

Phase Transitions on Nonamenable Graphs

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Abstract. We survey known results about phase transitions in various models of statistical physics when the underlying space is a nonamenable graph. Most attention is devoted to transitive graphs and trees.

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§1. Introduction.

We shall give a summary of some of the main results known about phase transitions on nonamenable graphs. All terms will be defined as needed beginning in Section 2. Among the graphs we consider, we pay special attention to transitive graphs and trees (regular or not), as these are the cases that arise most naturally. Both of these classes of graphs also have some feature that permits a satisfying analysis to be performed (or to be conjectured): transitive graphs look the same from each vertex, while trees lack cycles. Certain phenomena are known to occur for all transitive nonamenable graphs, others are conjectured to hold for all transitive nonamenable graphs, while still others depend on

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different aspects of the graph. The subject is quite rich because of the interplay between probabilistic models and geometry. In particular, there is a greater variety of probabilistic behavior possible on nonamenable graphs than on amenable graphs. The area is developing vigorously, but a great deal remains to be discovered. A number of parallels among different processes will be evident to the reader, and consequently, a number of questions will suggest themselves. We have, however, omitted all discussion of critical exponents.

The models we consider all involve a parameter. Changing the parameter leads to qualitative changes of behavior. When such a change occurs, we shall say that there is a **phase transition**. (Note: in some publications, a phase transition is said to occur for a *fixed* parameter value when there is more than one Gibbs measure at that value. By contrast, our term is not precisely defined.) There is usually at least one **critical value** for the parameter, i.e., a value separating two intervals of the parameter where there are different qualitative behaviors on each side of the critical value. For the most basic phase transitions, those that usually occur on amenable graphs, Häggström (2000) showed that a phase transition occurs simultaneously in all or none of the following models on any given graph, assuming only that the graph has bounded degree: bond percolation, site percolation, the Ising model, the Widom-Rowlinson model, and the beach model. However, what makes nonamenable graphs truly distinctive is often the presence of a second critical value that does not occur on amenable graphs. The extent to which such behavior is understood varies widely from model to model and from graph to graph.

There are various probabilistic characterizations known of nonamenability. The first such result was proved for the most basic probabilistic process, namely, random walk, in the thesis of Kesten (1959a, 1959b). He showed that a countable group Γ is amenable iff the spectral radius is 1 for some (or every) symmetric group-invariant random walk whose support generates Γ . The extension of Kesten's theorem to the setting of invariant random walks on transitive graphs involves unimodularity and has been studied by Soardi and Woess (1990), Salvatori (1992), and Saloff-Coste and Woess (1996). We shall return to random walks, now with a parameter, in Section 9.

Due to lack of time, we were unable to survey results concerning branching random walk, which has many similarities to results here and, indeed, has inspired many of them. We mention just one example: A group Γ is amenable iff for some (or every) symmetric group-invariant random walk with support generating Γ and for some (or every) tree T with branching number larger than 1, the associated T -indexed random walk on Γ is recurrent. In particular, this is the case for branching random walk corresponding to any Galton-Watson branching process with mean larger than 1. See Benjamini and Peres (1994) for definitions and a proof (which depends on Kesten's theorem above). This result inspired

Conjecture 3.8 by means of an intuitive analogy between the range of a branching random walk and an infinite percolation cluster; see the proof of Thm. 4 in Benjamini and Schramm (1996) for a direct relationship between branching random walk and percolation.

We now give a somewhat more detailed preview of some of the results to be surveyed. For ordinary Bernoulli percolation on transitive amenable graphs, it is well known that when there is a.s. an infinite cluster, then there is a.s. a unique infinite cluster. This is now known to fail in many cases of transitive nonamenable graphs, and has been conjectured to fail in all transitive nonamenable graphs. Moreover, it is known that the uniqueness and nonuniqueness phases, if not empty, determine single intervals of the parameter. This leads to the study of two critical parameters, the usual one at the top of the regime of nonexistence of infinite clusters and a possibly new one at the bottom of the uniqueness phase. It also leads to the study of the behavior of the infinite clusters when there are infinitely many and how they merge as the parameter is increased.

One of the important new tools for studying percolation is the Mass-Transport Principle and its use in invariant percolation. This provides some general results that allow one to manipulate the clusters of Bernoulli percolation in a rather flexible fashion. In particular, nonamenability turns out to be more of an asset than a liability, as it provides for new thresholds that are trivial in the amenable case.

The Ising model is one of a natural family of models that includes Bernoulli percolation. Additional complications, such as boundary conditions and the optional parameter of an external field, lead to questions that do not arise for Bernoulli percolation. Sometimes, they can be used to characterize exactly amenability. But the number of different phase transitions that are possible for Potts models and the related random cluster models is sufficiently great that it has so far precluded the kind of unified picture that is at least conjectured for percolation.

The contact process is now reasonably well understood on the euclidean lattices \mathbb{Z}^d . However, some fundamental results there are still not known in the more general setting of amenable transitive graphs. For example, analogous to the number of infinite clusters in Bernoulli percolation are the phases in the contact process of extinction, weak survival, and strong survival. It might be that weak survival is impossible iff the transitive graph is amenable. Some results in this direction are known.

When we consider trees, we are often able to calculate precisely many critical values for various processes, even for completely general trees without any regularity. In almost all instances, these critical values turn out to be functions of a single number associated to the tree, its average branching number.

§2. Background on Graphs.

The basic definitions of the terms pertaining to graphs are as follows. Let $G = (V, E)$ be an unoriented graph with vertex set V and symmetric edge set $E \subseteq V \times V$. We write edges as $[x, y]$. If x and y are the endpoints of an edge, we call them **adjacent** or **neighbors** and write $x \sim y$. All graphs are assumed without further comment to be connected, denumerable, and locally finite. The only exception is that random subgraphs of a given graph may well be disconnected. Given $K \subset V$, set $\partial_V K := \{y \notin K; \exists x \in K, x \sim y\}$ and $\partial_E K := \{[x, y] \in E; x \in K, y \notin K\}$. Define the **vertex-isoperimetric constant** of G by

$$\iota_V(G) := \inf \left\{ \frac{|\partial_V K|}{|K|}; K \subset V \text{ is finite and nonempty} \right\},$$

and let the **edge-isoperimetric constant** of G be

$$\iota_E(G) := \inf \left\{ \frac{|\partial_E K|}{|K|}; K \subset V \text{ is finite and nonempty} \right\}.$$

A graph G is called **amenable** if $\iota_E(G) = 0$. If G has bounded degree, then this is equivalent to $\iota_V(G) = 0$. An **automorphism** of G is a bijection of V that induces a bijection of E . The set of automorphisms of G forms a group denoted $\text{Aut}(G)$. We say that a group $\Gamma \subseteq \text{Aut}(G)$ is **transitive** or **acts transitively** if V has only one orbit under Γ , i.e., if for all $x, y \in V$, there is some $\gamma \in \Gamma$ such that $\gamma x = y$. We say that Γ is **quasi-transitive** if Γ splits V into finitely many orbits. We call the graph G itself **(quasi-)transitive** if $\text{Aut}(G)$ is. Most results concerning quasi-transitive graphs can be deduced from corresponding results for transitive graphs or can be deduced in a similar fashion but with some additional attention to details. For simplicity, we shall therefore ignore quasi-transitive graphs in the sequel. (The extension of results to quasi-transitive graphs is important, however. Not only do they arise naturally, but they are crucial to the study of planar transitive graphs.)

