1},\ldots,\gamma_o\theta_1(K_{n-1},\beta_n)^{-1}\} . \end{cases}$$

Now put

$$(3.9)\quad \theta_o(K,\gamma) = \begin{cases} \theta_1(K,\gamma) & \text{if } K \neq K_o \text{ or } \gamma \notin \{\beta_n\}_1^\infty , \\ \theta_1(K_{n-1},\beta_n) & \text{if } K = K_o \text{ and } \gamma = \beta_n . \end{cases}$$

By (3.8), $\{\beta_n\}_1^\infty$ are distinct, hence tend to ∞. It is also clear that $\theta_o(K,\gamma)$ satisfies (3.6) and, if $K \neq K_o$, (3.7). If $\gamma \in K_o$ and $\gamma_1 \in \hat{G}\setminus(L(K_o)\cup\{\beta_n\}_1^\infty)$, then again (3.7a) and (3.7b) clearly hold. Finally, $\theta_o(K_o,\beta_n)$ satisfies (3.7a) and (3.7b) for all $\gamma \in K_o$ because, first, $\theta_1(K_{n-1},\beta_n)$ satisfies (3.7a) and (3.7b) for all $\gamma \in K_{n-1}$ and, second, $K_o\subset K_{n-1}$ and $\beta_n \notin L(K_{n-1})$. Thus, $\theta_o(K,\gamma)$, $L(K)$ are SD functions.

It remains to show that $\{\theta_o(K_o,\beta_n)\}_1^\infty$ are distinct and hence tend to ∞. Let $m < n$. Then $\beta_n \notin L(K_{n-1})$, $\gamma_o \in K_{n-1}$, $\gamma_o\beta_n \in \Omega(\Phi)$, and $\gamma_o\theta_1(K_{m-1},\beta_m)^{-1} \in K_{n-1}$. By Lemma 3.4, $\theta_1(K_{n-1},\beta_n) \neq \theta_1(K_{m-1},\beta_m)$. That is, by (3.9), $\theta_o(K_o,\beta_n) \neq \theta_o(K_o,\beta_m)$, as desired. □

We also have the following freedom.

LEMMA 3.6. If Φ is superdissociate, an SD function $\theta_o(K,\gamma)$ can be chosen so that $\theta_o(K,\gamma) = \gamma$ for $\gamma \in \Phi\cup\overline{\Phi}$.

PROOF. If $\theta_1(K,\gamma)$, $L_1(K)$ are SD functions for Φ, put

$$L(K) = L_1(K\cup\{id\}) ,$$

$$\theta_o(K,\gamma) = \begin{cases} \theta_1(K\cup\{id\},\gamma) & \text{if } \gamma \notin L(K) , \\ \gamma & \text{if } \gamma \in L(K) . \end{cases}$$

Then $\theta_o(K,\gamma)$, $L(K)$ are certainly SD functions. Also, if $\gamma \notin L(K)$, then (3.7a) implies that $\gamma = id\cdot\gamma = \theta\omega$ for some $\omega \in \Omega(\Phi_\theta)$, where $\theta = \theta_1(K\cup\{id\},\gamma)$. If $\gamma \in \Phi\cup\overline{\Phi}$, then dissociativity implies $\theta = \gamma$, as desired. □

We are ready to prove the analogue of Theorem III.7.6.

THEOREM 3.7. Let Φ be superdissociate and suppose a satisfies (3.3). Let μ be the Riesz product based on Φ and a. Then for all $\nu \ll \mu$,

$$(3.10)\qquad R(\nu) = R(\mu)\cdot\|\hat{\nu}\|_\infty .$$

In particular, if $\nu \geq 0$, then

$$(3.11)\qquad R(\nu) = R(\mu)\cdot\|\nu\| .$$

PROOF. First consider the case $d\nu = P\,d\mu$, where P is a trigonometric polynomial. Let the finite set $K_o\subset\hat{G}$ be the support of $\hat{P}$. If K_o does not contain id, adjoin id to K_o. Then we have

$$\hat{\nu}(\gamma_1) = \int_G P(t)\,\overline{\gamma_1}(t)\,d\mu(t) = \sum_{\gamma\in K_o} \hat{P}(\gamma^{-1})\,\hat{\mu}(\gamma\gamma_1) .$$

If $\gamma_1 \in \Psi(K_o)$, then it follows that $\hat{\nu}(\gamma_1) = 0$. For $\gamma_1 \notin \Psi(K_o)$, let $\gamma_1' = \gamma_1\theta_o(K_o,\gamma_1)^{-1}$, where $\theta_o(K,\gamma)$, $L(K)$ are any SD functions for $\mathfrak{B}$. We claim that for $\gamma_1 \notin \Psi(K_o) \cup L(K_o)$ and $\gamma \in K_o$,

$$\hat{\mu}(\gamma\gamma_1) = \hat{\mu}(\theta_o(K_o,\gamma_1))\ \hat{\mu}(\gamma\gamma_1') . \tag{3.12}$$

