Seventy Years of Rajchman Measures

By Russell Lyons

Dedicated to Jean-Pierre Kahane.

Abstract. Rajchman measures are those Borel measures on the circle (say) whose Fourier transform vanishes at infinity. Their study proper began with Rajchman, but attention to them can be said to have begun with Riemann's theorem on Fourier coefficients, later extended by Lebesgue. Most of the impetus for the study of Rajchman measures has been due to their importance for the question of uniqueness of trigonometric series. This motivation continues to the present day with the introduction of descriptive set theory into harmonic analysis. The last ten years have seen the resolution of several old questions, some from Rajchman himself. We give a historical survey of the relationship between Rajchman measures and their common null sets with a few of the most interesting proofs.

Sommaire. Les mesures de Rajchman sont les mesures boréliennes sur le cercle dont la transformée de Fourier s'annulle à l'infini. Leur étude proprement dite a commencé avec Rajchman, mais on peut dire qu'elles ont retenu l'attention des mathématiciens depuis le théorème de Riemann sur les coefficients de Fourier, étendu plus tard par Lebesgue. La raison majeure de l'étude des mesures de Rajchman a toujours éte leur importance pour la question de l'unicité des séries trigonométriques. Cette motivation persiste avec l'introduction de la théorie descriptive des ensembles en analyse harmonique. Les dix dernières années ont vu la résolution de plusieurs questions anciennes, parmi elles quelques-unes de Rajchman lui-même. Nous donnons ici une revue historique des rapports entre les mesures de Rajchman et les ensembles négligeables correspondants communs avec quelques-unes des preuves les plus intéressantes.

1991 Mathematics Subject Classification. Primary 43-03. Secondary 42-03, 01-02, 43A25, 43A10, 42A63, 43A46, 42A55

Key words and phrases. Rajchman, null sets, Baire category, Fourier coefficients.

This is a slightly expanded version of a talk given at the Conference in Honor of Jean-Pierre Kahane, June 1993.

§1. Early History.

In his *Habilitation* of 1854 in Göttingen, Riemann (1990, Chapter XII, §10) proved that given any periodic function which is integrable in Riemann's sense, its Fourier coefficients tend to zero at infinity. Lebesgue (1903) extended this to functions which are Lebesgue integrable. To phrase the Riemann-Lebesgue theorem in terms of measures, recall that given a finite Borel measure on the circle, μ, its Fourier transform is defined as

$$\widehat{\mu}(n) := \int_{\mathbf{T}} e^{-2\pi i n t}\, d\mu(t)\,.$$

Here, $\mathbf{T}$ denotes the circle, $\mathbf{R}/\mathbf{Z}$, or $[0,1)$. Let Lebesgue measure be λ. Then the Fourier transform of the measure $f\lambda$ is the same as the Fourier transform of f. Combining the Radon-Nikodym theorem with the theorem of Riemann-Lebesgue, we see that every measure μ which is absolutely continuous with respect to λ has its Fourier transform vanishing at infinity.

Define the class of measures

$$R := M_0(\mathbf{T}) := \{\mu \in M(\mathbf{T})\,;\ \lim_{|n|\to\infty} \widehat{\mu}(n) = 0\}\,,$$

where $M(\mathbf{T})$ denotes the set of all finite Borel measures on $\mathbf{T}$. This is the class of **Rajchman measures**, named after Rajchman because he was the first to begin their proper study. The Riemann-Lebesgue theorem thus says that all absolutely continuous[1] measures are Rajchman measures.

Menshov (1916) was the first to construct a singular Rajchman measure. His purpose, in fact, was to give an example of a set of Lebesgue measure zero that was also of multiplicity for trigonometric series. That such a set exists stunned the mathematical world of the time. While the study of sets of multiplicity and of uniqueness have always provided the primary motivation for the study of Rajchman measures, we shall not pursue that theme here (see Kechris and Louveau (1987) for a fairly current treatment of this subject).

Menshov's example of a singular Rajchman measure was a modification of the usual Cantor-Lebesgue middle-thirds measure, μ_{CL}. On the other hand, since for all n,

$$\widehat{\mu}_{\mathrm{CL}}(n) = \widehat{\mu}_{\mathrm{CL}}(3n)\,,$$

we see that the Cantor-Lebesgue measure is a continuous[2] measure that is *not* a Rajchman measure. However, and to our benefit, this was not observed by F. Riesz (1918), who,

---

[1] with respect to Lebesgue measure

[2] This means that the measure of every singleton is zero.

instead, constructed his famous Riesz products for precisely this purpose: that is, to provide an example of a continuous measure that is not a Rajchman measure. Riesz asked about the converse, i.e., whether every Rajchman measure is continuous. This was proved true by Neder (1920). Thus, if we denote the absolutely continuous measures by $L^1(\mathbf{T})$ and the continuous ones by $M_c(\mathbf{T})$, we may write

$$L^1(\mathbf{T}) \subsetneq R \subsetneq M_c(\mathbf{T})\,.$$

Of course, Wiener (1924) gave an exact characterization of continuous measures as those whose Fourier coefficients vanish at infinity in mean size:

$$\mu \in M_c(\mathbf{T}) \iff \lim_{|n|\to\infty} \frac{1}{2n+1} \sum_{|k|\le n} |\widehat{\mu}(k)| = 0\,.$$

Now both $L^1(\mathbf{T})$ and $M_c(\mathbf{T})$ are defined in terms of their common null sets: $L^1(\mathbf{T})$ consists of those measures which annihilate (give measure zero to) all sets of Lebesgue measure zero and $M_c(\mathbf{T})$ consists of those measures which annihilate all countable sets. The question naturally arises[3] whether there is not some class of sets intermediate between those of Lebesgue measure zero and the countable ones such that a measure is a Rajchman measure iff it annihilates all the sets in this class. This question will be a focusing element of the present survey.