Let Γ be a finitely generated group and S a finite symmetric generating set for Γ . The (right) **Cayley graph** $G = G(\Gamma, S)$ of Γ is the graph with vertex set $V := \Gamma$ and edge set $E := \{[v, vs]; v \in \Gamma, s \in S\}$. Note that Γ acts transitively on G by the **translations** $\gamma : x \mapsto \gamma x$.

A **tree** is a graph without cycles or loops. A branching process with one initial progenitor gives rise naturally to a random tree, its genealogical tree. When the branching process is a Galton-Watson process, we call the resulting tree a **Galton-Watson tree**.

We now review the modular function. Each compact group has a unique left-invariant Radon probability measure, called Haar measure. It is also the unique right-invariant

Radon probability measure. A locally compact group Γ has a left-invariant σ -finite Radon measure $|\bullet|$; it is unique up to a multiplicative constant. For every $\gamma \in \Gamma$, the measure $A \mapsto |A\gamma|$ is left invariant, whence there is a positive number $m(\gamma)$ such that $|A\gamma| = m(\gamma)|A|$ for all measurable A . The map $\gamma \mapsto m(\gamma)$ is a homomorphism from Γ to the multiplicative group of the positive reals and is called the **modular function** of Γ . If $m(\gamma) = 1$ for every $\gamma \in \Gamma$, then Γ is called **unimodular**. In particular, this is the case if Γ is countable, where Haar measure is counting measure. See, e.g., Royden (1988) for more on Haar measure.

We give the automorphism group $\text{Aut}(G)$ of a graph G the topology of pointwise convergence. By Corollary 6.2 of Benjamini, Lyons, Peres, and Schramm (1999b), if there is a transitive unimodular closed subgroup of $\text{Aut}(G)$, then $\text{Aut}(G)$ is also unimodular. In particular, this is the case if G is the Cayley graph of a group Γ . For this reason and for simplicity, we shall not generally consider subgroups of $\text{Aut}(G)$. However, the reader may wish instead to concentrate on translation-invariant measures on Cayley graphs, i.e., on the subgroup Γ of automorphisms of a Cayley graph G of Γ . We call a graph G **unimodular** if $\text{Aut}(G)$ is.

The stabilizer

$$S(x) := \{\gamma \in \text{Aut}(G) ; \gamma x = x\}$$

of any vertex x is compact and so has finite Haar measure. Note that if $\gamma u = y$, then $S(y) = \gamma S(u) \gamma^{-1}$, whence

$$|S(y)| = |S(u) \gamma^{-1}| = m(\gamma)^{-1} |S(u)|.$$

Thus, G is unimodular iff for all x and y in the same orbit, $|S(x)| = |S(y)|$. In particular, if G is transitive, then G is unimodular iff $|S(x)| = |S(y)|$ for all neighbors x and y .

Unimodularity of $\text{Aut}(G)$ is a simple and natural combinatorial property, as shown by Schlichting (1979) and Trofimov (1985). Namely, if $|\bullet|$ denotes cardinality (for subsets of G) as well as Haar measure (for subsets of $\text{Aut}(G)$), then for any vertices $x, y \in G$,

$$|S(x)y|/|S(y)x| = |S(x)|/|S(y)|;$$

thus, G is unimodular iff for all x and y in the same orbit,

$$|S(x)y| = |S(y)x|. \tag{2.1}$$

If G is transitive, then G is unimodular iff (2.1) holds for all neighbors x, y .

An **end** of a graph G is an equivalence class of infinite nonself-intersecting paths in G , with two paths equivalent if for all finite $A \subset G$, the paths are eventually in the same connected component of $G \setminus A$.

EXAMPLE 2.1. Let G be the regular tree of degree 3. Fix an end ξ of G and let Γ be the set of automorphisms preserving ξ . Then Γ is a closed transitive subgroup of $\text{Aut}(G)$ that is not unimodular. For an example of a transitive graph G whose full automorphism group is not unimodular, add to the above tree, for each vertex x , the edge between x and its ξ -grandparent. These examples were described by Trofimov (1985).

Next, we review amenability. Let Γ be any locally compact group and $L^\infty(\Gamma)$ be the Banach space of measurable real-valued functions on Γ that are essentially bounded with respect to Haar measure. A linear functional on $L^\infty(\Gamma)$ is called a **mean** if it maps the constant function $\mathbf{1}$ to the number 1 and nonnegative functions to nonnegative numbers. If $f \in L^\infty(\Gamma)$ and $\gamma \in \Gamma$, we write $L_\gamma f(h) := f(\gamma h)$. We call a mean μ **invariant** if $\mu(L_\gamma f) = \mu(f)$ for all $f \in L^\infty(\Gamma)$ and $\gamma \in \Gamma$. Finally, we say that Γ is **amenable** if there is an invariant mean on $L^\infty(\Gamma)$. Følner (see Paterson (1988), Theorem 4.13) showed that Γ is amenable iff for every nonempty compact $B \subset \Gamma$ and $\epsilon > 0$, there is a nonempty compact set $A \subset \Gamma$ such that $|BA \Delta A| \leq \epsilon|A|$. In this case, one often refers informally to A as a Følner set.

Now let $G = (V, E)$ be a graph. Given a set $K \subseteq V$, let

$$|K|_* := \sum_{x \in K} |S(x)|.$$

Note that $|\cdot|_*$ is just counting measure if G is unimodular and Haar measure is normalized so that $|S(o)| = 1$. Say that a transitive graph G is **$|\cdot|_*$ -amenable** if for all $\epsilon > 0$, there is a finite $K \subset V$ such that $|\partial_V K|_* < \epsilon|K|_*$. If G is unimodular, then this concept is the same as amenability of G . A mean on $\ell^\infty(V)$ is called **invariant** if every $f \in \ell^\infty(V)$ has the same mean as does $L_\gamma f$ (defined as the function taking $x \mapsto f(\gamma x)$) for every $\gamma \in \text{Aut}(G)$.

For automorphism groups of graphs, amenability has the following interpretations:

THEOREM 2.2. (BENJAMINI, LYONS, PERES, AND SCHRAMM 1999B) *Let G be a transitive graph. The following are equivalent:*

- (i) $\text{Aut}(G)$ is amenable;
- (ii) G has an invariant mean;
- (iii) G is $|\cdot|_*$ -amenable.

