

Szegő Limit Theorems

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Abstract. The first Szegő limit theorem has been extended by Bump-Diaconis and Tracy-Widom to limits of other minors of Toeplitz matrices. We use a more geometric method to extend their results still further. Namely, we allow more general measures and more general determinants. We also give a new extension to higher dimensions, which extends a theorem of Helson and Lowdenslager.

§1. Introduction.

Let λ denote Lebesgue measure on the unit-length circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For a finite positive measure μ on \mathbb{T} , its Fourier coefficients are $\widehat{\mu}(n) := \int_{\mathbb{T}} e^{-2\pi i n t} d\mu(t)$. For $n \geq 0$, define

$$D_n(\mu) := \det[\widehat{\mu}(k - j)]_{0 \leq j, k \leq n}.$$

The Szegő limit theorems determine the asymptotics of $D_n(\mu)$. The first Szegő limit theorem (Grenander and Szegő (1984), p. 44), which is the one of concern here, determines $\lim_{n \rightarrow \infty} D_{n+1}(\mu)/D_n(\mu)$. To state this beautiful result of Szegő's in its extended form due to Kolmogorov and Kreĭn, define, for $f \geq 0$ and $\log^+ f \in L^1(\lambda)$, the **geometric mean** of f by

$$\text{GM}(f) := \exp \int_{\mathbb{T}} \log f d\lambda.$$

According to the arithmetic mean-geometric mean inequality, if $0 \leq f \in L^1(\lambda)$, then $\log^+ f \in L^1(\lambda)$ and $0 \leq \text{GM}(f) \leq \int f d\lambda$.

THEOREM 1.1. (SZEGŐ LIMIT THEOREM) *Let μ be a finite positive measure on \mathbb{T} with infinite support. Let $f := [d\mu/d\lambda]$ be the Radon-Nikodým derivative of the absolutely continuous part of μ . Then*

$$\lim_{n \rightarrow \infty} D_{n+1}(\mu)/D_n(\mu) = \text{GM}(f). \tag{1.1}$$

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This result has been extended in various ways, three of which we consider here. The first two extensions were proved by Bump and Diaconis (2002) and Tracy and Widom (2002), while the third was proved by Helson and Lowdenslager (1958). Our theorems extend theirs further yet. These extensions concern quotients of determinants where, instead of the determinant in the numerator having one more particular row and column than the determinant in the denominator, as in (1.1), the numerator contains finitely many more rows and columns arising from inner products among arbitrary vectors. In addition, we shall consider the case where μ is complex, as did Bump and Diaconis and Tracy and Widom. Theorems 3.1 and 5.2 of this paper are used in Lyons and Steif (2003) for studying entropy and phase multiplicity in stationary determinantal processes.

The approach of Bump and Diaconis relies on certain identities for symmetric functions and representations of the symmetric group, while Tracy and Widom proceed via factorization and analysis of Toeplitz operators. Our approach is different, although it bears some similarities to that of Tracy and Widom. Namely, we go to the level of vectors in Hilbert space and analyze projections on various subspaces. This more geometric method appears to be more flexible and leads to a more transparent proof and formulation of the results. Although we use row operations that leave determinants unchanged, one could instead use multivectors and continue a geometric approach. However, to maintain greater accessibility, we have omitted exterior algebra.

In a brief Section 2, we recall some general notation and facts from complex analysis. This allows us to state and prove our first result in Section 3, which concerns the case $\mu \geq 0$. We turn to the case of complex μ in Section 4. This requires a lemma about convergence of non-orthogonal linear projections in Hilbert space, whose proof is relegated to the appendix of the paper. Helson and Lowdenslager (1958) proved an analogue of Theorem 1.1 for higher dimensions. In Section 5, we extend their theorem by proving analogues of our results for higher dimensions; both the real and complex case are treated there. Actually, Helson and Lowdenslager stated their result not as a limit of quotients of determinants, but as an extremum problem, as in Grenander and Szegő (1984).

§2. General Notation and Hardy Spaces.

Write $e_n(t) := e^{2\pi int}$ and for $f \in L^1(\lambda)$, write $\widehat{f}(n) := \int_{\mathbb{T}} f \overline{e_n} d\lambda$. For $p \geq 1$, let $H^p(\mathbb{T})$ denote the Hardy space of those $f \in L^p(\lambda)$ with $\widehat{f}(n) = 0$ for all $n < 0$. Write P_{H^2} for the orthogonal projection from $L^2(\lambda) \rightarrow H^2(\mathbb{T})$. For all $f \in H^1(\mathbb{T})$, we have $\text{GM}(|f|) > 0$ (see Rudin (1987), Theorem 17.17, p. 344). For the converse, for any function $f \geq 0$ with $\log f \in L^1(\lambda)$, define

$$\Phi_f(z) := \exp \frac{1}{2} \int_{\mathbb{T}} \frac{e_1(t) + z}{e_1(t) - z} \log f(t) d\lambda(t) \quad (2.1)$$

for $|z| < 1$. The **outer** function

$$\varphi_f(t) := \lim_{r \uparrow 1} \Phi_f(re_1(t)) \quad (2.2)$$

exists for λ -a.e. $t \in \mathbb{T}$ and satisfies $|\varphi_f|^2 = f$ λ -a.e. Also, $\varphi_f \in H^p(\mathbb{T})$ iff $f \in L^{p/2}(\lambda)$. If $f \in L^2(\lambda)$, then the limit in (2.2) also holds in $L^1(\lambda)$. See Rudin (1987), Theorem 17.11, p. 340, and Theorem 17.16, p. 343. Cauchy's integral formula shows that if $f \in L^2(\lambda)$, then

$$\widehat{\varphi_f}(0) = \Phi_f(0) = \sqrt{\text{GM}(f)}. \quad (2.3)$$

By the factorization Theorem 17.17, p. 344, of Rudin (1987), if $g \in H^p(\mathbb{T})$ and φ is an outer function such that $g\varphi \in L^q(\lambda)$, then $g\varphi \in H^q(\mathbb{T})$. Define $\Phi_f := \varphi_f := \mathbf{0}$ if $\log f \notin L^1(\lambda)$.

§3. Positive Measures.

Note that the determinant $D_n(\mu)$ is the same when j, k take values in any index set of $n + 1$ consecutive integers. We shall, in fact, use the index set $\{-1, -2, \dots, -n\}$ for $D_{n-1}(\mu)$.

In this section, we extend Theorem 1.1 to more general determinants as follows.