In order to formulate this question and related ideas concisely, we introduce the following notation. For a class $\mathcal{E}$ of subsets of the circle, let $\mathcal{E}^\perp$ denote the set

$$\mathcal{E}^\perp := \{\mu \in M(\mathbf{T})\,;\ \forall E \in \mathcal{E} \quad \mu(E) = 0\}\,.$$

We want to know if there is a class $\mathcal{E}$ such that $R = \mathcal{E}^\perp$.

[3] As we shall see, although various answers were proposed to this question over the years, it seems that the question itself did not appear in print until Lyons (1983), who heard it from his Ph. D. advisor, Allen L. Shields.

§2. Conjectures.

The proper study of the class R began with Rajchman. He (1928, 1929) and his student Milicer-Grużewska (1928a) proved that the class R is a band, i.e., is closed under absolute continuity:

$$\nu \ll \mu \in R \quad \Longrightarrow \quad \nu \in R\,. \tag{2.1}$$

This is easy to prove: By, say, the Stone-Weierstrass theorem, the trigonometric polynomials are uniformly dense in the set of all continuous functions on $\mathbf{T}$. They, in turn, are norm-dense in $L^1(\mu)$. Therefore, the trigonometric polynomials are norm-dense in $L^1(\mu)$. By the Radon-Nikodym theorem, there is an $f \in L^1(\mu)$ such that $\nu = f\mu$. For any trigonometric polynomial P, it is easy to see that $P\mu \in R$, whence

$$\limsup_{|n|\to\infty} |\widehat{f\mu}(n)| = \limsup_{|n|\to\infty} \left|(f\mu - P\mu)\widehat{\ }(n)\right| \le \|f - P\|_{L^1(\mu)}\,.$$

Since the right-hand side can be made arbitrarily small, we conclude that $\nu \in R$.

It follows, in particular, that $\mu \in R \iff |\mu| \in R$. Of course, the property of being a band is necessary in order that there be some class $\mathcal{E}$ such that $R = \mathcal{E}^\perp$. Note, too, that this property generalizes the Riemann-Lebesgue theorem, which is the special case of (2.1) where $\mu = \lambda$.

Rajchman (1922) showed that the Cantor middle-thirds set is annihilated by every Rajchman measure. In fact, define[4] the class of **H-sets**:

$$H := \left\{ E \subset \mathbf{T}\,;\ \exists n_k \uparrow \infty \quad \overline{\bigcup_k n_k E} \ne \mathbf{T} \right\}\,.$$

Thus, Cantor's middle-thirds set satisfies the definition when $n_k = 3^k$. Rajchman actually showed the more general fact that every H-set is annihilated by every Rajchman measure, or, in other words,

$$R \subseteq H^\perp\,. \tag{2.2}$$

Define the class of $\mathcal{U}_0$**-sets**[5]

$$\mathcal{U}_0 := \{E \subset \mathbf{T}\,;\ \forall \mu \in R \quad \mu(E) = 0\}\,.$$

This is the class of common null sets for R. Then another way to write (2.2) is

$$H \subseteq \mathcal{U}_0\,.$$

[4] We use the notation $nE := \{nx\,;\ x \in E\}$. Here, multiplication is in $\mathbf{T}$, i.e., mod 1.

[5] These are known as sets of uniqueness in the wide sense.

Rajchman asked whether (2.2) could be strengthened to

$$R = H^{\perp} \tag{2.3}$$

(see Milicer-Grużewska (1928c), p. 167n and (1928b), pp. 158–159). Bari (1951, pp. 85–86) said that, indeed, this was a conjecture of Rajchman's. This is consistent with the fact that his student Milicer-Grużewska (1928b, c) tried to establish its truth. Another question from this period was whether every Borel $\mathcal{U}_0$-set was meager (i.e., first category).

Shreĭder (1948) (see also Bari 1951, pp. 85–86) claimed that Rajchman's conjecture (2.3) was true, but Shreĭder never published a proof. In fact, he later (1950) made another claim about characterizing R without mentioning this first claim. Namely, he defined a new class of sets, W-sets, and asserted that

$$R = W^{\perp}\,. \tag{2.4}$$

Shreĭder made some comments about the proof, but, again, never published a true proof.

W-sets are defined as follows. A sequence $\{x_k\} \subset \mathbf{T}$ is said to have an **asymptotic distribution** $\nu \in M(\mathbf{T})$ if

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \delta(x_k) = \nu$$

in the weak* topology $\sigma(M(\mathbf{T}), C(\mathbf{T}))$, where $\delta(x)$ denotes the Dirac point mass at x. In case $\nu = \lambda$, the sequence is said to be **uniformly distributed**. A Borel set $E \subset \mathbf{T}$ is called a **W-set** if there is an increasing sequence of integers $\{n_k\}$ such that for every $x \in E$, the sequence $\{n_k x\}$ has an asymptotic distribution but is not uniformly distributed.