THEOREM 2.3. (SOARDI AND WOESS 1990) *Let G be a transitive graph. Then G is amenable iff $\text{Aut}(G)$ is amenable and unimodular.*

As usual, for any set A , we write 2^A for $\{0, 1\}^A$ and identify it with the collection of subsets of A . It is given the usual product topology and Borel σ -field.

Let o be a fixed vertex in G . In case G is a tree, then o will always designate the **root** of G . We denote by $|x|$ the graph distance between o and x in G for $x \in V$. Let $B(x, n)$ denote the set of vertices in G within distance n of x . Write

$$\text{gr}(G) := \liminf_{n \rightarrow \infty} |B(o, n)|^{1/n}$$

for the (lower exponential) **growth rate** of G . When G is transitive, we could replace \liminf by \lim because

$$|B(o, m+n)| \leq |B(o, m)| \cdot |B(o, n)|.$$

For any graph G , the fact that $\iota_V(G) \leq |\partial_V B(o, n)|/|B(o, n)|$ for each n implies that $1 + \iota_V(G) \leq \text{gr}(G)$. In particular, if G is nonamenable of bounded degree, then $\text{gr}(G) > 1$.

We shall sometimes have processes indexed by elements of a graph as well as by time. In order to distinguish between invariance under graph automorphisms and under time, we shall reserve the term **invariant** for the former and use **stationary** for the latter.

§3. Bernoulli Percolation on Transitive Graphs.

In **Bernoulli(p) bond percolation** on a graph, each edge is **open** (or **occupied** or **retained**) with probability p independently. Those edges that are not open are **closed** (or **vacant** or **removed**). The corresponding product measure on 2^E is denoted \mathbf{P}_p . The **percolation subgraph** is the random graph whose vertices are V and whose edges are the open edges. Let $K(x)$ be the **cluster** of x , that is, the connected component of x in the percolation subgraph. We write

$$\theta_x(p) := \mathbf{P}_p[K(x) \text{ is infinite}].$$

On a transitive graph, the value of $\theta_x(p)$ is independent of the choice of x , whence the subscript x is dropped. The event that $K(x)$ is infinite is often written $x \leftrightarrow \infty$. We also write $x \leftrightarrow y$ for $y \in K(x)$ and

$$\tau_p(x, y) := \mathbf{P}_p[x \leftrightarrow y].$$

Let

$$p_c := p_c(G) := \inf \{p; \theta(p) > 0\}$$

be the critical probability for percolation.

Bernoulli site percolation is defined similarly with vertices replacing edges. We shall use the superscripts “bond” and “site” when needed to distinguish the two models. See Grimmett (1999) for more information about Bernoulli percolation.

If G is a regular tree, then $K(o)$ is a Galton-Watson tree (except for the first generation), so its analysis is easy and well known. The first analysis of percolation on a nonamenable graph that is not a tree was carried out by Grimmett and Newman (1990) on the cartesian product of the integers and a regular tree of sufficiently high degree. They proved that for some $p > p_c$, multiple infinite clusters coexist, while for other p , there is a unique infinite cluster. As a consequence of a method for studying random walks, Lyons (1995) gave a threshold for Bernoulli percolation on transitive graphs of exponential growth (Theorem 3.1 below). There followed the paper of Benjamini and Schramm (1996), which has spawned a considerable amount of continuing research.

The results that follow are valid for both bond and site percolation when not otherwise stated. The only relations we shall state between site and bond percolation follow from the usual coupling of the two processes (see, e.g., Grimmett and Stacey (1998) for the coupling): $p_c^{\text{bond}}(G) \leq p_c^{\text{site}}(G)$ for every graph G , with strict inequality for most transitive G proved by Grimmett and Stacey (1998); and $p_u^{\text{bond}}(G) \leq p_u^{\text{site}}(G)$ for transitive G , where p_u is defined in (3.1) and Theorem 3.7(i) is being used to establish the inequality.

It is well known that if G is any infinite graph with the degree of each vertex at most d , then $p_c(G) \geq 1/(d-1)$. In the other direction, Lyons (1995) observed:

THEOREM 3.1. *If G is any transitive graph, then $p_c(G) \leq 1/\text{gr}(G)$.*

This also follows immediately from

THEOREM 3.2. (AIZENMAN AND BARSKY 1987) *If G is any transitive graph and $p < p_c(G)$, then $\mathbf{E}_p[|K(o)|] < \infty$.*

[Aizenman and Barsky (1987) worked only on \mathbb{Z}^d , but their proof works in greater generality.]

In particular, if G is nonamenable and transitive, then it has exponential growth, so that $0 < p_c(G) < 1$. The fact that $p_c(G) < 1$ was extended to nonamenable nontransitive graphs by Benjamini and Schramm (1996), who showed

THEOREM 3.3. *For any graph G , we have $p_c^{\text{bond}}(G) \leq 1/(1 + \iota_E(G))$ and $p_c^{\text{site}}(G) \leq 1/(1 + \iota_V(G))$.*

REMARK 3.4. The proof of Theorem 3.3 actually gives better bounds, with $\iota_E(G)$ and $\iota_V(G)$ replaced by

$$\iota_E^*(G) := \lim_{n \rightarrow \infty} \inf \left\{ \frac{|\partial_E K|}{|K|}; o \in K \subset V, K \text{ is connected, } n \leq |K| < \infty \right\}$$

and

$$\iota_V^*(G) := \lim_{n \rightarrow \infty} \inf \left\{ \frac{|\partial_V K|}{|K|}; o \in K \subset V, K \text{ is connected, } n \leq |K| < \infty \right\},$$

respectively, the **anchored expansion constants** introduced in Benjamini, Lyons, and Schramm (1999).

The next question concerns the number of infinite clusters when there is at least one. When G is transitive, the argument of Newman and Schulman (1981) shows that for any p , the number of infinite clusters in Bernoulli(p) percolation is an a.s. constant, either 0, 1, or ∞ . As p increases from 0 to 1, this constant goes from 0 to ∞ to 1, possibly skipping ∞ , as was shown by Häggström and Peres (1999) in the unimodular case and by Schonmann (1999b) in general:

THEOREM 3.5. *Let G be a transitive graph. Let $p_1 < p_2$. If there is a unique infinite cluster \mathbf{P}_{p_1} -a.s., then there is a unique infinite cluster \mathbf{P}_{p_2} -a.s. Furthermore, in the standard coupling of Bernoulli percolation processes, if there exists an infinite cluster \mathbf{P}_{p_1} -a.s., then a.s. every infinite p_2 -cluster contains an infinite p_1 -cluster.*

Here, we refer to the **standard coupling** of Bernoulli(p) percolation for all p where, for bond percolation, say, each edge $e \in E$ is assigned an independent uniform $[0, 1]$ random variable $U(e)$ and the edges where $U(e) \leq p$ are retained for Bernoulli(p) percolation.