For set $\theta = \theta_o(K_o,\gamma_1)$. If $\gamma\gamma_1 \in \Omega(\mathfrak{B})$, then $\gamma\gamma_1' = \gamma\gamma_1\ \theta^{-1} \in \Omega(\mathfrak{B}_\theta)$ by (3.7a), so that (3.12) holds by "multiplicativity" of $\hat{\mu}$. If $\gamma\gamma_1 \notin \Omega(\mathfrak{B})$, then we claim $\gamma\gamma_1' \notin \Omega(\mathfrak{B})$, whence both sides of (3.12) are 0 . For if $\gamma\gamma_1' \in \Omega(\mathfrak{B})$ but $\gamma\gamma_1 \notin \Omega(\mathfrak{B})$, then since $\gamma\gamma_1 = \gamma\gamma_1'\ \theta \in \theta\ \Omega(\mathfrak{B})$, it must be that $\gamma\gamma_1' \in \theta\ \Omega(\mathfrak{B}_\theta)$. But then $\gamma\gamma_1\ \theta^{-2} = \gamma\gamma_1'\ \theta^{-1} \in \Omega(\mathfrak{B}_\theta)$, contradicting (3.7b). This establishes (3.12).

Therefore for $\gamma_1 \notin \Psi(K_o) \cup L(K_o)$,

$$\sum_{\gamma\in K_o} \hat{P}(\gamma^{-1})\ \hat{\mu}(\gamma\gamma_1) = \hat{\mu}(\theta_o(K_o,\gamma_1)) \sum_{\gamma\in K_o} \hat{P}(\gamma^{-1})\ \hat{\mu}(\gamma\gamma_1') ,$$

whence if $\gamma_1 \notin L(K_o)$,

$$\hat{\nu}(\gamma_1) = \begin{cases} 0 & \text{if } \gamma_1 \in \Psi(K_o) , \\ \hat{\mu}(\theta_o(K_o,\gamma_1))\ \hat{\nu}(\gamma_1') & \text{if } \gamma_1 \notin \Psi(K_o) . \end{cases} \tag{3.13}$$

Hence for $\gamma_1 \notin L(K_o)$,

$$|\hat{\nu}(\gamma_1)| \le \begin{cases} 0 & \text{if } \gamma_1 \in \Psi(K_o) , \\ |a(\theta_o(K_o,\gamma_1))|\cdot\|\hat{\nu}\|_\infty & \text{if } \gamma_1 \notin \Psi(K_o) . \end{cases} \tag{3.14}$$

We claim that this implies $R(\nu) \le R(\mu)\cdot\|\hat{\nu}\|_\infty$. For choose an infinite set $B = \{\alpha_n\}_1^\infty \subset \hat{G}\setminus(\Psi(K_o) \cup L(K_o))$ such that $|\hat{\nu}(\alpha_n)| \ge R(\nu) - \frac{1}{n}$. Choose SD functions $\theta_o(K,\gamma)$, $L(K)$ and $\{\beta_n\}_1^\infty \subset B$ according to Lemma 3.5. Since (3.14) holds for any choice of SD functions, we have

$$R(\nu) = \lim_{n\to\infty}|\hat{\nu}(\beta_n)| \le \overline{\lim_{n\to\infty}}|a(\theta_o(K_o,\beta_n))|\cdot\|\hat{\nu}\|_\infty$$

$$\le R(\mu)\ \|\hat{\nu}\|_\infty ,$$

as claimed. On the other hand, we may instead choose $\theta_o(K,\gamma)$ according to Lemma 3.6. Let $\{\gamma_n\}_1^\infty \subset \mathfrak{B}\setminus L(K_o)$ be such that $|\hat{\mu}(\gamma_n)| \to R(\mu)$. Since $\mathrm{id} \in K_o$, it follows that $\gamma_n \notin \Psi(K_o)$, whence by (3.13), $\hat{\nu}(\gamma_n) = \hat{\mu}(\gamma_n)\ \hat{\nu}(\mathrm{id})$ and

$$R(\nu) \ge \lim_{n\to\infty}|\hat{\nu}(\gamma_n)| = R(\mu)|\hat{\nu}(\mathrm{id})| .$$

If $\gamma \in \hat{G}$, then substitution of $\gamma^{-1}\nu$ for ν in this relation yields $R(\nu) \ge R(\mu)|\hat{\nu}(\gamma)|$. Therefore $R(\nu) \ge R(\mu)\ \|\hat{\nu}\|_\infty$ and the theorem is proved for the case $d\nu = P\ d\mu$, P a trigonometric polynomial.