THEOREM 3.1. *Let μ be a finite positive measure on \mathbb{T} with infinite support. Let $f := [d\mu/d\lambda]$ be the Radon-Nikodým derivative of the absolutely continuous part of μ . Given any functions $f_0, \dots, f_r, g_0, \dots, g_r \in L^2(\mu)$, let $F_j := P_{H^2}(f_j \overline{\varphi_f})$ and $G_j := P_{H^2}(g_j \overline{\varphi_f})$. Define*

$$p_j := \begin{cases} f_j & \text{if } 0 \leq j \leq r, \\ e_j & \text{if } -1 \geq j \geq -n \end{cases}$$

and

$$q_j := \begin{cases} g_j & \text{if } 0 \leq j \leq r, \\ e_j & \text{if } -1 \geq j \geq -n. \end{cases}$$

We have

$$\lim_{n \rightarrow \infty} D_{n-1}(\mu)^{-1} \det \left[\int p_j \overline{q_k} d\mu \right]_{-n \leq j, k \leq r} = \det \left[\int F_j \overline{G_k} d\lambda \right]_{0 \leq j, k \leq r}.$$

REMARK 3.2. If $\text{GM}(f) = 0$, then $\varphi_f = \mathbf{0}$, so $F_j = G_j = \mathbf{0}$. Otherwise, $f > 0$ λ -a.e., so that $\lambda \ll f\lambda$ and there is no ambiguity about the equivalence class of f_j or g_j with respect to λ . Also, $\int |f_j \overline{\varphi_f}|^2 d\lambda = \int |f_j|^2 f d\lambda \leq \int |f_j|^2 d\mu < \infty$, so that $f_j \overline{\varphi_f} \in L^2(\lambda)$ and F_j is well defined. Likewise G_j is well defined.

REMARK 3.3. Since μ has infinite support, $[\widehat{\mu}(k-j)]_{-1 \geq j, k \geq -n}$ is non-singular. Indeed, if it were singular, then since it is a Gram matrix $[(e_j, e_k)_\mu]$, where the subscript μ indicates that the inner product is taken in $L^2(\mu)$, it would follow that the vectors e_{-1}, \dots, e_{-n} would be linearly dependent, i.e., there would be scalars a_j such that $\sum_j a_j e_j = 0$ μ -a.e. This would imply that μ would have support contained in the zero set of this trigonometric polynomial, i.e., μ would have support of cardinality at most $n-1$.

REMARK 3.4. The case considered by Bump and Diaconis (2002) and Tracy and Widom (2002) is that where all functions f_j and g_j are of the form e_n for various $n \geq 0$. These give minors of the Toeplitz matrix other than merely $D_n(\mu)$. In this case, when $f_j := e_j$ and $g_k := e_k$, the limiting matrix entries $\int F_j \overline{G_k} d\lambda$ become

$$\int P_{H^2}(e_j \overline{\varphi_f}) \overline{P_{H^2}(e_k \overline{\varphi_f})} d\lambda = \sum_{l=0}^{\min(j,k)} \overline{\widehat{\varphi}_f(j-l)} \widehat{\varphi}_f(k-l). \quad (3.1)$$

Bump and Diaconis gave a different formula than (3.1); Tracy and Widom gave the same formula as ours. Both sets of authors assumed that μ was absolutely continuous. In addition, Bump and Diaconis assumed that $f = e^g$ for some g satisfying $\sum_{n \in \mathbb{Z}} (|\widehat{g}(n)| + |n\widehat{g}(n)|^2) < \infty$, while Tracy and Widom assumed that f was bounded above and bounded away from 0. On the other hand, Bump and Diaconis showed the strong Szegő limit theorem, which gives finer asymptotics.

REMARK 3.5. The special case $r := 0$, $f_0 := g_0 := e_j$ for any fixed $j > 0$ and μ is absolutely continuous is due to Kolmogorov and Wiener (see Grenander and Szegő (1984), Section 10.9).

REMARK 3.6. In case one of F_j or G_k is easier to calculate than the other, one could use instead of $\int F_j \overline{G_k} d\lambda$ either of the equivalent expressions $\int F_j \overline{g_k} \varphi_f d\lambda$ or $\int f_j \overline{G_k} \varphi_f d\lambda$.

Proof of Theorem 3.1. For the ease of the reader, we treat first the case $r = 0$, $f_0 = g_0 = \mathbf{1}$, when Theorem 3.1 becomes the Szegő limit theorem. Since $\widehat{\mu}(k-j) = (e_j, e_k)_\mu$, we have

that

$$D_n(\mu)/D_{n-1}(\mu) = \|P_n e_0\|_\mu^2,$$

where P_n is the orthogonal projection onto $\{e_{-1}, \dots, e_{-n}\}^\perp$ in $L^2(\mu)$. (This is sometimes called ‘‘Gram’s formula’’.) This quotient therefore tends (monotonically) to $\|P_\infty e_0\|_\mu^2$, where P_∞ is the orthogonal projection of $L^2(\mu)$ onto

$$H_\infty := \{g \in L^2(\mu); \forall n < 0 \ (g, e_n)_\mu = 0\}.$$

Now $g \in H_\infty$ iff $g \in L^2(\mu)$ and $g\mu$ is an analytic measure. By the F. and M. Riesz theorem (Rudin (1987), Theorem 17.13, p. 341), it follows that $g \in H_\infty$ iff $g = 0$ a.e. with respect to the singular part of μ , $g \in L^2(f)$ and $gf \in H^1(\mathbb{T})$. In particular, we may from now on disregard the singular part of μ . That is, $P_\infty e_0$ is the same as the orthogonal projection of e_0 in $L^2(f)$ onto $H_\infty := \{g \in L^2(f); \forall n < 0 \ (g, e_n)_f = 0\}$ and its norm in $L^2(\mu)$ is the same as its norm in $L^2(f)$. Write $h_0 := f \cdot P_\infty e_0$ and $\varphi := \varphi_f$.

If $\text{GM}(f) > 0$, then $h_0/f \in L^2(f)$, $h_0 \in H^1(\mathbb{T})$ and $h_0/\varphi \in H^2(\mathbb{T})$. Also, for all $g \in H_\infty$, we have

$$(h_0/f, g)_\mu = (e_0, g)_\mu.$$

For all $m \geq 0$, we have $e_m/\overline{\varphi} \in H_\infty$, whence

$$(h_0/f, e_m/\overline{\varphi})_\mu = (e_0, e_m/\overline{\varphi})_\mu,$$

or in other words, $\widehat{(h_0/f)}(m) = \widehat{\varphi}(m)$. Since $\widehat{\varphi}(m) = \sqrt{\text{GM}(f)}\delta_{0,m}$ for $m \geq 0$, we obtain that

$$\widehat{(h_0/f)}(m) = \sqrt{\text{GM}(f)}\delta_{0,m} \tag{3.2}$$

for $m \geq 0$. Since $h_0/\varphi \in H^2(\mathbb{T})$, (3.2) holds for all m . That is, $h_0 = \sqrt{\text{GM}(f)}\varphi$. Therefore,

$$\|P_\infty e_0\|_\mu^2 = \|h_0/f\|_\mu^2 = \int \frac{|h_0|^2}{f} d\lambda = \text{GM}(f).$$

If $\text{GM}(f) = 0$, then for all $g \in H_\infty$,

$$\text{GM}(|gf|)^2 = \text{GM}(|gf|^2) = \text{GM}(|g|^2 f)\text{GM}(f) = 0$$

since $\text{GM}(|g|^2 f) \leq \int |g|^2 f d\lambda < \infty$ as $g \in L^2(f)$. Since $gf \in H^1(\mathbb{T})$, this means that $gf = \mathbf{0}$ as noted in Section 2. In other words, $g = 0$ μ -a.e. Therefore $H_\infty = 0$ and so the limit is 0.