If we define

$$p_u(G) := \inf \{p; \text{there is a unique infinite cluster in Bernoulli}(p) \text{ percolation}\}, \quad (3.1)$$

then it follows from Theorem 3.5 that when G is transitive,

$$p_u(G) = \sup \{p; \text{there is not a unique infinite cluster in Bernoulli}(p) \text{ percolation}\}.$$

It is not hard to show that when G is a transitive graph with at least 3 ends, then $p_u(G) = 1$. Since nonamenable transitive graphs cannot have only two ends, the remaining cases fall under the following conjecture, suggested in a question of Benjamini and Schramm (1996):

CONJECTURE 3.6. *If G is a transitive nonamenable graph with one end, then $p_u(G) < 1$.*

This conjecture has been confirmed in the following cases:

- G is a Cayley graph of a finitely presented group (Babson and Benjamini 1999);
- G is planar (Lalley (1998) for site percolation on co-compact Fuchsian groups of genus at least 2, and Benjamini and Schramm (2000) for percolation in general; the full result can also be deduced from the argument of Babson and Benjamini (1999));

- G is the cartesian product of two infinite graphs (Häggström, Peres, and Schonmann 1999);
- G is a Cayley graph of a Kazhdan group, i.e., a group with Kazhdan's property T (Lyons and Schramm 2000).

Some additional information about the uniqueness phase is contained in the following theorem:

THEOREM 3.7. *Let G be a transitive graph.*

(i) (SCHONMANN 1999B)

$$p_u(G) = \inf\{p; \sup_R \inf_x \mathbf{P}_p[B(o, R) \leftrightarrow B(x, R)] = 1\}. \quad (3.2)$$

(ii) (LYONS AND SCHRAMM 2000) *If G is unimodular and $\inf_x \tau_p(o, x) > 0$, then there is a unique infinite cluster \mathbf{P}_p -a.s. Therefore,*

$$p_u(G) = \inf\{p; \inf_x \tau_p(o, x) > 0\}. \quad (3.3)$$

Equation (3.3) implies (3.2), but it is unknown whether (3.3) holds in the nonunimodular case.

As is well known, when G is amenable and transitive, there can never be infinitely many infinite clusters (Burton and Keane (1989) for \mathbb{Z}^d and Gandolfi, Keane, and Newman (1992) in general), whence $p_c(G) = p_u(G)$. Behavior that is truly different from the amenable case arises when there *are* infinitely many infinite clusters. This has been conjectured always to be the case on nonamenable transitive graphs for an interval of p :

CONJECTURE 3.8. (BENJAMINI AND SCHRAMM 1996) *If G is a transitive nonamenable graph, then $p_c(G) < p_u(G)$.*

This has been confirmed in certain cases:

- if G is the product of any transitive graph with a regular tree of sufficiently high degree (Grimmett and Newman (1990) when the transitive graph is \mathbb{Z} and Benjamini and Schramm (1996) in general);
- if G is planar (Lalley (1998) for site percolation on co-compact Fuchsian groups of genus at least 2, and Benjamini and Schramm (2000) for percolation in general);
- for bond percolation if $\iota_E(G)/d \geq 1/\sqrt{2}$ and for site percolation if $\iota_V(G)/d \geq 1/\sqrt{2}$, where d is the degree of G (Schonmann 2000); this implies the first bulleted case above.
- if G is any Cayley graph of a group of cost larger than 1. This includes, first, free groups of rank at least 2 and fundamental groups of compact surfaces of genus larger than

1. Second, let Γ_1 and Γ_2 be two groups of finite cost with Γ_1 having cost larger than 1. Then every amalgamation of Γ_1 and Γ_2 over an amenable group has cost larger than 1. Third, every HNN extension of Γ_1 over an amenable group has cost larger than 1. For the definition of cost and proofs that these groups have cost larger than 1, see Gaboriau (1998, 2000). The proof that $p_c(G) < p_u(G)$ follows fairly easily from Theorem 3.11 below.

The third bulleted case above uses the following lower bound for $p_u(G)$ [or the weaker bound of Benjamini and Schramm (1996), Theorem 4]. Here, a **simple cycle** is a cycle that does not use any vertex or edge more than once.

THEOREM 3.9. (SCHRAMM 1997) *Let G be a transitive graph and let $a_n(G)$ be the number of simple cycles of length n in G that contain o . Then*

$$p_u(G) \geq \liminf_{n \rightarrow \infty} a_n(G)^{-1/n}. \tag{3.4}$$

Proof. We give the proof for site percolation, the proof for bond percolation being similar. Let $U(x)$ be independent uniform $[0, 1]$ random variables indexed by V . Take $p > p' > p_u \geq p_c$. In order to show that $p_u(G) \geq \liminf_{n \rightarrow \infty} a_n(G)^{-1/n}$, we shall show that $\sum_n a_n(G)p^n = \infty$. Let ω be the open subgraph formed by the vertices x with $U(x) \leq p$. First, observe that since ω contains a.s. a unique infinite cluster, that infinite cluster K has only one end, since otherwise removing a finite number of sites would create more than one infinite cluster.

Second, with positive probability, there are two (edge- and vertex-) disjoint infinite rays in K . Otherwise, by Menger's theorem, for any vertex $x \in K$, a.s. there would be infinitely many vertices x_n , each of whose removal would leave x in a finite open component. But given ω , given any such vertex x , and given any such vertices x_n , $U(x_n) > p'$ a.s. for infinitely many n . This means that $K(x)$ is finite $\mathbf{P}_{p'}$ -a.s. and contradicts $p' > p_c$.

Therefore, with positive probability there are two infinite rays in ω starting at o that are disjoint except at o . Since K has only one end, the two rays may be connected by paths in ω that stay outside arbitrarily large balls. In particular, there are an infinite number of simple cycles in ω through o , whence the expected number of simple cycles through o in ω must be infinite. That is, $\sum_n a_n(G)p^n = \infty$. ■

Additional evidence for Conjecture 3.8 is provided by

THEOREM 3.10. (PAK AND SMIRNOVA-NAGNIBEDA 2000) *For any finitely generated non-amenable group Γ , there exists some Cayley graph G of Γ with $p_c(G) < p_u(G)$.*

The proof of Theorem 3.10 shows that the Cayley graph can be found so as to satisfy Schonmann's condition above that $\iota(G)/d \geq 1/\sqrt{2}$.