Other types of sets were considered by Kahane and Salem (1964) and Kahane (1964). For example, recall that a number $x \in (0,1)$ is called **normal** in base 2 if in its base 2 expansion, every block of digits occurs as often as any other block of the same length; in other words, if $x = 0.x_1x_2x_3\cdots$ (base 2), $k \ge 1$, and $a_1, \ldots, a_k \in \{0,1\}$, then

$$\lim_{N\to\infty} \frac{1}{N} |\{n \in [1,N]\,;\; x_n = a_n,\; x_{n+1} = a_{n+1},\; \ldots,\; x_{n+k-1} = a_{n+k-1}\}| = 2^{-k}\,.$$

Borel (1909) showed[6] that with respect to Lebesgue measure, almost every number is normal. In other words, the set of nonnormal numbers base 2 has Lebesgue measure zero. Kahane and Salem (1964) and Kahane (1964) asked whether this extended to Rajchman measures, in other words, whether the set of nonnormal numbers base 2 is a $\mathcal{U}_0$-set. They also asked whether the more general nonnormal sets, called W*-sets, are $\mathcal{U}_0$-sets. It turns out that it is not hard to show that $(W^*)^{\perp} \subseteq R$ (Lyons 1983), so this question is equivalent to asking whether $R = (W^*)^{\perp}$.

[6] His proof was not complete, but this was the first statement of a strong law of large numbers.

It is well known that a number x is normal in base 2 iff $\{2^k x\}$ is uniformly distributed. This motivates the definition of W*-sets: A Borel set $E \subset \mathbf{T}$ is called a **W*-set** if there is an increasing sequence of integers $\{n_k\}$ such that for every $x \in E$, the sequence $\{n_k x\}$ is *not* uniformly distributed. Weyl (1916) showed the analogue of Borel's theorem, namely that every W*-set has Lebesgue measure zero.

Baker (1971, 1974) interpreted the Kahane-Salem question as a conjecture and attempted to establish it.

Let us also note that if there is any class $\mathcal{E}$ such that $R = \mathcal{E}^\perp$, then

$$R = \mathcal{U}_0^\perp . \tag{2.5}$$

To see this, note that $R \subseteq \mathcal{E}^\perp$ is equivalent to $\mathcal{E} \subseteq \mathcal{U}_0$, which, in turn, implies that $\mathcal{U}_0^\perp \subseteq \mathcal{E}^\perp$. Since by definition, $R \subseteq \mathcal{U}_0^\perp$, we get that if $\mathcal{E}^\perp = R$, then

$$R \subseteq \mathcal{U}_0^\perp \subseteq \mathcal{E}^\perp \subseteq R ,$$

as claimed. Thus, the weakest conjecture to make in this area would be that $R = \mathcal{U}_0^\perp$.

§3. Some Answers.

Some of the questions of the preceding section were answered by Lyons (1983); see also Lyons (1984, 1985, 1986, 1987). It turns out that the H-sets form too small a class to characterize R, so that Rajchman's conjecture (2.3) is not true. On the other hand, Shreĭder's claim (2.4) is correct, hence so is (2.5). However, the class W^* is too big to characterize R; in fact, even the set of nonnormal numbers base 2 do not form a $\mathcal{U}_0$-set. The details of these proofs cannot be entered into here, except to mention that Rajchman's conjecture was disproved by exhibiting a Riesz product that was not a Rajchman measure yet which annihilated all H-sets; and the set of nonnormal numbers base 2 was shown not to be a $\mathcal{U}_0$-set by exhibiting a Rajchman measure that was carried by this set.

The question of category was answered by Debs and Saint Raymond (1987): every Borel $\mathcal{U}_0$-set is indeed meager. The original proof of this remarkable fact relied on quite a bit of descriptive set theory. In total, it was intricate and sophisticated. However, Kechris and Louveau (1987), Chapter VIII, §3, and (1990) found a wonderfully simple proof which we present now. For this theorem, the important aspect of a $\mathcal{U}_0$-set being Borel is only that Borel sets have the property of Baire. Recall that E is said to have the **property of Baire** if there are an open set V and a meager set P such that $E = V \triangle P$. It is easy to check that the sets with the property of Baire form a σ-algebra and, hence, include all the Borel sets. We shall thus prove:

THEOREM 3.1. (DEBS AND SAINT RAYMOND) *Every $\mathcal{U}_0$-set with the property of Baire is meager.*

We shall see that Banach-space duality and the Baire category theorem take all the work out of proving this when appropriately utilized. Set

$$\mathsf{Prob}(\mathbf{T}) := \{\mu \in M(\mathbf{T})\,;\ \mu \geq 0,\ \|\mu\| = 1\}\,,$$

and

$$\mathcal{R} := R \cap \mathsf{Prob}(\mathbf{T})\,.$$

Now $\widehat{\mathcal{R}} := \{\widehat{\mu}\,;\ \mu \in \mathcal{R}\}$ is a subspace of $c_0(\mathbf{Z})$. The Banach-space dual of $c_0(\mathbf{Z})$ is isometrically isomorphic to $\ell^1(\mathbf{Z})$ which, in turn, is isometrically isomorphic to the space of functions with absolutely convergent Fourier series

$$A(\mathbf{T}) := \{f \in C(\mathbf{T})\,;\ \widehat{f} \in \ell^1(\mathbf{Z})\}\,.$$

The combination of these isomorphisms gives the duality relation

$$\int_{\mathbf{T}} f\, d\mu = \sum_{n \in \mathbf{Z}} \widehat{f}(n)\widehat{\mu}(-n)\,. \tag{3.1}$$

Also define

$$\mathcal{R}(E) := \{\mu \in \mathcal{R}\,;\ \mu(E) = 1\}\,,$$

for any set $E \subseteq \mathbf{T}$.