We have thus proved the Szegő formula. This proof also shows immediately that the linear span of $\{e_n; n \geq 0\}$ is dense in $L^2(\mu)$ iff $\text{GM}(f) = 0$, a theorem of Kolmogorov

and Kreĭn. (More precisely, as written, this proof decides the density of the linear span of $\{e_n; n \leq -1\}$ by deciding whether its orthocomplement is 0, but this is equivalent.)

Now we continue with the general case. Consider $j \geq 0$. Since

$$P_n f_j = f_j - \sum_{-1 \geq i \geq -n} a_i e_i$$

for some constants a_i , row operations can be used to change the j th row from its initial value $[(f_j, q_k)_\mu]_{-n \leq k \leq r}$ to $[(P_n f_j, q_k)_\mu]_{-n \leq k \leq r}$ without changing the determinant. Since P_n is an orthogonal projection, we have $(P_n f_j, q_k)_\mu = (P_n f_j, P_n q_k)_\mu$. If we change all rows $j \geq 0$ in this manner, we obtain a block diagonal matrix, which shows that

$$D_{n-1}(\mu)^{-1} \det [(p_j, q_k)_\mu]_{-n \leq j, k \leq r} = \det [(P_n f_j, P_n g_k)_\mu]_{0 \leq j, k \leq r}.$$

Thus, the limit is

$$\det [(P_\infty f_j, P_\infty g_k)_\mu]_{0 \leq j, k \leq r}.$$

As before, if $\text{GM}(f) = 0$, then $H_\infty = 0$ and the limit is 0. Otherwise, the reasoning that led to (3.2) now leads to

$$[f(P_\infty f_j)/\varphi]^\wedge(m) = \widehat{f_j \overline{\varphi}}(m)$$

for all $m \geq 0$, whence $f(P_\infty f_j)/\varphi = F_j$. Likewise, $f(P_\infty g_k)/\varphi = G_k$. This gives the formula since

$$(P_\infty f_j, P_\infty g_k)_\mu = \int P_\infty f_j \cdot \overline{P_\infty g_k} d\mu = \int P_\infty f_j \cdot \overline{P_\infty g_k} \cdot f d\lambda = \int F_j \overline{G_k} d\lambda. \quad \blacksquare$$

REMARK 3.7. A bivariate generating function for the matrix entries of (3.1) is

$$\sum_{j, k \geq 0} z^j \bar{\zeta}^k \sum_{l=0}^{\min(j, k)} \overline{\widehat{\varphi}_f(j-l)} \widehat{\varphi}_f(k-l) = \frac{\Phi_f(z) \overline{\Phi_f(\bar{\zeta})}}{1 - \bar{\zeta} z}.$$

§4. Complex Measures.

We now consider the case of absolutely continuous complex measures, μ . The proof of the main result in this section, Theorem 4.2, could be modified so as to allow a positive singular part to μ and to include all of Theorem 3.1. However, the proof would become less elegant.

Let Poly_n denote the linear span of $\{e_0, e_1, \dots, e_n\}$. Given a pair of functions $\varphi, \psi \in L^2(\lambda)$, consider the condition

$$\exists \epsilon > 0 \quad \exists n_0 \quad \forall n \geq n_0 \quad \forall S \in \text{Poly}_n \quad \exists T \in \text{Poly}_n \setminus \{0\} \quad (\varphi S, \psi T)_\lambda \geq \epsilon \|\varphi S\|_\lambda \|\psi T\|_\lambda. \quad (4.1)$$

Of course, this holds if $\varphi = \psi$, since we may then take $\epsilon := 1$ and $T := S$. Some readers may prefer the following restatement of (4.1). Given two subspaces H_1 and K_1 of a Hilbert space H , define

$$\epsilon(H_1, K_1; H) := \epsilon(H_1, K_1) := \inf_{\substack{x \in H_1 \\ \|x\|=1}} \sup_{\substack{y \in K_1 \\ \|y\|=1}} |(x, y)|.$$

The condition $\epsilon(H_1, K_1) > 0$ is weaker than $H_1 = K_1$ and stronger than $H_1 \cap K_1^\perp = 0$. Our condition (4.1) is equivalent to

$$\liminf_{n \rightarrow \infty} \epsilon(\varphi \cdot \text{Poly}_n, \psi \cdot \text{Poly}_n; L^2(\lambda)) > 0.$$

Condition (4.1) will be used via the following criterion. We write $H_n \uparrow H_\infty$ to mean that $H_n \subseteq H_{n+1}$ for all n and $\bigcup H_n$ is dense in H_∞ .

LEMMA 4.1. *Suppose that H is a Hilbert space, H_n, K_n are non-zero closed subspaces for $1 \leq n \leq \infty$ with $H = H_n + K_n^\perp$ and $H_n \cap K_n^\perp = 0$ for all $1 \leq n \leq \infty$. Suppose that $H_n \uparrow H_\infty$ and $K_n \uparrow K_\infty$. Let $T_n : H \rightarrow K_n^\perp$ be the linear projection along H_n ($1 \leq n \leq \infty$). Then $T_n \rightarrow T_\infty$ in the strong operator topology iff*

$$\liminf_{n \rightarrow \infty} \epsilon(H_n, K_n) > 0. \quad (4.2)$$

This lemma should be known, but we could not locate a reference. Thus, we include its proof in an appendix. Note that when $H_n = K_n$, which will correspond to the case $\varphi = \psi$ in our application, it is trivial that $T_n \rightarrow T_\infty$ in the strong operator topology.

As we have noted already, $\epsilon(H_n, K_n) > 0$ implies that $H_n \cap K_n^\perp = 0$. If $\dim H_n = \dim K_n < \infty$, as will be the case in our application of (4.2), this in turn implies that $H = H_n + K_n^\perp$.