In general, for $\nu \ll \mu$, let $d\nu = f\ d\mu$, $f \in L^1(G,\mu)$. Since Theorem 1.1 supplies, for any $\varepsilon > 0$, a trigonometric polynomial P such that $\|f-P\|_{L^1(\mu)} < \varepsilon$, the remainder of the proof is exactly like that for Theorem III.7.6. □

4. Notes

Baker [1] shows that Weyl groups are exactly those groups which satisfy an analogue of one of Weyl's theorems concerning W*-sets in $\mathbb{R}$.

APPENDIX

APPENDIX:

SOME OPEN QUESTIONS

In the appendix, "U" will denote the class of Borel U-sets, not all U-sets.

We consider the most interesting question to be: is $R = U^{\perp}$? This was already asked in Section III.9. The other questions of that section will not be repeated here.

Section III.2

1. Is Lemma 2.8 true without the hypothesis (i)? Is it true for $f_{m,n} \in L^1(\mu)$ converging in the weak topology from $L^\infty(\mu)$?

2. Kakutani extended Lemma 2.6 to uniformly convex Banach spaces (Theorem III.10.1). Does the rate of norm convergence extend?

3. If $\mu \notin R$ and $\hat{\mu}(n_k) \to \alpha \neq 0$, then the proof of Proposition 2.3 shows that for some subsequence $\{n_k'\} \subset \{n_k\}$, $|\mu|(W(\{n_k'\})) \neq 0$. Is $|\mu|(W(\{n_k\})) \neq 0$?

4. What is $\sup_{E \in W} |\mu|E$? According to Proposition 2.3, it is at least $R(\mu)$. What if $\mu \in J$? What is $\sup_{E \in W^*} |\mu|E$?

Note that if $|\nu| \leq |\mu|$, then $\sup_{E \in W} |\mu|E \geq R(\nu)$. This raises the related question, what is $\sup R(\nu)$ over all such ν ?

Section III.4

1. Do Theorems 4.4 and 4.5 hold for <u>complex</u> measures μ ?

Section III.7

1. Is (7.15) necessary for Theorem 7.6; that is, is it enough to have $n_{k+1}/n_k \geq 3$?

2. When does μ have the property that if $\nu \ll \mu$, then $\frac{R(\mu)}{\|\nu\|} \leq \frac{R(\mu)}{\|\mu\|}$? When does μ have the property that if $0 \leq \nu \ll \mu$, then $\frac{R(\nu)}{\|\nu\|} = \frac{R(\mu)}{\|\mu\|}$? When does there correspond to μ a fixed sequence $\{n_k\}_{k=1}^{\infty}$ such that whenever $0 \leq \nu \ll \mu$, we have $|\hat{\nu}(n_k)| \to R(\nu)$? How are these properties related to each other? See Theorem 7.6.

Section III.8

1. Is (8.3) necessary for Theorem 8.3? Is (8.14) necessary for Theorem 8.8?

2. Does there exist a measure concentrated on an $H^{(m)}$-set which puts no mass on any $H^{(m-1)}$-set ? This is likely to be true. (However, if it is false for every m, then every $H^{(m)}$-set is almost an H_σ-set .)

Section IV.2

1. Given $0 < \gamma < 1$, a non-trivial arc $I \subset \mathbb{T}$, and a lacunary sequence $\{n_k\}_{k=1}^{\infty}$, is it true that

$$\{x: \forall K \ \frac{1}{K} \sum_{k=1} \chi_I(n_k x) \leq \gamma \cdot mI\}$$

is not a U-set ? Theorem 2.2 says this is true for $n_k = 2^{k-1}$ and $I = [\frac{1}{2},1)$. Is every maximal A-set corresponding to a lacunary sequence not a U-set ?

Section IV.3

1. If (3.16) fails for all i, are μ and μ' mutually absolutely continuous (i.e., equivalent)?

Section IV.4

1. Call $\mu \in M(\mathbb{T})$ absolutely pure if μ satisfies the conclusion of Theorem 3.2. (Van Kampen [1, p. 444] calls such measures "pure.") Are Riesz products absolutely pure? Riesz products are pure with respect to the classes of countable sets, of Lebesgue-measure-zero sets (Zygmund [1, I, Chapter V, §7]), and of U_0-sets (Theorem 4.4).

2. Suppose $\{q^k x\} \sim \nu$ a.e. $[\mu]$, $q \in \mathbb{Z}^+$. What conditions on μ, ν, and q are necessary or sufficient to imply that for all subsequences $\{k_\ell\}$, $\{q^{k_\ell} x\} \sim \nu$ a.e. $[\mu]$? See Theorem 4.6, Corollary 4.7, and the examples following Theorem 3.10. The question can be extended from $\{q^k\}$ to sequences $\{n_k\}$. Compare Theorem 4.5.

3. Suppose that $\{q^k x\} \sim \nu$ for some $x \in \mathbb{T}$ and some $q \in \mathbb{Z}^+$. It follows easily from Weyl's criterion that $\hat{\nu}(qn) = \hat{\nu}(n)$ for all $n \in \mathbb{Z}$, so that $\nu \notin R$. Is $\nu \in J$?