We begin with a simple observation:

LEMMA 3.2. *If K is a closed subset of $\mathbf{T}$, then $\widehat{\mathcal{R}(K)}$ is norm-closed in $c_0(\mathbf{Z})$.*

Proof. Let $\mu_n \in \mathcal{R}(K)$ and $S \in c_0(\mathbf{Z})$ be such that

$$\|\widehat{\mu}_n - S\| \to 0\,.$$

Then for every $k \in \mathbf{Z}$, $\widehat{\mu}_n(k) \to S(k)$. By Herglotz's theorem (Katznelson 1976, p. 38) or the Banach-Alaoglu theorem (applied to the unit ball of $M(\mathbf{T})$), it follows that there is a positive measure μ such that $\widehat{\mu} = S$. Since $\widehat{\mu}(0) = S(0) = 1$, we know that μ is, in fact, a probability measure. By (3.1), we have that $\int f\, d\mu = 0$ for all $f \in A(\mathbf{T})$ which vanish on K. Since $A(\mathbf{T})$ contains all sufficiently smooth functions, the set of $f \in A(\mathbf{T})$ vanishing on K is dense in $\{f \in C(\mathbf{T})\,;\ f = 0 \text{ on } K\}$. Therefore, μ is supported by K. ∎

The following key lemma transfers the topology of the circle to that of certain normed vector spaces.

LEMMA 3.3. *Let K be a closed arc in $\mathbf{T}$ and $V \subseteq K$ be open and dense in K. Then $\widehat{\mathcal{R}(V)}$ is a norm-dense G_δ in $\widehat{\mathcal{R}(K)}$.*

Proof. We first show that $\widehat{\mathcal{R}(V)}$ is dense in $\widehat{\mathcal{R}(K)}$. Since $\widehat{\mathcal{R}(V)}$ is convex, it suffices (by the Hahn-Banach theorem) to prove only weak density. But the weak topology on $\widehat{\mathcal{R}(K)}$, which is generated by integration of measures against functions in $A(\mathbf{T})$, is weaker than the weak* topology on $\mathcal{R}(K)$ induced by $M(\mathbf{T})$ being the dual of $C(\mathbf{T})$. Thus, it suffices to prove that $\mathcal{R}(V)$ is weak* dense in $\mathcal{R}(K)$ or, better, in $\mathsf{Prob}(\mathbf{T})$. Again by convexity, it suffices to prove that for every $x \in K$, the Dirac measure $\delta(x)$ lies in the weak* closure of $\mathcal{R}(V)$. Let λ_n be normalized Lebesgue measure cut to $(x - 1/n, x + 1/n) \cap V$. By the Riemann-Lebesgue theorem, $\lambda_n \in \mathcal{R}(V)$. Clearly the weak* limit of λ_n is $\delta(x)$. This completes the proof of density.

Now we show that $\widehat{\mathcal{R}(V)}$ is a G_δ in $\widehat{\mathcal{R}(K)}$. In fact, we can see that it is a countable intersection of weakly open sets:

$$\widehat{\mathcal{R}(V)} = \bigcap_{n\geq 1} \bigcup_{\substack{f\in A(\mathbf{T}) \\ 0\leq f\leq \mathbf{1}_V}} \left\{\widehat{\mu} \in \widehat{\mathcal{R}(K)}\,;\ \int f\,d\mu > 1 - \frac{1}{n}\right\}. \quad \blacksquare$$

Proof of Theorem 3.1. Let E be a set with the property of Baire and which is not meager. We want to show that $E \notin \mathcal{U}_0$, i.e., that $\mathcal{R}(E) \neq \emptyset$. Now E is comeager (the complement of a meager set) in some closed arc K of positive length. That is, there are sets V_n open and dense in K such that

$$E \supseteq \bigcap_n V_n\,.$$

Therefore,

$$\mathcal{R}(E) \supseteq \mathcal{R}\left(\bigcap_n V_n\right) = \bigcap_n \mathcal{R}(V_n)\,. \tag{3.2}$$

We may apply the Baire category theorem to $\widehat{\mathcal{R}(K)}$ by virtue of Lemma 3.2. On doing so, we get, because of Lemma 3.3, that the right-hand side of (3.2) is non-empty. ∎

This result combining measure, category, and Fourier coefficients is not only amazing in itself, but can be used to give several quick proofs of seemingly unrelated results. We shall give a few examples from Kechris and Louveau (1987), Chapter VIII, §3, and (1990).

THEOREM (MENSHOV). *There are singular Rajchman measures.*[7]

[7] We can also easily reach Menshov's goal of producing a set of Lebesgue measure zero that is of multiplicity for trigonometric series: Given a singular Rajchman measure, μ, let E be a set of Lebesgue measure zero that carries μ. Regularity of μ provides a closed subset $F \subseteq E$ of positive μ-measure. Since F carries a Rajchman measure, it is of multiplicity.

Proof. If not, then we would obviously have $\mathcal{U}_0$ being the class of sets of Lebesgue measure zero. But we know that these are not all meager. ∎

THEOREM (LYONS). *The set of nonnormal numbers base 2 is not a $\mathcal{U}_0$-set.*

Proof. It is well known that this set is comeager: Its complement is contained in the meager F_σ-set

$$\bigcup_{N\geq 1}\bigcap_{n\geq N}\left\{x = 0.x_1x_2\ldots(\text{base }2)\,;\ \left|\frac{1}{n}\sum_{k\leq n}x_k - \frac{1}{2}\right| \leq \frac{1}{4}\right\}.$$ ∎

Being of Hausdorff dimension zero, or even of Hausdorff measure zero for some gauge function, cannot force a set to be a $\mathcal{U}_0$-set:

THEOREM (IVASHEV-MUSATOV 1962). *If $h : (0,\infty) \to (0,\infty)$ is non-decreasing with $h(0^+) = 0$, then there is a closed set of Hausdorff h-measure zero that is not in $\mathcal{U}_0$.*

Proof. We first construct E, a dense G_δ of Hausdorff h-measure zero. Choose a countable dense set $\{x_n\} \subset \mathbf{T}$. Given $\epsilon > 0$, let $I_n(\epsilon)$ be non-empty open intervals about x_n such that $\sum_n h(|I_n(\epsilon)|) < \epsilon$. Then

$$E := \bigcap_{r\geq 1}\bigcup_n I_n\left(\frac{1}{r}\right)$$

works.