THEOREM 4.2. Suppose that $\mu = \psi\bar{\varphi}\lambda$ for some pair of outer functions $\varphi, \psi \in H^2(\mathbb{T})$ that satisfies condition (4.1). Given any functions $f_0, \dots, f_r, g_0, \dots, g_r \in L^2(|\varphi|^2 + |\psi|^2)$, let $F_j := P_{H^2}(f_j\bar{\varphi})$ and $G_j := P_{H^2}(g_j\bar{\psi})$. Define

$$p_j := \begin{cases} f_j & \text{if } 0 \leq j \leq r, \\ e_j & \text{if } -1 \geq j \geq -n \end{cases}$$

and

$$q_j := \begin{cases} g_j & \text{if } 0 \leq j \leq r, \\ e_j & \text{if } -1 \geq j \geq -n. \end{cases}$$

We have

$$\lim_{n \rightarrow \infty} D_{n-1}(\mu)^{-1} \det \left[\int p_j \bar{q}_k d\mu \right]_{-n \leq j, k \leq r} = \det \left[\int F_j \bar{G}_k d\lambda \right]_{0 \leq j, k \leq r}.$$

Note that $L^2(|\varphi|^2 + |\psi|^2) \subseteq L^2(|\mu|)$ by the Cauchy-Schwarz inequality.

Proof. Let $H_n(\varphi) := e_1\varphi\text{Poly}_n$. By virtue of (4.1), we have for $n \geq n_0$,

$$\overline{H_n(\varphi)} \cap \overline{H_n(\psi)}^\perp = 0,$$

and so

$$L^2(\mathbb{T}) = \overline{H_n(\varphi)} + \overline{H_n(\psi)}^\perp.$$

A consequence of Beurling's theorem (Rudin (1987), Theorem 17.23, p. 350) is that

$$H_n(\varphi) \uparrow H_0^2(\mathbb{T}) := e_1 H^2(\mathbb{T}). \quad (4.3)$$

Thus the projection along $\overline{H_n(\varphi)}$ to $\overline{H_n(\psi)}^\perp$ tends to the orthogonal projection $P_{\overline{H_0^2(\mathbb{T})}^\perp} = P_{H^2}$.

Now $\int p_j \bar{q}_k d\mu = (\bar{\varphi}p_j, \bar{\psi}q_k)_\lambda$. Let $F_j^{(n)}$ be the projection of $\bar{\varphi}f_k$ along $\overline{H_n(\varphi)}$ to $\overline{H_n(\psi)}^\perp$. Row operations show that for $n \geq n_0$,

$$D_{n-1}(\mu)^{-1} \det \left[\int p_j \bar{q}_k d\mu \right]_{-n \leq j, k \leq r} = \det [(F_j^{(n)}, \bar{\psi}g_k)_\lambda]_{0 \leq j, k \leq r}.$$

Because of our assumption (4.1) and Lemma 4.1, the limit is $\det[(F_j, \bar{\psi}g_k)_\lambda]_{0 \leq j, k \leq r}$, which is the same as $\det[(F_j, G_k)_\lambda]_{0 \leq j, k \leq r}$. \blacksquare

REMARK 4.3. The limit (4.3) is often used to prove Beurling's theorem and it has a simple direct proof: If $g \in H_0^2(\mathbb{T})$ and $g \perp H_n(\varphi)$ for all $n \geq 0$, then $\widehat{g\bar{\varphi}}(k) = 0$ for all $k \geq 1$, i.e., $g\bar{\varphi} \in \overline{H^1(\mathbb{T})}$. Dividing by $\bar{\varphi}$, we get that $g \in \overline{H^2(\mathbb{T})} = (H_0^2(\mathbb{T}))^\perp$, so that $g = \mathbf{0}$.

The case considered by Bump and Diaconis (2002) and Tracy and Widom (2002) is that where all functions f_j and g_j are of the form e_n for various $n \geq 0$. In addition, Bump and Diaconis assumed that $f = e^g$ for some g satisfying $\sum_{n \in \mathbb{Z}} (|\widehat{g}(n)| + |n\widehat{g}(n)|^2) < \infty$, while Tracy and Widom assumed that φ and ψ are bounded above and that the Toeplitz matrix corresponding to $\psi\bar{\varphi}$ has uniformly invertible finite sections.

The assumption of Bump and Diaconis implies that of Tracy and Widom. Indeed, write $g = g_1 + g_2$, where $g_1 := \sum_{n \geq 0} \widehat{g}(n)e_n$ and $g_2 := \sum_{n < 0} \widehat{g}(n)e_n$. Set $f_1 := e^{g_1 + \bar{g}_1}$ and $f_2 := e^{g_2 + \bar{g}_2}$. Then $f = \psi\bar{\varphi}$ with $\psi := e^{g_1} = \varphi_{f_1}$ and $\varphi := e^{\bar{g}_2} = \varphi_{f_2}$, so that ψ and φ are bounded outer functions. Furthermore, since g is continuous, Kreĭn's Theorem (Böttcher and Silbermann (1999), Theorem 1.15, p. 18) in combination with a theorem of Gohberg and Feldman (Böttcher and Silbermann (1999), Theorem 2.11, p. 39) shows that the Toeplitz matrix of f has uniformly invertible finite sections.

Our theorem covers that of Tracy and Widom since boundedness of ψ and uniform invertibility of finite sections implies uniform boundedness of the projections along $\overline{H_n(\varphi)}$ to $\overline{H_n(\psi)}^\perp$, as we see by simply writing the equations: If $g \in L^2(\lambda)$ is written as $g = u + v$ with $u \in \overline{H_n(\varphi)}$ and $v \in \overline{H_n(\psi)}^\perp$, then write $\bar{u} = \sum_{k=1}^n a_k e_k \varphi$. The coefficients a_k are determined by the requirement that $g - u \perp \overline{H_n(\psi)}$, i.e., by the equations

$$\forall k \in [1, n] \quad \widehat{\psi g}(-k) = \sum_{j=1}^n a_j \widehat{\psi \bar{\varphi}}(j - k).$$

Now observe that $\left[\sum_{k=1}^n |\widehat{\psi g}(-k)|^2 \right]^{1/2} \leq \|\psi g\|_\lambda \leq \|\psi\|_\infty \|g\|_\lambda$.