4. For a general Riesz product, can an explicit W-set be given on which it is concentrated? See the remarks preceding Theorem 4.5. For example, if μ is as in (4.1) and $\gamma_k \to \gamma$, let

$$d\nu(x) = \prod_{k=1}^{\infty} [1 + \mathrm{Re}\{\gamma e(n_k x)\}] \ dm(x) .$$

Is $\{n_k x\} \sim \nu$ a.e. $[\mu]$? If $n_k = q^{k-1}$, this is true by Theorem 4.6. Compare Question 3 for Section III.2.

Section IV.5

1. Is the union of W-sets a W-set ? Is the union of A-sets an A-set ? Is the union of W-sets an A-set ?

2. Are $H^{(m)}$-sets A-sets ? This is true for $m = 1$.

3. Are W-sets U-sets ? What about those W-sets of the form $\{x: \{n_k x\} \sim \nu\}$ for fixed $\nu \neq m$?

4. Is $U \subset W^*$? Is $U \subset W_\sigma$? Is $U \subset A_\sigma$? Is $U = (\bigcup_{m=1}^{\infty} H^{(m)})_\sigma$? (Since $H^{(m)} \subset H^{(m+1)}$, this latter class equals $\bigcup_{m=1}^{\infty} H^{(m)}$.)

5. Is $U_0 \subset W_\sigma$? See Proposition 5.13'. If some U_0-set is non-meager, then this cannot be true because of Theorem 6.1. Is $U_0 \subset W^*$?

6. If E is almost a W^*-set , is E a W^*-set ? If the answer is "yes," then by Proposition 5.16, $U_0 \subset W^*$, hence, for example, $N \subset W^*$ and $U \subset W^*$.

7. Is $H \subset W$? See Proposition 5.14. Are $H^{(m)}$-sets almost W-sets ?

8. By Theorem 5.20, $s_\infty(E) = 1 \Leftrightarrow s^+(E) = 1$. If $s^+(E) > 0$, is $s_\infty(E) > 0$?

9. What else can be said about sets E for which $s^+(E) > 0$? for which $s_\infty(E) > 0$?

10. Are all N-sets translates of W^*-sets ? See Theorem 5.34.

Section IV.6

1. Are all maximal W^*-sets corresponding to lacunary sequences co-meager? See Theorems 6.2, 6.3.

2. If $s^+(E) > 0$, is E meager?

Section IV.7

1. The bounds for $|R_+(\mu) - R_-(\mu)|$ given by Theorem 7.11 are effective only when $R(|\mu|)$ or $R(\mu)$ is close to 1 . Are there bounds effective for other ranges of $R(|\mu|)$ or $R(\mu)$?

2. The Hausdorff dimension of any lacunary maximal W^*-set , but not of any maximal W^*-set , is 1 (Erdös and Taylor [1]). What about maximal W-sets ? See also Bari [2, II, Chapter XIV, §23, p. 404].

3. Is the translate of a W-set a W_σ-set ? What about A-sets , W^*-sets ? What about homotheties (see Zygmund [1, I, (VI.2.13), p. 238, and (IX.6.18), p. 350] for homotheties of N-sets and of U-sets) ? It is not difficult to prove that a rational translate of a W^*-set is a W^*-set .

Section V.1

1. For non-abelian locally compact groups G , it is still possible to define $M_0(G)$ and to prove Theorem 1.2 (Dunkl and Ramirez [1]). Is $M_0(G)$ characterized by its common null sets U_0 ?

Section V.3

1. Does Theorem 3.7 hold if $\circledast$ is merely dissociate? Compare Question 1 for Section III.7.

GLOSSARY

GLOSSARY

We do not give here those terms defined in Chapters I or II. The section number where a term is first introduced is indicated after its definition.

$\mathcal{N}$ add $\mathcal{A}$	See Section IV.5.
almost	If C is a class of sets, a Borel set E is almost in C if for all Borel measures μ concentrated on E, $\exists F \subset E$ such that $F \in C$ and $\mu(E \setminus F) = 0$ (IV.5).
almost 1 - 1	If (X, F, μ) is a probability space and Y is a set, $\phi: X \to Y$ is almost 1 - 1 if $\exists E \in F$ such that $\phi \vert E$ is 1 - 1, $\phi[E] \cap \phi[E^c] = \emptyset$, and $\mu E = 1$ (III.5).
almost torsion-free	A group is almost torsion-free if $\forall m \in \mathbb{Z}^+$, it contains at most finitely many elements of order m (V.2).
A-set	A Borel set E in an LCA group G is an A-set if there exists a sequence $\gamma_k \to \infty$ in $\hat{G}$ such that $\forall x \in E$, $(\gamma_k(x))$ is badly distributed (V.1).