Now by Theorem 3.1, there is some $\mu \in \mathcal{R}(E)$. By regularity of μ, there is a closed subset $F \subseteq E$ of positive μ-measure. This set F satisfies all the conditions. ∎

§4. Locally Compact Groups.

This section may be skipped on first reading. The characterization (2.4) holds when $\mathbf{T}$ is replaced by any locally compact abelian group and when W-sets are given the proper definition (see Lyons 1983, 1985). The situation of nonabelian groups is considerably more complicated. There, one has several reasonable definitions of Rajchman measure and of W-set. Nevertheless, Blümlinger (1989, 1991) has proved appropriate analogues to (2.4) for locally compact groups. In this section, we shall show how to extend Theorem 3.1 of Debs and Saint Raymond to the setting of locally compact abelian groups.

Given a locally compact abelian group G, let $M(G)$ denote the regular Borel finite measures on G and $\mathsf{Prob}(G)$ denote the regular Borel probability measures on G. Define the probability **Rajchman measures**

$$\mathcal{R} := \{\mu \in \mathsf{Prob}(G)\,;\ \widehat{\mu} \in C_0(\widehat{G})\}\,,$$

where $C_0(\widehat{G})$ denotes the continuous functions on the dual group $\widehat{G}$ that vanish at infinity. The Riemann-Lebesgue theorem in this setting (Hewitt and Ross 1970, p. 81, Remark (28.42)) guarantees that

$$f\lambda_G \subseteq \mathcal{R}$$

for any $f \ge 0$ with $\int f\, d\lambda_G = 1$, where λ_G is a left Haar measure on G.

LEMMA 4.1. *If $K \subseteq V \subseteq G$ with K compact and V open, then there is a $\rho \in M(\widehat{G})$ such that $\mathbf{1}_K \le \widehat{\rho} \le \mathbf{1}_V$.*

Proof. Let U be a symmetric compact neighborhood of the identity in G such that $KU^2 \subseteq V$. Then

$$\begin{aligned}\mathbf{1}_{KU} * \mathbf{1}_U(x) &= \int \mathbf{1}_{KU}(y)\mathbf{1}_U(y^{-1}x)\, d\lambda_G(y)\\ &= \int \mathbf{1}_{KU}(y)\mathbf{1}_U(x^{-1}y)\, d\lambda_G(y) = \lambda_G(xU \cap KU)\,.\end{aligned}$$

Now $\lambda_G(xU \cap KU) \le \lambda_G(xU) = \lambda_G(U)$ and

$$\lambda_G(xU \cap KU) = \begin{cases} \lambda_G(xU) & \text{if } x \in K,\\ 0 & \text{if } x \notin KU^2 \end{cases} = \begin{cases} \lambda_G(U) & \text{if } x \in K,\\ 0 & \text{if } x \notin KU^2, \end{cases}$$

whence

$$\mathbf{1}_K \le \lambda_G(U)^{-1}\mathbf{1}_{KU} * \mathbf{1}_U \le \mathbf{1}_{KU^2} \le \mathbf{1}_V\,.$$

Furthermore, $\mathbf{1}_{KU}$ and $\mathbf{1}_U$ lie in $L^2(G)$ since K and U are compact, whence $\check{\mathbf{1}}_{KU}$ and $\check{\mathbf{1}}_U$ lie in $L^2(\widehat{G})$, where $\check{}$ indicates inverse Fourier transform. Thus, $\check{\mathbf{1}}_{KU}\check{\mathbf{1}}_U \in L^1(\widehat{G})$. This allows us to define

$$\rho := \lambda_G(U)^{-1}\check{\mathbf{1}}_{KU}\check{\mathbf{1}}_U\lambda_{\widehat{G}} \in M(\widehat{G})\,.$$

Then $\widehat{\rho} = \lambda_G(U)^{-1}\mathbf{1}_{KU} * \mathbf{1}_U$. ∎

COROLLARY 4.2. *For $\mu \in \mathsf{Prob}(G)$ and $V \subseteq G$ open,*

$$\mu(V) = \sup\left\{\int_G \widehat{\rho}\,d\mu\,;\ \rho \in M(\widehat{G}),\ 0 \le \widehat{\rho} \le \mathbf{1}_V\right\}.$$

Proof. This follows by regularity of μ and the lemma. ∎

LEMMA 4.3. *If $F \subseteq G$ is closed, then $\widehat{\mathcal{R}(F)}$ is closed in $C_0(\widehat{G})$.*