We next give some additional cases when (4.1) holds. We begin with a reformulation of (4.1), for which we are grateful to Doron Lubinsky. Let $w := |\psi|^2$ and $\sigma := \varphi/\psi$. Then

$$(\varphi S, \psi T)_\lambda = \int \sigma S \bar{T} w \, d\lambda,$$

so that

$$\sup_{T \in \text{Poly}_n} \frac{|(\varphi S, \psi T)_\lambda|}{\|\psi T\|_\lambda} = \sup_{T \in \text{Poly}_n} \frac{|(\sigma S, T)_w|}{\|T\|_w} = \|P_{\text{Poly}_n}(\sigma S)\|_w,$$

where the orthogonal projection onto Poly_n takes place in $L^2(w)$. Note that $\sigma \in L^2(w)$ since $\varphi \in L^2(\lambda)$. Let \mathbf{p}_n be the standard orthogonal polynomials for the weight w , i.e., $\mathbf{p}_n \in \text{Poly}_n$ with positive leading coefficient and $(\mathbf{p}_m, \mathbf{p}_n)_w = \delta_{m,n}$. If we write

$$\sigma S = \sum_{k \geq 0} a_k \mathbf{p}_k, \tag{4.4}$$

then $\|\sigma S\|_w^2 = \sum_{k \geq 0} |a_k|^2$ and $\|P_{\text{Poly}_n}(\sigma S)\|_w^2 = \sum_{k=0}^n |a_k|^2$. Thus, (4.1) is equivalent to

$$\exists \epsilon > 0 \quad \exists n_0 \quad \forall n \geq n_0 \quad \forall S \in \text{Poly}_n \quad \sum_{k=0}^n |a_k|^2 \geq \epsilon \sum_{k=0}^{\infty} |a_k|^2, \quad (4.5)$$

where a_k are defined by (4.4).

PROPOSITION 4.4. *If $\varphi, \psi \in L^2(\mathbb{T})$ and $\sigma := \varphi/\psi$ is an analytic polynomial that has no zeroes in the closed unit disc, then (4.1) holds.*

Of course, this means that Theorem 4.2 applies to the pair φ, ψ if, in addition, they are outer functions.

Proof. We use the notation above and show that (4.5) holds. Let the zeroes of σ be z_1, \dots, z_m , all outside the closed unit disc, and suppose first that each zero is simple. Let $S \in \text{Poly}_n$. Since $(\sigma S)(z_j) = 0$, we have

$$\sum_{l=1}^m a_{n+l} \mathbf{p}_{n+l}(z_j) = - \sum_{k=0}^n a_k \mathbf{p}_k(z_j)$$

for $1 \leq j \leq m$ in the notation of (4.4). Write these equations as

$$\sum_{l=1}^m a_{n+l} \tilde{\mathbf{p}}_{n+l}(z_j) / z_j^{n+1} = - \sum_{k=0}^n a_k \tilde{\mathbf{p}}_k(z_j) / z_j^{n+1} =: \zeta_j, \quad (4.6)$$

where

$$\tilde{\mathbf{p}}_k(z) := \mathbf{p}_k(z) \left(\overline{\Phi_w(\bar{z}^{-1})} \right)^{-1}.$$

Recall Szegő's asymptotics

$$\text{for all } \rho > 1, \quad \lim_{k \rightarrow \infty} z^{-k} \tilde{\mathbf{p}}_k(z) = 1 \text{ uniformly for } |z| > \rho \quad (4.7)$$

(see Grenander and Szegő (1984), p. 51). Because $|z_j| > 1$, it follows that

$$C := \sup_{j,n} \sum_{k=0}^n |\tilde{\mathbf{p}}_k(z_j) / z_j^{n+1}|^2 < \infty.$$

By the Cauchy-Schwarz inequality, we obtain that

$$|\zeta_j|^2 \leq C \sum_{k=0}^n |a_k|^2.$$

Let $M_n := [\tilde{\mathbf{p}}_{n+l}(z_j)/z_j^{n+1}]_{1 \leq j, l \leq m}$ be the matrix of coefficients in the system of equations (4.6) for a_{n+l} , considered as variables. If M_n is not singular, then

$$[a_{n+l}]_{1 \leq l \leq m} = M_n^{-1}[\zeta_j]_{1 \leq j \leq m}.$$

Now by (4.7), the matrix M_n tends to the Vandermonde matrix determined by z_1, \dots, z_m . Hence, for all large n , we have not only that M_n is nonsingular, but also that its inverse has ℓ^2 -norm bounded by some constant D that depends only on σ . Therefore, for all large n , we have

$$\sum_{l=0}^m |a_{n+l}|^2 \leq D^2 \sum_{j=1}^m |\zeta_j|^2 \leq CD^2 m \sum_{k=0}^n |a_k|^2.$$

This clearly implies (4.5) for $\epsilon := 1/(1 + CD^2 m)$ and finishes the proof for the case of simple zeroes.

Now suppose that the zero z_j of σ has multiplicity r_j , so that $s := \sum_{j=1}^m r_j$ is the degree of σ . Multiply (4.7) by z^p and take the r th derivative (Rudin (1987), Theorem 10.28, p. 214) to obtain that

$$\lim_{k \rightarrow \infty} \mathbf{q}_{k,p}^{(r)}(z) = z^{p-r} \prod_{t=0}^{r-1} (p-t) \quad \text{for } |z| > 1, \quad (4.8)$$

where

$$\mathbf{q}_{k,p}^{(r)}(z) := \left(\frac{d}{dz} \right)^r (z^{-k+p} \tilde{\mathbf{p}}_k(z)).$$

Since $z \mapsto z^{-n-1} \sigma(z) S(z) \left(\overline{\Phi_w(\bar{z}^{-1})} \right)^{-1}$ has a zero at z_j of order at least r_j , it follows that

$$\sum_{l=1}^s a_{n+l} \mathbf{q}_{n+l, l-1}^{(r)}(z_j) = - \sum_{k=0}^n a_k \mathbf{q}_{k, l-1}^{(r)}(z_j)$$

for $0 \leq r < r_j$ and $1 \leq j \leq m$. We may now follow the same reasoning as for the case of simple zeroes, but instead of finding a coefficient matrix tending to a Vandermonde matrix, we find instead a limit matrix that has r_j columns corresponding to each z_j , namely, for each $r = 0, 1, \dots, r_j - 1$, it has the column

$$\left[z_j^{l-1-r} \prod_{t=0}^{r-1} (l-1-t) \right]_{1 \leq l \leq s}.$$

Thus, by reasoning analogous to before, it suffices to establish that these columns form a nonsingular matrix. To do this, we show that there is no nontrivial linear relation among the rows. Indeed, if b_1, \dots, b_s are constants such that for $1 \leq j \leq m$ and $0 \leq r < r_j$,

$$\sum_{l=1}^s b_l z_j^{l-1-r} \prod_{t=0}^{r-1} (l-1-t) = 0,$$

then the polynomial $\sum_{l=1}^s b_l z^{l-1}$ has a zero at z_j of order at least r_j for each $1 \leq j \leq m$. But since this polynomial has degree at most $s - 1$, this implies that the polynomial is identically zero, i.e., all $b_l = 0$, as desired. \blacksquare

It would be interesting to have a good characterization of those φ, ψ such that (4.1) holds.

§5. Higher Dimensions.