asymptotic H-set	A Borel set $E \subset \mathbb{T}$ is an asymptotic H-set if there is a non-empty open arc $I \subset \mathbb{T}$ and a sequence of integers $n_k \uparrow \infty$ such that $\forall x \in E$ $\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \chi_I(n_k x) = 0$ (III.8).
atom	A set E is an atom of μ if $\|\mu\|(E) \neq 0$ and if whenever $F \subset E$ is measurable either $\mu F = 0$ or $\mu F = \mu E$ (III.2).
band	A set $A \subset M(G)$ is a band if $\nu << \mu \in A \Rightarrow \nu \in A$ (IV.1).
$C^{\perp}$	If C is a class of sets in an LCA group G, $C^{\perp} = \{\mu \in M(G): \|\mu\|E = 0 \ \forall E \in C\}$ (III.9).
Dirichlet set	A Borel set $E \subset \mathbb{T}$ is a Dirichlet set if $\underline{\lim}_{\|n\|\to\infty} \\|e(nx) - 1\\|_{L^\infty(E)} = 0$ (III.7).
dissociate	See Section V.3.
$\hat{G}$	$\hat{G}$ is the dual of an LCA group G (V.1).

Helson constant	$s(E)$ (III.7).
Helson set	If $s(E) > 0$, E is a Helson set (III.7).
$H^{(m)}$-set	A Borel set $E \subset \mathbb{T}$ is an $H^{(m)}$-set if there is a quasi-independent sequence $\{V_k\}_{k=1}^{\infty}$ and a non-empty open set $B \subset \mathbb{T}^m$ such that $\forall x \in E$ $\forall k$ $(n_k^{(1)}x, \ldots, n_k^{(m)}x) \notin B$, where $V_k = (n_k^{(1)}, \ldots, n_k^{(m)})$ (III.8).
hyperlacunary	A sequence $(n_k) \subset \mathbb{Z}^+$ is hyperlacunary if $n_{k+1}/n_k \to \infty$ as $k \to \infty$ (III.6).
J	If G is an LCA group, $J = \{\mu \in M(G): \mu E = 0 \ \forall E \in U_0\}$ (IV.1).
lacunary	A sequence $(n_k) \subset \mathbb{Z}^+$ is lacunary if $\exists q > 1$ such that $\forall k$ $n_{k+1}/n_k \geq q$ (III.4).
lacunary W*-set	A W*-set E corresponding to a lacunary sequence is a lacunary W*-set (III.4).

less-than-exponential growth	A sequence $\{n_k\} \subset \mathbb{Z}^+$ is of less-than-exponential growth if $\lim_{k\to\infty} n_{k+1}/n_k = 1$ (III.4).
$M(E)$	For Borel $E \subset \mathbb{T}$, $M(E) = \{\mu \in M(\mathbb{T}): |\mu|(E) = \|\mu\|\}$ (III.7).
$M^+(E)$	$M^+(E) = \{\mu \in M(E): \mu \geq 0\}$ (III.7).
measure-copreserving	If (X,F,μ), (Y,G,ν) are measure spaces, then $\phi: X \to Y$ is measure-copreserving if $\forall E \in F$ $\phi[E]$ is ν-measurable and $\nu(\phi[E]) = \mu E$ (III.5).
$M(G)$	$M(G)$ is the set of (finite) complex regular Borel measures on an LCA group G (V.1).
$M^+(G)$	$M^+(G) = \{\mu \in M(G): \mu \geq 0\}$ (V.1).
$M_o(G)$	$M_o(G) = \{\mu \in M(G): \lim_{\substack{\gamma\to\infty \\ \gamma\in\hat{G}}} \hat{\mu}(\gamma) = 0\}$ (V.1).
$\mathrm{mix}(N_1,N_2,\dots)$, $\mathrm{Mix}(N_1,N_2,\dots)$, N mix M, N Mix M	See Section IV.5.

$\|\hat{\mu}\|_\infty$	For $\mu \in M(G)$, $\|\hat{\mu}\|_\infty = \sup_{\gamma\in\hat{G}} |\hat{\mu}(\gamma)|$ (III.7, V.3).
μ^m	$\mu^m = \mu*\mu*\dots*\mu$ m times (IV.3).
$N(k)$	$N(k)$ is the k-th element of the sequence N.
$(n_1,n_2,\dots,n_m)x$	If $(n_1,n_2,\dots,n_m) \in \mathbb{Z}^m$ and $x \in \mathbb{T}$, $(n_1,n_2,\dots,n_m)x = (n_1x,\dots,n_mx) \in \mathbb{T}^m$ (III.8).
N-set	A Borel set $E \subset \mathbb{T}$ is an N-set if $\exists a_k, b_k \in \mathbb{R}$ such that $\forall x \in E$ $\Sigma_{k=1}^\infty |a_k \cos 2\pi kx + b_k \sin 2\pi kx| < \infty$ but $\Sigma_{k=1}^\infty (a_k^2 + b_k^2)^{1/2} = \infty$ (IV.5).
N_o-set	A Borel set $E \subset \mathbb{T}$ is an N_o-set if $\exists n_k \uparrow \infty$ such that $\forall x \in E$ $\Sigma_{k=1}^\infty |\sin \pi n_k x| < \infty$ (IV.5).
$\Omega(\mathbb{D})$	See Section V.3.
quasi-independent	If $m \in \mathbb{Z}^+$, a sequence $(V_k)_1^\infty \subset (\mathbb{Z}^+)^m$ is quasi-independent if for all $\Lambda \in \mathbb{Z}^m$, $\Lambda \neq 0$, $|\sum_{i=1}^m n_k^{(i)} \ell_i| \to \infty$ as $k \to \infty$, where $V_k = (n_k^{(1)},\dots,n_k^{(m)})$ and $\Lambda = (\ell_1,\dots,\ell_m)$ (III.8).