Proof. Let $\mu_n \in \mathcal{R}(F)$ and $S \in C_0(G)$ be such that

$$\|\widehat{\mu}_n - S\| \to 0\,.$$

It follows from the Banach-Alaoglu theorem (as in the proof of Lemma 3.2) that there is a $\mu \in \mathsf{Prob}(G)$ such that $S = \widehat{\mu}$. To see that $\mu(F) = 1$, let $V := G \setminus F$. If $\rho \in M(\widehat{G})$ is such that $0 \le \widehat{\rho} \le \mathbf{1}_V$, then by the bounded convergence theorem,

$$\int_G \widehat{\rho}\,d\mu = \int_{\widehat{G}} \widehat{\mu}\,d\rho = \lim_{n\to\infty}\int_{\widehat{G}} \widehat{\mu}_n\,d\rho = \lim_{n\to\infty}\int_G \widehat{\rho}\,d\mu_n = \lim_{n\to\infty} 0 = 0\,.$$

It follows from the preceding corollary that $\mu(V) = 0$, as desired. ∎

LEMMA 4.4. *If $V \subseteq G$ is open, then $\widehat{\mathcal{R}(V)}$ is a dense G_δ in $\widehat{\mathcal{R}\left(\overline{V}\right)}$.*

This follows just as for Lemma 3.3. (It also holds in greater generality than indicated: Namely, if $F \subseteq G$ is closed and $V \subseteq F$ is relatively open, then $\widehat{\mathcal{R}(V)}$ is a dense G_δ in $\widehat{\mathcal{R}(F)}$. But this requires additional argument.)

We now get our desired result:

THEOREM 4.5. *If $E \subseteq G$ has the property of Baire and is annihilated by every Rajchman measure, then E is meager.*

Proof. Write E as $E = V \triangle P$ with V open and P meager. If E is not meager, then $V \ne \emptyset$. Since E is comeager in $\overline{V}$, it follows that $\widehat{\mathcal{R}(E)} \ne \emptyset$, just as in the proof of Theorem 3.1. ∎

§5. Further Developments.

The results of Lyons mentioned in Section 3 developed afterwards in several directions. One naturally wonders whether it is possible to prove (2.5) directly, that is, without proving a stronger result about some special subclass of $\mathcal{U}_0$ such as W. Indeed, Louveau (Kechris and Louveau 1987, Theorem IX.1.2) was able to do this by establishing a general criterion for characterizing classes of measures by null sets. In order to describe this criterion, let X be a compact metric space, $M(X)$ be the space of finite Borel measures on X, the subset of probability measures being $\mathsf{Prob}(X)$, and $\mathcal{C} \subseteq M(X)$. Denote by $\mathcal{C}^\perp$ the class of common null sets of $\mathcal{C}$:

$$\mathcal{C}^\perp := \{E \subseteq X \,;\, \forall \mu \in \mathcal{C} \quad \mu(E) = 0\}\,.$$

We are interested in when $\mathcal{C} = (\mathcal{C}^\perp)^\perp$.

Now it is clearly the case that any class of measures of the form $\mathcal{E}^\perp$ is a norm-closed band. In order to see one more necessary condition on classes of the form $\mathcal{E}^\perp$, give $\mathsf{Prob}(X)$ the weak* topology and suppose that Λ is a weak* Borel probability measure *on* $\mathsf{Prob}(X)$. We call $\nu \in \mathsf{Prob}(X)$ the **barycenter** of Λ if for all $f \in C(X)$,

$$\int_X f\,d\nu = \int_{\mathsf{Prob}(X)} \int_X f\,d\mu\,d\Lambda(\mu)\,. \tag{5.1}$$

A class $\mathcal{C}$ is called **measure convex** if it contains the barycenter of every probability measure carried by $\mathcal{C}$. Now if (5.1) holds for all $f \in C(X)$, then it holds also for every bounded Borel-measurable function on X since the smallest class containing $C(X)$ and closed under bounded pointwise limits is this class of functions. In particular, we get

$$\nu(E) = \int_{\mathsf{Prob}(X)} \mu(E)\,d\Lambda(\mu)$$

for every Borel subset E of X. Thus, we see that every class of the form $\mathcal{E}^\perp$ is measure-convex. Louveau's result is a remarkable converse:

Theorem (Louveau). *Let $\mathcal{C} \subseteq M(X)$ be a norm-closed measure convex band. If $\mathcal{C} \cap \mathsf{Prob}(X)$ is weak* analytic in $\mathsf{Prob}(X)$, then $\mathcal{C} = \mathcal{C}^{\perp\perp}$.*

In particular, this holds for $\mathcal{R}$, which is weak* closed and, by the bounded convergence theorem, measure convex.

It would be interesting to know what definability conditions could replace the hypothesis of weak* analyticity in this theorem. As we shall see by an example, weak* analyticity is not necessary.

Louveau's theorem is a consequence of work of Mokobodzki.[7] We include a proof taken from Lyons (1988).

Proof. Of course, $\mathcal{C} \subseteq \mathcal{C}^{\perp\perp}$. In order to show the converse, it suffices to show that for $\mu \in \mathcal{C}^{\perp\perp}$ and $\mu \geq 0$, we have $\mu \in \mathcal{C}$. Now since $\mathcal{C}$ is a norm-closed convex band, we may decompose μ as $\mu = \mu_1 + \mu_2$, with $\mu_1 \in \mathcal{C}$ and $\mu_2 \perp \mathcal{C}$: simply let $\mu_1 := \mathbf{1}_E \mu$, where E is a Borel set such that $\mu(E) = \sup\{\mu(F)\,;\; F$ is Borel and $\mathbf{1}_F \mu \in \mathcal{C}\}$. Because $\mathcal{C} \cap \mathsf{Prob}(X)$ is analytic and measure convex, there is (Dellacherie and Meyer 1983, p. 191, Remark 37) a Borel set $E \subset X$ such that E carries every $\nu \in \mathcal{C}$ and $\mu_2(E) = 0$. Thus $X \setminus E \in \mathcal{C}^{\perp}$, whence $\mu(X \setminus E) = 0$, and so $\mu_2(X \setminus E) = 0$. Therefore $\mu_2 = 0$ and $\mu \in \mathcal{C}$. ∎