The technology we use to replace the theory of Hardy spaces for higher dimensions was provided by Helson and Lowdenslager (1958), who proved the extension of Theorem 1.1 to higher dimensions. We review the relevant definitions and facts from their theory before giving our theorems, which extend Theorems 3.1 and 4.2. Fix a positive integer d and let λ be Lebesgue measure on $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$.

For $k \in \mathbb{Z}^d$ and $x \in \mathbb{T}^d$, let $e_k(x) := e^{2\pi i k \cdot x}$. For $f \in L^2(\mathbb{T}^d)$, write $\widehat{f}(k) := (f, e_k)_\lambda := \int_{\mathbb{T}^d} f \overline{e_k} d\lambda$. Let $S \subset \mathbb{Z}^d$ have the properties $S \cup (-S) = \mathbb{Z}^d \setminus \{\mathbf{0}\}$, $S \cap (-S) = \emptyset$, and $S + S \subset S$. The associated **ordering** of \mathbb{Z}^d is that where $k \prec l$ iff $l - k \in S$. For example, we could have $(k_1, k_2, \dots, k_d) \prec \mathbf{0}$ if $k_i < 0$ when i is the first index such that $k_i \neq 0$, which we call the **lexicographic ordering**. The replacement for the Hardy spaces $H^p(\mathbb{T})$ ($p \geq 1$) are the **Helson-Lowdenslager spaces**

$$\mathbf{HL}^p := \mathbf{HL}^p(\mathbb{T}^d, S) := \left\{ \varphi \in L^p(\mathbb{T}^d); \text{supp } \widehat{\varphi} \subset S \cup \{\mathbf{0}\} \right\}.$$

Let $P_{\mathbf{HL}^2} : L^2(\mathbb{T}^d) \rightarrow \mathbf{HL}^2$ be the orthogonal projection $\sum_{k \in \mathbb{Z}^d} a_k e_k \mapsto \sum_{k \in S \cup \{\mathbf{0}\}} a_k e_k$. For $0 \leq f \in L^1(\lambda)$ and any set $R \subseteq \mathbb{Z}^d$, let $[R]$ be the linear span of $\{e_k; k \in R\}$ and $[R]_f$ be its closure in $L^2(f)$. In place of outer functions, we use **spectral factors**, which are the functions $\varphi \in \mathbf{HL}^2$ with the properties

$$\widehat{\varphi}(\mathbf{0}) > 0 \tag{5.1}$$

and

$$1/\varphi \in [S \cup \{\mathbf{0}\}]_{|\varphi|^2}. \tag{5.2}$$

Helson and Lowdenslager (1958) show that for $0 \leq f \in L^1(\mathbb{T}^d)$, the condition $\mathbf{GM}(f) > 0$ is equivalent to the existence of a spectral factor φ such that $|\varphi|^2 = f$. (More precisely, they prove $\mathbf{GM}(f) > 0$ iff $\exists \varphi \in \mathbf{HL}^2$ satisfying (5.1) and $|\varphi|^2 = f$. Their proof shows that in this case, φ can be chosen so that also (5.2) holds.)

LEMMA 5.1. *Suppose that φ satisfies (5.2), $h \in \text{HL}^1$, and $h/\varphi \in L^2(\mathbb{T}^d)$. Then $h/\varphi \in \text{HL}^2$.*

Proof. Let $f := |\varphi|^2$. By (5.2), there exist trigonometric polynomials p_n with $\text{supp } \widehat{p_n} \subset S \cup \{\mathbf{0}\}$ and such that $p_n \rightarrow 1/\varphi$ in $L^2(f)$. Since $h/\varphi \in L^2(\mathbb{T}^d)$, we have $h/f \in L^2(f)$. Thus,

$$\begin{aligned} \widehat{h/\varphi}(k) &= \int \frac{h}{\varphi} \overline{e_k} d\lambda = \int \frac{h\overline{e_k}}{f} \frac{1}{\varphi} f d\lambda = (h\overline{e_k}/f, 1/\overline{\varphi})_f = \lim_{n \rightarrow \infty} (h\overline{e_k}/f, \overline{p_n})_f \\ &= \lim_{n \rightarrow \infty} \int h p_n \overline{e_k} d\lambda = \lim_{n \rightarrow \infty} \widehat{h p_n}(k). \end{aligned}$$

Since $h p_n \in \text{HL}^1$, we have $\widehat{h p_n}(k) = 0$ for $k \notin S$, whence $\widehat{h/\varphi}(k) = 0$ for $k \notin S$. That is, $h/\varphi \in \text{HL}^2$. \blacksquare

For $A \subseteq \mathbb{Z}^d$, let (A) denote the set of corresponding complex exponentials $\{e_k; k \in A\}$. For any two finite ordered sets of functions $\mathcal{F}, \mathcal{G} \subset L^2(\mu)$ of the same cardinality, let

$$(\mathcal{F}, \mathcal{G})_\mu := \det [(p, q)_\mu]_{p \in \mathcal{F}, q \in \mathcal{G}}.$$

Here, the ordering of the sets \mathcal{F}, \mathcal{G} is used to order the rows and columns of the matrix whose determinant appears in this equation. Also, we write $(p, q)_\mu := \int p \overline{q} d\mu$ even if μ is a complex measure. Write $\mathcal{F} \uplus \mathcal{G}$ for the set $\mathcal{F} \cup \mathcal{G}$ ordered by concatenating \mathcal{G} after \mathcal{F} .

We are now ready to state and prove our extension of Theorem 3.1.

THEOREM 5.2. *Let $w : \mathbb{T}^d \rightarrow [0, \infty)$ be measurable with $\text{GM}(w) > 0$. Let φ be a spectral factor for w . Given any two finite ordered sets of functions $\mathcal{F}, \mathcal{G} \subset L^2(w)$ of the same cardinality, let $\mathcal{F}' := \langle P_{\text{HL}^2}(f\overline{\varphi}); f \in \mathcal{F} \rangle$ and define \mathcal{G}' likewise. Let $S_n \subset -S$ be finite ordered sets increasing to $-S$. We have*

$$\lim_{n \rightarrow \infty} \frac{(\mathcal{F} \uplus (S_n), \mathcal{G} \uplus (S_n))_w}{((S_n), (S_n))_w} = (\mathcal{F}', \mathcal{G}')_\lambda. \quad (5.3)$$

The case $\mathcal{F} = \mathcal{G} = \langle \mathbf{1} \rangle$ is the theorem of Helson and Lowdenslager (1958). Actually, there are special considerations in that case that allow Helson and Lowdenslager to make the same conclusion when $w = [d\mu/d\lambda]$ and the matrix entries are given by inner products in $L^2(\mu)$.