R — $M_o(G)$ (V.1).

$R(\mu)$ — For $\mu \in M(G)$, $R(\mu) = \overline{\lim}|\hat{\mu}(\gamma)|$ as $\gamma \to \infty$ in $\hat{G}$ (V.3).

R-set — A Borel set $E \subset \mathbb{T}$ is an R-set if $\exists a_k, b_k \in \mathbb{R}$ such that $\forall x \in E$ $\sum_{k=1}^{\infty}(a_k \cos 2\pi kx + b_k \sin 2\pi kx)$ converges but $a_k^2 + b_k^2 \not\to 0$ (IV.5).

SD function — See Section V.3.

s(E) — For Borel $E \subset \mathbb{T}$,

$$s(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|} : 0 \neq \mu \in M(E)\right\}$$

(III.7).

$s^+(E)$ — For Borel $E \subset \mathbb{T}$,

$$s^+(E) = \inf\left\{\frac{R(\mu)}{\|\mu\|} : 0 \neq \mu \in M^+(E)\right\}$$

(III.7).

$s_\infty(E)$ — For Borel $E \subset \mathbb{T}$,

$$s_\infty(E) = \inf\left\{\frac{R(\mu)}{\|\hat{\mu}\|_\infty} : 0 \neq \mu \in M(E)\right\}$$

(IV.5).

$S_N(K,x,\ell)$ — $S_N(K,x,\ell) = \sum_{k=1}^{K} e(-\ell N(k)x)$ (IV.5).

superdissociate — See Section V.3.

U-set, set of uniqueness — A set $E \subset \mathbb{T}$ is a U-set if the only trigonometric series $\Sigma_{-\infty}^{\infty} c_n e(nt)$ which converges to 0 for all $t \notin E$ is $c_n \equiv 0$ (III.8).

U_o-set — If E is a Borel set in an LCA group G, then E is a U_o-set if $\mu E = 0$ $\forall \mu \in M_o(G)$ (V.1).

weak Dirichlet set — A Borel set $E \subset \mathbb{T}$ is a weak Dirichlet set if $\forall \mu \in M^+(E)\ \forall \varepsilon > 0\ \exists E_1 \subset E$ such that E_1 is a Dirichlet set and $\mu(E \setminus E_1) < \varepsilon$ (III.7).

Weyl group — An LCA group G is a Weyl group if $\forall m \neq 0$ and for all sequences $\gamma_k \to \infty$ in $\hat{G}$, we have $\gamma_k^m \to \infty$ as $k \to \infty$ (V.1).

word in $\mathbb{B}$ — See Section V.3.

W-set — A Borel set E in an LCA group G is a W-set if $\exists \{\gamma_k\}_1^\infty \subset \hat{G}$ tending to ∞ such that $\forall x \in E$ $\{\gamma_k(x)\}_{k=1}^{\infty}$ is Weyl-distributed (V.1).

W_1-set — A Borel set E in an LCA group G is a W_1-set if $\exists (\gamma_k)_1^\infty \subset \hat{G}$ tending to ∞ such that $\forall x \in E$ $\exists \nu \in M(\mathbb{T})$ such that $\{\gamma_k(x)\}_{k=1}^\infty \sim \nu$ and $\hat{\nu}(1) \neq 0$ (V.1).

$W^*(N)$ — $W^*(N) = \{x \in \mathbb{T} : (\exists \ell \neq 0)\ S_N(K,x,\ell) \neq o(K)\}$ (IV.5).

$z^{(\varepsilon)}$ — For $z \in \mathbb{C}$, $z^{(\varepsilon)} = z$, 1, or $\bar{z}$ if $\varepsilon = 1$, 0, or -1 respectively (IV.4).

REFERENCES

REFERENCES

J. Arbault
[1] Sur l'ensemble de convergence absolue d'une série trigonométrique, Bull. Soc. Math. France (Paris) 80(1952), 253-317.

William Arveson
[1] An Invitation to C*-algebras. New York: Springer-Verlag, 1976.

R.C. Baker
[1] A diophantine problem on groups. III, Proc. Camb. Phil. Soc. 70(1971), 31-47.
[2] A diophantine problem on groups. IV, Illinois J. of Math. 18(1974), 552-564.