We have seen, in a sense, how far off W*-sets are from characterizing Rajchman measures: they are not even meager sets. How far off was Rajchman's conjecture (2.3)? One answer to this question was provided by Kechris and Lyons (1988), who introduced a natural transfinite hierarchy on $H^{\perp}$ dividing $H^{\perp}$ into ω_1 levels, where ω_1 is the first uncountable ordinal. Kechris and Lyons proved that there were indeed ω_1 non-empty levels in $H^{\perp}$, whereas the Rajchman measures were all at level 1. Furthermore, this hierarchy had such definability properties that it forced $H^{\perp} \cap \mathsf{Prob}(\mathbf{T})$ to be a weak* coanalytic *non-Borel* set in $\mathsf{Prob}(\mathbf{T})$. This quantifies how far Rajchman's conjecture was from the truth. Also, it shows that $H^{\perp}$ is an example of a set whose intersection with $\mathsf{Prob}(\mathbf{T})$ is not weak* analytic yet $H^{\perp} = (H^{\perp})^{\perp\perp}$.

Similar results were proved for Dirichlet sets by Host, Louveau, and Parreau (unpublished) and independently by Kechris and Lyons (1988). These results on H-sets and on Dirichlet sets were simultaneously generalized in a new direction by Sylvain Kahane[8] (1993). To define Dirichlet sets, let $\|x\| := |x - \mathbf{Z}|$ for $x \in \mathbf{T}$. A set $E \subset \mathbf{T}$ is called a **Dirichlet set** if there is an increasing sequence of integers $\{n_k\}$ such that $\lim_{k\to\infty} \sup\{\|n_k x\|\,;\; x \in E\} = 0$. (One could write this as $n_k E \to 0$.) Denote by D the class of Dirichlet sets. Define the **Dirichlet measures** as the class

$$\mathsf{Dir} := \left\{ \mu \in M(\mathbf{T})\,;\; \mu \geq 0,\ \exists E_k \in D \quad E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots \right.$$
$$\left. \text{and } \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \|\mu\| \right\}.$$

Note that Dirichlet sets are H-sets. Also note that if E is an H-set with I an arc in the complement of $n_k E$, then $E \subset n_k^{-1} I^c$, where $n_k^{-1} I^c$ denotes the inverse image of I^c

[7] A complete development of the necessary work of Mokobodzki, including the Choquet capacitability theorem, can be found in Kechris and Louveau (1987), Chapter IX.

[8] A cousin of Jean-Pierre Kahane.

under the map $x \mapsto n_k x$. This is the union of n_k equally spaced small arcs. We relax this to define the class of **$\mathbf{L}_0$-sets** as those which can be covered by a collection of well-spaced small arcs:

$$L_0 := \Big\{ E \subset \mathbf{T}\,;\ \exists \epsilon_n \downarrow 0\ \exists \alpha > 0\ \forall n\ \exists \text{ a finite collection } I_k^n \text{ of arcs such that}$$
$$E \subset \bigcup_k I_k^n \text{ and } \forall k \quad |I_k^n| \le \epsilon_n \text{ and } \forall l \ne k \quad d(I_k^n, I_l^n) \ge \alpha \epsilon_n \Big\}\,.$$

Thus $D \subset H \subset L_0$, so that

$$D^\perp \supset H^\perp \supset L_0^\perp\,. \tag{5.2}$$

Also, it is clear that no sum of two Dirichlet measures can annihilate all Dirichlet sets:

$$(\mathsf{Dir} + \mathsf{Dir}) \cap D^\perp = \emptyset\,. \tag{5.3}$$

S. Kahane's theorem is as follows.

THEOREM (S. KAHANE). $\mathsf{Dir} + \mathsf{Dir}$ *cannot be separated from* $L_0^\perp$ *by a weak* Borel set. In other words, if* $L_0^\perp \cap \mathsf{Prob}(\mathbf{T}) \subseteq \mathcal{C} \subseteq \mathsf{Prob}(\mathbf{T}) \setminus (\mathsf{Dir} + \mathsf{Dir})$, *then* $\mathcal{C}$ *is not weak* Borel in* $\mathsf{Prob}(\mathbf{T})$.

In view of (5.2) and (5.3), this theorem implies the results on H-sets and on Dirichlet sets, insofar as they relate to their annihilators not being Borel.

Another exceedingly interesting subclass of $\mathcal{U}_0$, the sets of uniqueness, was shown in an ingenious way by Kaufman (1992) to be too small to characterize Rajchman measures: $R \ne U^\perp$, where U denotes the class of sets of uniqueness.

REFERENCES

Baker, R. C. (1971). A diophantine problem on groups. III, *Proc. Cambridge Philos. Soc.* **70**, 31–47.

Baker, R. C. (1974). A diophantine problem on groups. IV, *Illinois J. Math.* **18**, 552–564.

Bari, N. (1951). The uniqueness problem of the representation of functions by a trigonometric series, *Amer. Math. Soc. Transl. no. 52*, 1–89. Supplement to "The uniqueness problem . . .," (in Russian), *Uspehi Matem. Nauk* (N. S.) **7**, no. 5 (51) (1952), 193–196.