We next discuss behavior of percolation at the critical values p_c and p_u . It has been long conjectured that there are no infinite clusters at the critical value $p_c(G)$ when G is a euclidean lattice, i.e., $\theta(p_c(G)) = 0$. Benjamini and Schramm (1996) extended this conjecture to all transitive G . This was confirmed in the unimodular nonamenable case:

THEOREM 3.11. (BENJAMINI, LYONS, PERES, AND SCHRAMM (1999B, 1999A)) *If G is a unimodular nonamenable transitive graph, then $\theta(p_c(G)) = 0$.*

It follows from Theorems 3.5, 3.11, and a result of van den Berg and Keane (1984) that $\theta(p)$ is continuous in p on each nonamenable unimodular transitive G . (This was proved earlier by Wu (1997) for a graph that is not transitive but is similar to the hyperbolic plane.)

It is unknown how many infinite clusters there are at p_u . It is known that there is a unique infinite cluster at p_u when G is planar, nonamenable and transitive (Benjamini and Schramm 2000). On the other hand, there cannot be exactly one infinite cluster at p_u when G is a cartesian product (of infinite transitive graphs) with a nonamenable automorphism group (Schonmann (1999a) in the case of a tree cross \mathbb{Z} and Peres (2000) in general) or when G is a Cayley graph of a Kazhdan group (due to Peres; see Lyons and Schramm (2000)).

Finally, we discuss briefly the nature of the infinite clusters when there are infinitely many of them; see Benjamini, Lyons, and Schramm (1999) and Häggström, Schonmann, and Steif (2000) for more on this topic. A basic result is that when there are infinitely many infinite clusters, they are “indistinguishable” from each other:

THEOREM 3.12. *Let G be a transitive unimodular graph. Let \mathcal{A} be a Borel measurable set of subgraphs of G that is invariant under the automorphism group of G . Then either \mathbf{P}_p -a.s. all infinite clusters are in \mathcal{A} , or \mathbf{P}_p -a.s. they are all outside of \mathcal{A} .*

Theorem 3.12 was proved by Häggström and Peres (1999) for increasing sets \mathcal{A} and for all but possibly one value of p , while it was proved in general (and for certain other percolation processes) by Lyons and Schramm (2000).

For example, \mathcal{A} might be the collection of all transient subgraphs of G , or the collection of all subgraphs that have a given asymptotic rate of growth, or the collection of all subgraphs that have no vertex of degree 5.

If \mathcal{A} is the collection of all transient subgraphs of G , then Theorem 3.12 shows that almost surely, either all infinite clusters of ω are transient [meaning that simple random walk on them is transient], or all clusters are recurrent. In fact, Lyons and Schramm (2000) show that if G is nonamenable, then a.s. all infinite clusters are transient if Bernoulli percola-

tion produces more than one infinite component. (Benjamini, Lyons, and Schramm (1999) show that the same is true if Bernoulli percolation produces a single infinite component.)

We illustrate some uses of Theorem 3.12 by proving two theorems (though it should be noted that the original direct proofs of these theorems are simpler than the proof of Theorem 3.12). Theorem 3.12 is also used to prove Theorem 3.7(ii).

Proof of Theorem 3.5 in the unimodular case. Suppose that there exists an infinite cluster \mathbf{P}_{p_1} -a.s. Let ω be the open subgraph of the \mathbf{P}_{p_2} process and let η be an independent \mathbf{P}_{p_1/p_2} process. Thus, $\omega \cap \eta$ has the law of \mathbf{P}_{p_1} and, in fact, $(\omega \cap \eta, \omega)$ has the same law as the standard coupling of \mathbf{P}_{p_1} and \mathbf{P}_{p_2} . By assumption, $\omega \cap \eta$ has an infinite cluster a.s. Thus, for some cluster C of ω , we have $C \cap \eta$ is infinite with positive probability, hence, by Kolmogorov's 0-1 law, with probability 1. By Theorem 3.12, this holds for every cluster C of ω . ■

An extension to Theorem 3.5 in the unimodular case is as follows. It is unknown whether it holds in the nonunimodular case.

THEOREM 3.13. (HÄGGSTRÖM, PERES, AND SCHONMANN 1999) *Let G be a transitive unimodular graph. Let $p_1 < p_2$ be such that there are infinitely many infinite clusters \mathbf{P}_{p_1} -a.s. and \mathbf{P}_{p_2} -a.s. In the standard coupling of Bernoulli percolation processes on G , a.s. every infinite p_2 -cluster contains infinitely many infinite p_1 -clusters.*

Proof. (due to R. Schonmann) The number of infinite p_1 -clusters contained in a p_2 -cluster is a random variable whose distribution is the same for each infinite p_2 -cluster by Theorem 3.12. In fact, by an extension of Theorem 3.12 involving random scenery that is stated by Lyons and Schramm (2000) (and that has the same proof), this random variable is constant a.s. Thus, each infinite p_2 -cluster has the same number of infinite p_1 -clusters a.s. Since two infinite p_2 -clusters could merge through the addition of finitely many edges, the number of infinite p_1 -clusters contained in an infinite p_2 -cluster could change unless that number were infinite. ■

Theorem 3.12 does not hold for nonunimodular graphs (Lyons and Schramm 2000). However, Häggström, Peres, and Schonmann (1999) have found a replacement that does hold without the unimodularity assumption (as long as $p > p_c$; presumably, this caveat is not important since presumably there are no infinite clusters at p_c). Define \mathcal{A} to be **robust** if for every infinite connected subgraph C of G and every edge $e \in C$, we have $C \in \mathcal{A}$ iff there is an infinite connected component of $C \setminus \{e\}$ that lies in \mathcal{A} . For example, transience is a robust property.

THEOREM 3.14. (HÄGGSTRÖM, PERES, AND SCHONMANN 1999) *Let G be a transitive graph. Let $p > p_c(G)$ and let \mathcal{A} be a robust Borel measurable set of subgraphs of G . Assume that \mathcal{A} is invariant under the automorphism group of G . Then either \mathbf{P}_p -a.s. all infinite percolation components are in \mathcal{A} , or \mathbf{P}_p -a.s. they are all outside of \mathcal{A} .*

Finally, it should be noted that Benjamini and Schramm (1996) contains several interesting questions and conjectures about various families of graphs, including nontransitive and amenable graphs. One may consult Benjamini and Schramm (1999) for updates concerning progress on Bernoulli percolation on general graphs.

§4. Invariant Percolation on Transitive Graphs.

As we have mentioned in the introduction, there are interesting and useful results about invariant percolation, especially on transitive nonamenable graphs. We give a sample of these results here that show their nature and how they can be used. In addition, we illustrate the powerful mass-transport technique. All formally stated results in this section are from Benjamini, Lyons, Peres, and Schramm (1999b), which will be referred to simply as [BLPS99] throughout this section.