S. Banach and S. Saks
[1] Sur la convergence forte dans les champs L^p, Studia Math. 2(1930), 51-57.

Nina K. Bari
[1] The uniqueness problem of the representation of functions by a trigonometric series, Amer. Math. Soc. Transl. no. 52(1951), 1-89. Supplement to "The uniqueness problem ...," (in Russian), Uspehi Matem. Nauk (N.S.) 7, no. 5(51)(1952), 193-196.
[2] A Treatise on Trigonometric Series. Translated from the Russian by M.F. Mullins. Volumes I, II. New York: Macmillan, 1964.

Patrick Billingsley
[1] Convergence of Probability Measures. New York: Wiley, 1968.

J.R. Blum and B. Epstein
[1] On the Fourier transform of an interesting class of measures, Israel J. Math. 10(1971), 302-305.

Émile Borel
[1] Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Mat. Palermo 27(1909), 247-271.

Gavin Brown
[1] Singular infinitely divisible distribution functions whose characteristic functions vanish at infinity, Math. Proc. Camb. Phil. Soc. 82(1977), 277-287.

Gavin Brown and William Moran
[1] Sums of random variables in groups and the purity law, Z. Wahrsch. und Verw. Gebeite 30(1974), 227-234.

J.L. Brown, Jr. and R.L. Duncan
[1] Asymptotic distribution of real numbers modulo one, Amer. Math. Monthly 78(1971), 367-372.

H. Davenport, P. Erdős and W.J. LeVeque
[1] On Weyl's criterion for uniform distribution, Mich. Math. J. 10(1963), 311-314.

François Dress
[1] Intersections d'ensembles normaux, J. Number Theory 2(1970), 352-362.

Charles F. Dunkl and Donald E. Ramirez
[1] Translation in measure algebras and the correspondence to Fourier transforms vanishing at infinity, Mich. Math. J. 17(1970), 311-319.

P. Erdős and S.J. Taylor
[1] On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences, Proc. London Math. Soc. (3) 7(1957), 598-615.

H. Federer and A.P. Morse
[1] Some properties of measurable functions, Bull. Amer. Math. Soc. 49(1943), 270-277.

R.R. Goldberg and A.B. Simon
[1] Characterization of some classes of measures, Acta Sci. Math. (Szeged) 27(1966), 157-161.

Colin C. Graham and O. Carruth McGehee
[1] Essays in Commutative Harmonic Analysis. New York: Springer-Verlag, 1979.

Edwin Hewitt and Kenneth A. Ross
[1] Abstract Harmonic Analysis. Volume I, second edition, 1979. Volume II, 1970. Berlin: Springer-Verlag.

Edwin Hewitt and Karl Stromberg
[1] Real and Abstract Analysis. Berlin: Springer-Verlag, 1965.

N.N. Holščevnikova
[1] The sum of less-than-continuum many closed U-sets, Engl. trans., Moscow Univ. Math. Bull. 36, no. 1 (1981), 60-64.

B. Host and F. Parreau
[1] Sur les mesures dont la transformée de Fourier-Stieltjes ne tend pas vers zéro à l'infini, Colloq. Math. 41, no. 2 (1979), 285-289.

V. Hutson and J.S. Pym
[1] Applications of Functional Analysis and Operator Theory. London: Academic Press, 1980.

B. Jessen and A. Wintner
[1] Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38(1935), 48-88.

J.-P. Kahane
[1] Sur les mauvaises répartitions modulo 1, Ann. Inst. Fourier (Grenoble) 14, 2(1964), 519-526.

J.-P. Kahane and R. Salem
[1] Distribution modulo 1 and sets of uniqueness, Bull. Amer. Math. Soc. 70(1964), 259-261.

Shizuo Kakutani
[1] Weak convergence in uniformly convex spaces, Tôhoku Math. J. 45(1938), 188-193.

Yitzhak Katznelson
[1] An Introduction to Harmonic Analysis. Second corrected edition. New York: Dover, 1976.

Tatsuo Kawata
[1] Fourier Analysis in Probability Theory. New York: Academic Press, 1972.

John L. Kelley
[1] General Topology. Princeton: Van Nostrand, 1955.

J.F. Koksma and L. Kuipers, eds.
[1] Asymptotic Distribution Modulo 1. Groningen: Noordhoff, 1965(?). Same as Comp. Math. 16 (1964), 1-203.

T.W. Körner
[1] Some results on Kronecker, Dirichlet, and Helson sets, Ann. Inst. Fourier (Grenoble) 20, fasc. 2 (1970), 219-324.
[2] Some results on Kronecker, Dirichlet, and Helson sets, II, J. d'Analyse Math. 27(1974), 260-388.

L. Kuipers and H. Niederreiter
[1] Uniform Distribution of Sequences. New York: Wiley, 1974.

L.-Å. Lindahl and F. Poulsen, eds.
[1] Thin Sets in Harmonic Analysis. New York: Marcel Dekker, 1971.