Blümlinger, M. (1989). Rajchman measures on compact groups, *Math. Ann.* **284**, 55–62.

Blümlinger, M. (1991). Characterization of measures in the group C*-algebra of a locally compact group, *Math. Ann.* **289**, 393–402.

Borel, E. (1909). Les probabilités dénombrables et leurs applications arithmétiques, *Rend. Circ. Mat. Palermo* **27**, 247–271.

Debs, G. and Saint Raymond, J. (1987). Ensembles boréliens d'unicité et d'unicité au sens large, *Ann. Inst. Fourier* (Grenoble) **37**, 3, 217–239.

Dellacherie, C. and Meyer, P.-A. (1983). *Probabilités et Potentiel. Chapitres IX à XI.* Hermann, Paris.

Hewitt, E. and Ross, K. A. (1970). *Abstract Harmonic Analysis.* Vol. II. Springer, Berlin.

Ivashev-Musatov, O. S. (1962). M-sets and Hausdorff measure, *Soviet Math. Doklady* **3**, 213–216.

Kahane, J.-P. (1964). Sur les mauvaises répartitions modulo 1, *Ann. Inst. Fourier* (Grenoble) **14**, 2, 519–526.

Kahane, J.-P. and Salem, R. (1964). Distribution modulo 1 and sets of uniqueness, *Bull. Amer. Math. Soc.* **70**, 259–261.

Kahane, S. (1993). On the complexity of sums of Dirichlet measures, *Ann. Inst. Fourier* (Grenoble) **43**, 1, 111-123.

Katznelson, Y. (1976). *An Introduction to Harmonic Analysis.* 2nd corrected ed. Dover, New York.

Kaufman, R. (1992). M-sets and measures, *Ann. of Math.* **135**, 125–130.

Kechris, A. S. and Louveau, A. (1987). *Descriptive Set Theory and the Structure of Sets of Uniqueness.* London Math. Soc. Lecture Note Ser. 128, Cambridge Univ. Press, Cambridge.

Kechris, A. S. and Louveau, A. (1990). Covering theorems for uniqueness and extended uniqueness sets, *Colloq. Math.* **59**, 63–79.

Kechris, A. S. and Lyons, R. (1988). Ordinal rankings on measures annihilating thin sets, *Trans. Amer. Math. Soc.* **310**, 747–758.

Lebesgue, H. (1903). Sur les séries trigonométriques, *Ann. Sci. de l'École Normale (Paris)* (3) **20**, 453–485.

Lyons, R. (1983). *A Characterization of Measures Whose Fourier-Stieltjes Transforms Vanish at Infinity.* Ph. D. thesis, Univ. of Michigan.

Lyons, R. (1984). Characterizations of measures whose Fourier-Stieltjes transforms vanish at infinity, *Bull. Amer. Math. Soc.* (N.S.) **10**, 93–96.

Lyons, R. (1985). Fourier-Stieltjes coefficients and asymptotic distribution modulo 1, *Ann. of Math.* **122**, 155–170.

Lyons, R. (1986). The measure of non-normal sets, *Invent. Math.* **83**, 605–616.

Lyons, R. (1987). The size of some classes of thin sets, *Studia Math.* **86**, 59–78.

Lyons, R. (1988). Topologies on measure spaces and the Radon-Nikodym theorem, *Studia Math.* **91**, 125–129.

Menshov, D. E. (1916). Sur l'unicité du développement trigonométrique, *C. R. Acad. Sc. Paris*, Sér. A-B **163**, 433–436.

Milicer-Grużewska, H. (1928a). Sur les fonctions à variation bornée et à l'écart Hadamardien nul, *C. R. Soc. Sci. Varsovie [Sprawozdonia posiedzén Towarzystwa naukowego warszawskiego]* **21**, 67–78.

Milicer-Grużewska, H. (1928b). Une contribution à la question des variations sur les ensembles du type (H), *C. R. Soc. Varsovie* **21**, 157–165.

Milicer-Grużewska, H. (1928c). Sur la continuité de la variation, *C. R. Soc. Varsovie* **21**, 166–177.

Neder, L. (1920). Über die Fourierkoeffizienten der Funktionen von beschränkter Schwankung, *Math. Zeit.* **6**, 270–273.

Rajchman, A. (1922). Sur l'unicité du développement trigonométrique, *Fund. Math.* (Varsovie) **3**, 287–302; Rectification et addition à ma note "Sur l'unicité ...," *Fund. Math.* (Varsovie) **4** (1923), 366-367.

Rajchman, A. (1928). Sur une classe de fonctions à variation bornée, *C. R. Acad. Sci. Paris* **187**, 1026–1028.

Rajchman, A. (1929). Une classe de séries trigonométriques qui convergent presque partout vers zéro, *Math. Ann.* **101**, 686–700.

Riemann, B. (1990). *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachtrage : collected papers.* Springer, Berlin.

Riesz, F. (1918). Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung, *Math. Zeit.* **2**, 312–315.

Shreĭder, Yu. A. (1948). On the ring of functions of bounded variation, (in Russian), *Uspehi Mat. Nauk.*, t. III, **2** (24), 224–225.

Shreĭder, Yu. A. (1950). On the Fourier-Stieltjes coefficients of functions of bounded variation, (in Russian), *Dokl. Acad. Nauk SSSR* **74**, 663–664.

Weyl, H. (1916). Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77**, 313–352.

Wiener, N. (1924). The quadratic variation of a function and its Fourier coefficients, *J. of Math. and Phys.* **3**, 72–94.

Russell Lyons
Department of Mathematics
Indiana University
Bloomington, IN 47405-5701