A **bond percolation process** is a pair (\mathbf{P}, ω) , where ω is a random element in 2^E and \mathbf{P} denotes the distribution (law) of ω . We shall say that ω is the **configuration** of the percolation. A **site percolation process** (\mathbf{P}, ω) is given by a probability measure \mathbf{P} on $2^{V(G)}$, while a (mixed) **percolation** is given by a probability measure on $2^{V(G) \cup E(G)}$ that is supported on subgraphs of G . If ω is a bond percolation process, then $\hat{\omega} := V(G) \cup \omega$ is the associated mixed percolation. In this case, we shall not distinguish between ω and $\hat{\omega}$, and think of ω as a subgraph of G . Similarly, if ω is a site percolation, there is an associated mixed percolation $\hat{\omega} := \omega \cup (E(G) \cap (\omega \times \omega))$, and we shall not bother to distinguish between ω and $\hat{\omega}$.

If $x \in V(G)$ and ω is a percolation on G , the **cluster** (or **component**) $K(x)$ of x in ω is the set of vertices in $V(G)$ that can be connected to x by paths contained in ω . We shall not distinguish between the cluster $K(x)$ and the graph $(K(x), (K(x) \times K(x)) \cap \omega)$ whose vertices are $K(x)$ and whose edges are the edges in ω with endpoints in $K(x)$.

A percolation process (\mathbf{P}, ω) in a graph G is called **invariant** if \mathbf{P} is invariant under $\text{Aut}(G)$. Invariant percolation has proved useful for the study of Bernoulli percolation as well as other processes such as the random cluster model, as we shall see below. It is also interesting in itself.

We first present the very useful Mass-Transport Principle. Early forms of the mass-transport method were used by Adams (1990) and van den Berg and Meester (1991). It

was introduced in the study of percolation by Häggström (1997) and developed further in [BLPS99]. Let ξ be some (automorphism-)invariant process on G , such as invariant percolation, and let $F(x, y; \xi) \in [0, \infty]$ be a function of $x, y \in V$ and ξ . Suppose that F is invariant under the diagonal action of $\text{Aut}(G)$; that is, $F(\gamma x, \gamma y; \gamma \xi) = F(x, y, \xi)$ for all $\gamma \in \text{Aut}(G)$. We think of giving each vertex $x \in V$ some initial mass, possibly depending on ξ , then redistributing it so that x sends y the mass $F(x, y; \xi)$. With this terminology, one hopes for “conservation” of mass, at least in expectation. Of course, the total amount of mass is usually infinite. Nevertheless, there is a sense in which mass is conserved; in the transitive unimodular setting, we have that the expected mass at a vertex before transport equals the expected mass at a vertex afterwards. More generally, mass needs to be weighted according to the Haar measure of the stabilizer. Since F enters only in expectation, it is convenient to set $f(x, y) := \mathbf{E}F(x, y; \xi)$. For the reader to whom this is new, it is recommended to consider only the unimodular case; then all factors of $|S(x)|$ become 1 and all $*$ ’s below can be omitted.

MASS-TRANSPORT PRINCIPLE. *If G is a transitive graph and $f : G \times G \rightarrow [0, \infty]$ is invariant under the diagonal action of $\text{Aut}(G)$, then*

$$\sum_{x \in V} f(o, x) = \sum_{x \in V} f(x, o) |S(x)| / |S(o)|.$$

For a subgraph $K \subset G$, let $\text{deg}_K(x)$ denote the degree of x in K . If K is finite and nonempty, put

$$\alpha_K^* := \frac{1}{|K|_*} \sum_{x \in K} \text{deg}_K(x) |S(x)|;$$

this is the average (internal) degree in K , appropriately weighted if the graph is not unimodular. Then define

$$\alpha^*(G) := \sup\{\alpha_K^* ; K \subset G \text{ is finite and nonempty}\}.$$

If G is a regular graph of degree d , then

$$\alpha^*(G) + \iota_E^*(G) = d, \tag{4.1}$$

where

$$\iota_E^*(G) := \inf \left\{ \frac{1}{|K|_*} \sum_{[x,y] \in \partial_E K} |S(x)| ; K \subset V \text{ is finite and nonempty} \right\}.$$

For a random subgraph ω of G and a vertex $x \in G$, define

$$D^*(x) := \sum_{[x,y] \in \omega} |S(y)|/|S(x)|.$$

Let

$$d^* := \sum_{[o,y] \in G} |S(y)|/|S(o)|.$$

We give two simple but useful applications of the Mass-Transport Principle to illustrate the method. The first is quantitative, while the second is qualitative. Both were proved earlier by Häggström (1997) for regular trees. (His paper was the original impetus for [BLPS99].) Write

$$\theta(\mathbf{P}) := \mathbf{P}[o \leftrightarrow \infty]. \tag{4.2}$$

THEOREM 4.1. *Let G be a nonamenable transitive graph and \mathbf{P} be an invariant bond percolation on G . Then*

$$\theta(\mathbf{P}) \geq [\mathbf{E}D^*(o) - \alpha^*(G)]/\iota_E^*(G). \tag{4.3}$$

In particular, if $\mathbf{E}D^(o) > \alpha^*(G)$, then $\theta(\mathbf{P}) > 0$.*

The intuition is that if the expected (weighted) degree of a vertex is larger than the average internal degree of finite subgraphs, then it must be carried by some infinite components.

Proof. Let I_x be the indicator that $K(x)$ is finite. We put mass $D^*(x)I_x$ at each $x \in V$. In each finite component, the masses are redistributed proportionally to the weights $|S(y)|$ (for y in the component) among the vertices in that component. Since \mathbf{P} is invariant, so is this mass transport. Formally, we use the function

$$f(x, y) := \mathbf{E} \left[I_x \mathbf{1}_{\{y \in K(x)\}} \frac{D^*(x)|S(y)|}{|K(x)|_*} \right],$$

which is automorphism invariant. We have

$$\sum_{z \in V} f(o, z) = \mathbf{E}[D^*(o)I_o].$$

On the other hand,

$$\begin{aligned} \sum_{y \in V} f(y, o)|S(y)|/|S(o)| &= \mathbf{E} \left[I_o \sum_{y \in K(o)} \frac{D^*(y)|S(o)|}{|K(o)|_*} \frac{|S(y)|}{|S(o)|} \right] \\ &= \mathbf{E}[\alpha_{K(o)}^* I_o] \leq \alpha^*(G)(1 - \theta(\mathbf{P})). \end{aligned}$$

Since $D^* \leq d$ everywhere, the Mass-Transport Principle implies that

$$\begin{aligned} \mathbf{E}D^*(o) - d\theta(\mathbf{P}) &\leq \mathbf{E}[D^*(o)I_o] = \sum_{z \in V} f(o, z) = \sum_{y \in V} f(y, o) |S(y)| / |S(o)| \\ &\leq \alpha^*(G)(1 - \theta(\mathbf{P})). \end{aligned}$$

A little algebra using (4.1) completes the proof. ■

Variations on this result have proved useful. For example [BLPS99], if in addition to the above hypotheses, G is unimodular, \mathbf{P} has the property that all components are trees a.s., and $\mathbf{E}D^*(o) \geq 2$, then $\theta(\mathbf{P}) > 0$.