Proof. Let P_n be the orthogonal projection onto $(S_n)^\perp$ in $L^2(w)$. Also, let P_∞ be the orthogonal projection of $L^2(w)$ onto

$$H_\infty := (-S)^\perp = \{g \in L^2(w); \forall n \in -S \ (g, e_n)_w = 0\}.$$

Define $\mathcal{F}'_n := \langle P_n f; f \in \mathcal{F} \rangle$ and likewise for \mathcal{G}'_n , \mathcal{F}'_∞ , and \mathcal{G}'_∞ . By row and column operations, we have

$$\frac{(\mathcal{F} \uplus (S_n), \mathcal{G} \uplus (S_n))_w}{((S_n), (S_n))_w} = (\mathcal{F}'_n, \mathcal{G}'_n)_w.$$

This therefore tends to $(\mathcal{F}'_\infty, \mathcal{G}'_\infty)_w$ (indeed, we have entry-wise convergence of the corresponding matrices). Now $g \in H_\infty$ iff $g \in L^2(w)$ and $\text{supp } \widehat{g\bar{w}} \subseteq S \cup \{\mathbf{0}\}$. Thus,

$$H_\infty = \{g \in L^2(w); gw \in \text{HL}^1\}.$$

Let $f \in L^2(w)$ and write $h := w \cdot P_\infty f$. Since $h/w \in H_\infty$, we have $h/w \in L^2(w)$ and $h \in \text{HL}^1$. From the first of these relations, we see that $h/\varphi \in L^2(\mathbb{T}^d)$. Also,

$$\forall g \in H_\infty \quad (h/w, g)_w = (f, g)_w. \quad (5.4)$$

For all $m \in S \cup \{\mathbf{0}\}$, we have $e_m \varphi \in \text{HL}^2 \subset \text{HL}^1$, so that $e_m/\bar{\varphi} \in H_\infty$. Therefore, (5.4) implies that

$$(h/w, e_m/\bar{\varphi})_w = (f, e_m/\bar{\varphi})_w,$$

or in other words, $(\widehat{h/\varphi})(m) = \widehat{f\bar{\varphi}}(m)$ for all $m \in S \cup \{\mathbf{0}\}$. Since $h/\varphi \in \text{HL}^2$ by Lemma 5.1, it follows that $h/\varphi = P_{\text{HL}^2}(f\bar{\varphi})$. Thus, we have proved that for all f , we have

$$P_\infty f = \frac{P_{\text{HL}^2}(f\bar{\varphi})}{\bar{\varphi}}. \quad (5.5)$$

This gives (5.3) since

$$(P_\infty f, P_\infty g)_w = \int P_\infty f \cdot \overline{P_\infty g} \cdot w \, d\lambda = \int P_{\text{HL}^2}(f\bar{\varphi}) \overline{P_{\text{HL}^2}(g\bar{\varphi})} \, d\lambda. \quad \blacksquare$$

Unlike in Theorem 3.1, the limit (5.3) is not necessarily 0 when $\text{GM}(w) = 0$. For example, let S be the lexicographic ordering on \mathbb{Z}^2 and $w(x_1, x_2)$ be a function that depends only on x_2 with $\text{GM}(w) = 0$. If $\mathcal{F} := \mathcal{G} := \{e_{(1,0)}\}$, then the left-hand side of (5.3) is equal to $\widehat{w}(\mathbf{0})$, which need not equal 0.

However, if the order is archimedean, such as if $S = \{k \in \mathbb{Z}^d; k \cdot x > 0\}$, where $x \in \mathbb{R}^d$ has at least two coordinates whose quotient is irrational, then the limit (5.3) is 0 when $\text{GM}(w) = 0$, as we now show. (All archimedean orders arise in this way; in fact, for a characterization of all orders, see Teh (1961), Zaiceva (1953), or Trevisan (1953).)

PROPOSITION 5.3. Let $w : \mathbb{T}^d \rightarrow [0, \infty)$ be measurable with $\text{GM}(w) = 0$. Suppose that the order induced by S is archimedean. Given any two finite ordered sets of functions $\mathcal{F}, \mathcal{G} \subset L^2(\mu)$ of the same cardinality, and any finite ordered sets $S_n \subset -S$ increasing to $-S$, we have

$$\lim_{n \rightarrow \infty} \frac{(\mathcal{F} \uplus (S_n), \mathcal{G} \uplus (S_n))_w}{((S_n), (S_n))_w} = 0.$$

Proof. It suffices to establish that $H_\infty = 0$ in the notation of the proof of Theorem 5.2. Now for all $g \in H_\infty$,

$$\text{GM}(|gw|)^2 = \text{GM}(|gw|^2) = \text{GM}(|g|^2w)\text{GM}(w) = 0$$

since $\text{GM}(|g|^2w) \leq \int |g|^2w d\lambda < \infty$ as $g \in L^2(w)$. Since $gw \in \text{HL}^1$, this means that $gw = \mathbf{0}$ by the main result of Arens (1957). Thus, $H_\infty = 0$. ■

REMARK 5.4. Suppose that $\text{GM}(w) > 0$ and that φ is a spectral factor for w . By (5.5), we have $wP_\infty \mathbf{1} = \varphi P_{\text{HL}^2}(\overline{\varphi}) = \overline{\widehat{\varphi}(\mathbf{0})}\varphi$, so that φ is uniquely determined by w . It is essentially by this formula that Helson and Lowdenslager (1958) proved the existence of a spectral factor. This method goes back to Szegő (1921).

The extension of Theorem 4.2 is relatively straightforward:

THEOREM 5.5. Let $S_n \subset -S$ be finite ordered sets increasing to $-S$. Let $\mu = \psi \overline{\varphi} \lambda$ for some pair of spectral factors φ, ψ that satisfy the condition

$$\liminf_{n \rightarrow \infty} \epsilon(\varphi \cdot [-S_n], \psi \cdot [-S_n]; L^2(\lambda)) > 0. \quad (5.6)$$

Given any two finite ordered sets of functions $\mathcal{F}, \mathcal{G} \subset L^2(|\varphi|^2 + |\psi|^2)$ of the same cardinality, let $\mathcal{F}' := \langle P_{\text{HL}^2}(f\overline{\varphi}); f \in \mathcal{F} \rangle$ and $\mathcal{G}' := \langle P_{\text{HL}^2}(g\overline{\psi}); g \in \mathcal{G} \rangle$. We have

$$\lim_{n \rightarrow \infty} \frac{(\mathcal{F} \uplus (S_n), \mathcal{G} \uplus (S_n))_\mu}{((S_n), (S_n))_\mu} = (\mathcal{F}', \mathcal{G}')_\lambda. \quad (5.7)$$

Proof. Let $H_n(\varphi) := \{\varphi e_k; k \in -S_n\}$. By virtue of (5.6), we have for $n \geq n_0$,

$$\overline{H_n(\varphi)} \cap \overline{H_n(\psi)}^\perp = 0,$$

and so

$$L^2(\mathbb{T}^d) = \overline{H_n(\varphi)} + \overline{H_n(\psi)}^\perp.$$

We claim that

$$H_n(\varphi) \uparrow \text{HL}_0^2 := \left(\overline{\text{HL}^2} \right)^\perp. \quad (5.8)$$

Indeed, if $g \in \text{HL}_0^2$ and $g \perp H_n(\varphi)$ for all $n \geq 0$, then $\widehat{g\bar{\varphi}}(k) = 0$ for all $k \in S$, i.e., $g\bar{\varphi} \in \overline{\text{HL}^1}$. By Lemma 5.1, we may divide by $\bar{\varphi}$ to obtain that $g \in \overline{\text{HL}^2} = (\text{HL}_0^2)^\perp$, so that $g = \mathbf{0}$. This proves (5.8).