M. Loève
[1] Probability Theory. Fourth ed. Vol. I. New York: Springer-Verlag, 1977.

Michel Mendès France
[1] Nombres normaux. Applications aux fonctions pseudo-aléatoires, J. d'Analyse Math. 20(1967), 1-56.

D.E. Menšov
[1] Sur l'unicité du développement trigonométrique, C.R. Acad. Sci. Paris Sér. A-B 163(1916), 433-436.

Halina Milicer-Grużewska
[1] Sur les fonctions à variation bornée et à l'écart Hadamardien nul, C.R. Soc. Sci. Varsovie [Sprawozdania z posiedzén Towarzystwa naukowego warszawskiego] 21(1928), 67-78.
[2] Une contribution à la question des variations sur les ensembles du type (H), C.R. Soc. Sci. Varsovie 21(1928), 157-165.
[3] Sur la continuité de la variation, C.R. Soc. Sci. Varsovie 21(1928), 166-177.

Jacques Peyrière
[1] Étude de quelques propriétés de produits de Riesz, Ann. Inst. Fourier (Grenoble) 25, no. 2(1975), 127-169.

I.I. Pjateckii-Šapiro
[1] On the problem of uniqueness of expansion of a function in a trigonometric series, (in Russian), Moscov. Gos. Univ. Uč. Zap. 155, Mat. 5(1952), 54-72.
[2] Supplement to the work "On the problem ...," (in Russian), Moscov. Gos. Univ. Uč. Zap. 165, Mat. 7(1954), 79-97.

Alexandre Rajchman
[1] Sur l'unicité du développement trigonométrique, Fund. Math. (Varsovie) 3(1922), 287-302; Rectification et addition à ma note "Sur l'unicité ...," Fund. Math. 4(1923), 366-367.
[2] Sur la multiplication des séries trigonométriques et sur une classe remarquable d'ensembles fermés, Math. Ann. 95(1925), 389-408.
[3] Une classe de séries trigonométriques qui convergent presque partout vers zéro, Math. Ann. 101(1929), 686-700.

Gérard Rauzy
[1] Caractérisation des ensembles normaux, Bull. Soc. Math. France 98(1970), 401-414.

F. Riesz
[1] Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung, Math. Zeit. 3(1918), 312-315. Also in "Gesammelte Arbeiten." Vol. II, pp. 1452-1455, Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1960.

H.L. Royden
[1] Real Analysis. Second ed. New York: Macmillan, 1968.

Walter Rudin
[1] Fourier Analysis on Groups. New York: Interscience, 1962.
[2] Real and Complex Analysis. Second ed. New York: McGraw-Hill, 1974.

Tibor Šalát
[1] A remark on normal numbers, Rev. Roumaine Math. Pure et Appl. 11(1966), 53-56.

Raphaël Salem
[1] On the absolute convergence of trigonometric series, Duke Math. J. 8(1941), 317-334.
[2] On trigonometrical series whose coefficients do not tend to zero, Bull. Amer. Math. Soc. 47(1941), 899-901.
[3] Uniform distribution and capacity of sets, Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.], Tome Supplémentaire (1952), 193-195.

J. Schreier
[1] Ein Gegenbeispiel zur Theorie der schwachen Konvergenz, Studia Math. 2, 2(1930), 58-62.

Ju. A. Šreĭder
[1] On the ring of functions with bounded variation, (in Russian), Uspehi Mat. Nauk, t. III, 2(24) (1948), 224-225.
[2] On the Fourier-Stieltjes coefficients of functions with bounded variation, (in Russian), Dokl. Akad. Nauk SSSR 74(1950), 663-664.

William F. Stout
[1] Almost Sure Convergence. New York: Academic Press, 1974.

E.R. van Kampen
[1] Infinite product measures and infinite convolutions, Amer. J. Math. 62(1940), 417-448.

E.R. van Kampen and Aurel Wintner
[1] On divergent infinite convolutions, Amer. J. Math. 59(1937), 635-654.

I.M. Vinogradov
[1] On an estimate of trigonometric sums with prime numbers, (in Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 12(1948), 225-248.

Hermann Weyl
[1] Über ein Problem aus dem Gebeite der diophantischen Approximationen, Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse (1914), 234-244. Also in "Gesammelte Abhandlungen," Vol. I, pp. 487-497, Springer-Verlag, Berlin, 1968.
[2] Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77(1916), 313-352. Also in "Gesammelte Abhandlungen," Vol. I, pp. 563-599, Springer-Verlag, Berlin, 1968.

Adriaan C. Zaanen
[1] An Introduction to the Theory of Integration. Amsterdam: North-Holland, 1958.

Antoni Zygmund
[1] Trigonometric Series. Second edition, reprinted. Volumes I, II. Cambridge: Cambridge University Press, 1979.