Our second application of the Mass-Transport Principle helps us to count the ends of the components in the configuration of a percolation that is invariant under a unimodular automorphism group:

PROPOSITION 4.2. *Let G be a unimodular transitive graph. Let ω be the configuration of an invariant percolation on G such that ω has infinite components with positive probability. Almost surely every component of ω with at least 3 ends has infinitely many ends.*

Proof. Let ω_1 be the union of the components K of ω whose number $n(K)$ of ends is finite and at least 3. Given a component K of ω_1 , there is a connected subgraph $A \subset K$ with minimal $|V(A)|$ such that $K \setminus A$ has $n(K)$ infinite components. Let $H(K)$ be the union of all such subgraphs A . It is easy to verify that any two such subgraphs A must intersect, and therefore $H(K)$ is finite. Let $H(\omega_1)$ be the union of all $H(K)$, where K ranges over the components of ω_1 .

Begin with unit mass at each vertex x that belongs to a component K of ω_1 , and transport it equally to the vertices in $H(K)$. Then the vertices in $H(\omega_1)$ receive infinite mass. By the Mass-Transport Principle, no vertex can receive infinite mass, which means that ω_1 is empty a.s. ■

Among the characterizations of amenability via invariant percolation that appear in [BLPS99], we single out one that relates to the absence of phase transition:

THEOREM 4.3. *Let G be a transitive graph. Then each of the following conditions implies the next one:*

- (i) G is amenable;
- (ii) there is an invariant random nonempty subtree of G with at most 2 ends a.s.;
- (iii) there is an invariant random nonempty connected subgraph ω of G that satisfies $p_c(\omega) = 1$ with positive probability;
- (iv) $\text{Aut}(G)$ is amenable.

If G is assumed to be unimodular, then all four conditions are equivalent.

To see one use of Theorem 4.3, we present the proof of part of Theorem 3.11. (In fact, here we do not need the assumption of unimodularity.)

COROLLARY 4.4. *If G is a transitive graph with a nonamenable automorphism group and Bernoulli(p) percolation produces a unique infinite cluster a.s., then $p > p_c(G)$.*

Proof. Suppose that $p = p_c(G)$ and that there is a unique infinite cluster a.s. Then the infinite cluster K has $p_c(K) = 1$ a.s. Hence $\text{Aut}(G)$ is amenable. ■

Next, we present a characterization of unimodularity in terms of the expected degree of vertices in infinite components. Since any connected finite graph with vertex set V has average degree at least $2 - 2/|V|$, one might expect that for invariant percolation on a transitive graph G with all components infinite a.s., the expected degree of a vertex is at least 2. This inequality is true when G is unimodular, but surprisingly, whenever G is not unimodular, there is an invariant percolation where the inequality fails.

THEOREM 4.5. *Let G be a transitive graph. Let m be the minimum of $|S(x)|/|S(y)|$ for x, y neighbors in G . Then for any invariant percolation that yields infinite components with positive probability, the expected degree of o given that o is in an infinite component is at least $1 + m$. This is sharp for all G in the sense that there is an invariant bond percolation on G with every vertex belonging to an infinite component and having expected degree $1 + m$.*

A **forest** is a graph all of whose components are trees. The following theorem concerning phase transition on percolation components was shown when G is a tree by Häggström (1997).

THEOREM 4.6. *Let G be a unimodular transitive graph. Let ω be the configuration of an invariant percolation on G such that ω has infinite components with positive probability. If*

- (i) *some component of ω has at least 3 ends with positive probability,*

then

- (ii) *some component of ω has $p_c < 1$ with positive probability and*
- (iii) $\mathbf{E}[D^*(o) \mid |K(o)| = \infty] > 2$.

If ω is a forest a.s., then the three conditions are equivalent.

To show how Theorem 4.6 can be used, we now complete the proof of Theorem 3.11. (A more direct proof of Theorem 3.11 is provided by Benjamini, Lyons, Peres, and Schramm (1999a).)

Proof of Theorem 3.11. Let ω be the configuration of critical Bernoulli percolation on G . Then every infinite cluster K of ω has $p_c(K) = 1$ a.s. As we have mentioned, the number of infinite clusters of ω is equal a.s. to 0, 1 or ∞ . Corollary 4.4 rules out a unique infinite cluster. If there were more than one infinite cluster, then by opening the edges in a large ball, we see that there would be, with positive probability, a cluster with at least 3 ends. In light of Theorem 4.6, this would mean that with positive probability, some infinite cluster K had $p_c(K) < 1$. This is a contradiction. ■

§5. Ising, Potts, and Random Cluster Models on Transitive Graphs.

Ising and Potts models on graphs are defined using interaction strengths along bonds (here assumed identically 1), Boltzmann’s constant k_B , and the temperature T . These last two quantities always appear together in the expression $\beta := 1/(k_B T)$, called the **inverse temperature**. Given a finite graph G and an integer $q \geq 2$, let $\omega \in \{1, 2, \dots, q\}^V$. Write $I_\omega(e)$ for the indicator that ω takes different values at the endpoints of the edge e . The **energy** (or **Hamiltonian**) of ω is

$$H(\omega) := 2 \sum_{e \in E} I_\omega(e).$$

The **Potts measure** $\text{FPt}(\beta) = \text{FPt}^G(\beta)$ is the probability measure on $\{1, 2, \dots, q\}^V$ that is proportional to $e^{-\beta H(\omega)}$. In the case $q = 2$, it is more customary to use $\{-1, 1\}^V$ in place of $\{1, 2\}^V$, and the measure is called the **Ising measure**.

To define such measures on infinite graphs G , one can proceed via **exhaustions** of G , i.e., sequences of finite subgraphs G_n that are increasing and whose union is all of G . There are several ways to do this, in fact, and crucial questions are whether some of the limits they give are the same. One way to take a limit is simply to define $\text{FPt}^G(\beta)$ to be the weak* limit of $\text{FPt}^{G_n}(\beta)$; this is called the **free Potts measure** on G . Another way is as follows. Let $\text{Pt}_k^{G_n}(\beta)$ be the probability measure $\text{FPt}^{G_n}(\beta)$ conditioned on having $\omega(x) = k$ for every $x \in \partial_V^{\text{int}} G_n$, where

$$\partial_V^{\text{int}} K := \{x \in K; \exists y \notin K \ x \sim y\}$$

denotes the **internal vertex boundary** of any subset $K \subset V$. Then define the **Potts measure** $\text{Pt}_k^G(\beta)$ to be the weak* limit of $\text{Pt}_k^{G_n}(\beta)$. These limits always exist (see, e.g., Aizenman, Chayes, Chayes, and Newman (1988), referred to later as [ACCN]). It will be convenient to define the **wired Potts measure** $\text{WPt}^G(\beta)$ to be $\sum_{k=1}^q \text{Pt}_k^G(\beta)/q$. Note

that if G_n^* denotes the graph obtained from G_n by identifying all of the vertices in $\partial_V^{\text{int}} G_n$ to a single vertex, then $\text{WPt}^G(\beta)$ is the weak* limit of $\text{FPt}^{G_n^*}(\beta)$.