Thus the projection along $\overline{H_n(\varphi)}$ to $\overline{H_n(\psi)}^\perp$ tends to the orthogonal projection $P_{\overline{\text{HL}_0^2}^\perp} = P_{\text{HL}^2}$.

Now $(f, g)_\mu = (\bar{\varphi}f, \bar{\psi}g)_\lambda$ for any $f, g \in L^2(|\varphi|^2 + |\psi|^2)$. Let \mathcal{F}_n be the image of \mathcal{F} under the projection along $\overline{H_n(\varphi)}$ to $\overline{H_n(\psi)}^\perp$. Let $\mathcal{G}'' := \{\bar{\psi}g; g \in \mathcal{G}\}$. Row operations show that for $n \geq n_0$,

$$\frac{(\mathcal{F} \uplus (S_n), \mathcal{G} \uplus (S_n))_\mu}{((S_n), (S_n))_\mu} = (\mathcal{F}_n, \mathcal{G}'')_\lambda.$$

Because of our assumption (5.6) and Lemma 4.1, the limit is $(\mathcal{F}', \mathcal{G}')_\lambda$, which is the same as $(\mathcal{F}', \mathcal{G}')_\lambda$. ■

§6. Appendix.

In order to prove Lemma 4.1, we first demonstrate the following lemma.

LEMMA 6.1. *Suppose that H is a Hilbert space, H_1 and K_1 are non-zero closed subspaces, $H = H_1 + K_1^\perp$, and $H_1 \cap K_1^\perp = 0$. Let $T : H \rightarrow K_1^\perp$ be the linear projection along H_1 and*

$$\epsilon := \epsilon(H_1, K_1).$$

Then

$$\|T\| \leq \frac{1}{2} + \sqrt{\frac{1}{2(1 - \sqrt{1 - \epsilon^2})} + \frac{1}{4}} \quad (6.1)$$

and

$$\epsilon \geq \frac{1}{\sqrt{1 + \|T\|^2}}. \quad (6.2)$$

Proof. Let $v \in H$ with $\|v\| = 1$. Write $w := Tv \in K_1^\perp$ and $u := v - w \in H_1$. Choose $y \in K_1$ such that $(u, y) \geq \epsilon\|u\|\|y\|$ and $\|y\| = \epsilon\|u\|$. We have

$$\begin{aligned} 1 &= \|v\|^2 = \|u + w\|^2 = \|u\|^2 + \|w\|^2 + 2\Re(u, w) \\ &= \|u\|^2 + \|w\|^2 + 2\Re(u - y, w) \geq \|u\|^2 + \|w\|^2 - 2\|u - y\|\|w\| \\ &= \|u\|^2 + \|w\|^2 - 2\|w\| [\|u\|^2 + \|y\|^2 - 2\Re(u, y)]^{1/2} \\ &\geq \|u\|^2 + \|w\|^2 - 2\|w\| [\|u\|^2 + \|y\|^2 - 2\epsilon\|u\|\|y\|]^{1/2} \end{aligned}$$

$$\begin{aligned}
&= \|u\|^2 + \|w\|^2 - 2\|w\|\|u\|\sqrt{1-\epsilon^2} \\
&= (\|u\| - \|w\|)^2 + 2\|w\|\|u\|(1 - \sqrt{1-\epsilon^2}) \\
&\geq 2\|w\|\|u\|(1 - \sqrt{1-\epsilon^2}) = 2\|w\|\|v - w\|(1 - \sqrt{1-\epsilon^2}) \\
&\geq 2\|w\|(\|w\| - 1)(1 - \sqrt{1-\epsilon^2}).
\end{aligned}$$

Simple algebra shows that this inequality implies

$$\|Tv\| = \|w\| \leq \frac{1}{2} + \sqrt{\frac{1}{2(1 - \sqrt{1-\epsilon^2})} + \frac{1}{4}}.$$

Since $\|v\| = 1$, this is equivalent to (6.1).

To prove (6.2), note that by definition of ϵ , we have that

$$\exists x \in H_1 \setminus \{0\} \quad \forall y \in K_1 \quad |(x, y)| \leq \epsilon \|x\| \|y\|. \quad (6.3)$$

Choose such an x with $\|x\| = 1$ and set $y := P_{K_1}x$, the orthogonal projection of x onto K_1 . Then $y \neq 0$ because $H_1 \cap K_1^\perp = 0$. Write $z := x - y \in K_1^\perp$. Since $y = x - z$ with $x \in H_1$ and $z \in K_1^\perp$, it follows that $Ty = -z$. By (6.3) applied to y , we have

$$\begin{aligned}
\|z\|^2 &= \|x - y\|^2 = 1 + \|y\|^2 - 2\Re(x, y) \\
&\geq 1 + \|y\|^2 - 2\epsilon \|y\| \\
&= 1 + (\|y\| - \epsilon)^2 - \epsilon^2 \\
&\geq 1 - \epsilon^2.
\end{aligned} \quad (6.4)$$

Furthermore, $\|z\|^2 = 1 - \|y\|^2$, whence comparison to (6.4) shows that $\|y\| \leq \epsilon$. Therefore,

$$\|T\| \geq \|Ty\|/\|y\| = \|z\|/\|y\| \geq \sqrt{1-\epsilon^2}/\epsilon,$$

which is equivalent to (6.2). ■

Proof of Lemma 4.1. For each n and any $v \in H_n$, we have $T_m v = 0 = T_\infty v$ for all $m \geq n$. Also, for each $v \in K_\infty^\perp$, we have $T_m v = v = T_\infty v$ for all m . Therefore, $T_n v \rightarrow T_\infty v$ for all v belonging to the dense set $\bigcup_n H_n + K_\infty^\perp$. It follows by continuity and the principle of uniform boundedness that $T_n \rightarrow T_\infty$ in the strong operator topology iff

$$\sup_n \|T_n\| < \infty. \quad (6.5)$$

If (4.2) holds, then (6.5) is a consequence of (6.1), while if (6.5) holds, then (4.2) is a consequence of (6.2). ■

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