To define Potts measures in general, write $\omega|_{V'}$ for the restriction of ω to $V' \subset V$. For a finite subset $V' \subset V$, let G' denote the subgraph of G induced by V' , i.e., $G' := (V', (V' \times V') \cap E)$. For $\omega' \in \{1, \dots, q\}^{V'}$, write $\partial\omega' := \omega'|_{\partial_V^{\text{int}} G'}$. We call \mathbf{P} a **Potts measure** on G at inverse temperature β if \mathbf{P} is a Markov random field and for all finite $V' \subset V$ and all $\omega' \in \{1, \dots, q\}^{V'}$,

$$\mathbf{P}\left[\omega|_{V'} = \omega' \mid \omega|_{\partial_V^{\text{int}} G'} = \partial\omega'\right] = \text{FPt}^{G'}(\beta)\left[\omega = \omega' \mid \omega|_{\partial_V^{\text{int}} G'} = \partial\omega'\right].$$

It is easy to verify that the measures $\text{FPt}^G(\beta)$ and $\text{Pt}_k^G(\beta)$ are Potts measures in this sense.

Potts measures are intimately connected to random cluster measures, introduced by Fortuin and Kasteleyn (1972) and Fortuin (1972a, 1972b). See Häggström (1998) for a survey of the relationships and Grimmett (1995) for more details on random cluster measures, especially on \mathbb{Z}^d . Random cluster measures depend on two parameters, $p \in (0, 1)$ and $q > 0$. We restrict ourselves to $q \geq 1$ since the measures with $q < 1$ behave rather differently and are poorly understood; they are also unrelated to Potts measures. Given a finite graph G and $\omega \in 2^E$, write $\|\omega\|$ for the number of components of ω . The **random cluster measure** with parameters (p, q) on G , denoted $\text{FRC}(p, q) = \text{FRC}^G(p, q)$, is the probability measure on E proportional to $q^{\|\omega\|} \mathbf{P}_p(\omega)$, i.e., the Bernoulli(p) percolation measure \mathbf{P}_p biased by $q^{\|\omega\|}$ (and renormalized). On infinite graphs G , there are again several ways to define random cluster measures. The ones that concern us are obtained by taking limits over exhaustions G_n of G . Namely, define $\text{FRC}^G(p, q)$ to be the weak* limit of $\text{FRC}^{G_n}(p, q)$; this is called the **free random cluster measure** on G . Define the **wired random cluster measure** $\text{WRC}^G(p, q)$ to be the weak* limit of $\text{FRC}^{G_n^*}(p, q)$. These limits always exist (see, e.g., [ACCN]). Furthermore, they have positive correlations and so the free random cluster measure is stochastically dominated by the wired random cluster measure [ACCN].

Note that there is another use of “wired” in the literature, although when G has only one end, the meaning is the same as the present one. In the terminology of Grimmett (1995), the above random cluster measures are “limit random cluster measures”. We do not examine whether they satisfy so-called Gibbs specifications.

Since all the above limits exist regardless of the exhaustion chosen, the limiting measures are invariant under all graph automorphisms.

The fundamental relation between Potts and random cluster measures is the following: Let G be a finite graph and $q \geq 2$ an integer. Suppose that $p = 1 - e^{-2/\beta}$.

- If $\omega \in \{1, \dots, q\}^V$ is chosen with distribution $\text{FPt}(\beta)$ and $\eta \in 2^E$ is chosen independently with distribution \mathbf{P}_p , then $(1 - I_\omega)\eta$ has the distribution $\text{FRC}(p, q)$.
- Choose $\eta \in 2^E$ with distribution $\text{FRC}(p, q)$. For each component of η , choose independently and uniformly an element of $\{1, \dots, q\}$, assigning this element to every vertex in that component. The resulting $\omega \in \{1, \dots, q\}^V$ has distribution $\text{FPt}(\beta)$.

See [ACCN] or Häggström (1998) for proofs. By taking weak* limits and using positive correlations, one obtains corresponding statements for infinite graphs (see the proof of Theorem 2.3(c) of [ACCN]): First, the two above statements hold as written for infinite graphs. Second:

- If $\omega \in \{1, \dots, q\}^V$ is chosen with distribution $\text{WPt}(\beta)$ and $\eta \in 2^E$ is chosen independently with distribution \mathbf{P}_p , then $(1 - I_\omega)\eta$ has the distribution $\text{WRC}(p, q)$.
- Choose $\eta \in 2^E$ with distribution $\text{WRC}(p, q)$. For each component of η , choose independently and uniformly an element of $\{1, \dots, q\}$, assigning this element to every vertex in that component, where all infinite components are regarded as a single component (“connected at infinity”). The resulting $\omega \in \{1, \dots, q\}^V$ has distribution $\text{WPt}(\beta)$.

Third:

- If $\omega \in \{1, \dots, q\}^V$ is chosen with distribution $\text{Pt}_k(\beta)$ and $\eta \in 2^E$ is chosen independently with distribution \mathbf{P}_p , then $(1 - I_\omega)\eta$ has the distribution $\text{WRC}(p, q)$.
- Choose $\eta \in 2^E$ with distribution $\text{WRC}(p, q)$. For each finite component of η , choose independently and uniformly an element of $\{1, \dots, q\}$, assigning this element to every vertex in that component. Assign each vertex in an infinite component the color k . The resulting $\omega \in \{1, \dots, q\}^V$ has distribution $\text{Pt}_k(\beta)$.

Recall the notation (4.2). From the preceding relations, we obtain:

PROPOSITION 5.1. *Let G be any graph and $q \geq 2$ an integer. Let $\beta > 0$ and $p := 1 - e^{-2\beta}$. Then*

- (i) (JONASSON 1999) $\text{FPt}(\beta) = \text{WPt}(\beta)$ iff $\text{FRC}(p, q) = \text{WRC}(p, q)$;
- (ii) $\text{Pt}_k^G(\beta)$ is the same for all k iff $\theta(\text{WRC}(p, q)) = 0$.

Proof. Part (ii) is obvious, but part (i) needs some explanation. One implication of (i) is also obvious from the above relations, namely, that if $\text{FPt}(\beta) = \text{WPt}(\beta)$, then $\text{FRC}(p, q) = \text{WRC}(p, q)$. Conversely, if $\text{FRC}(p, q) = \text{WRC}(p, q)$, then a.s. there cannot be more than one infinite component. For if there were, then with positive probability there would be neighbors x, y belonging to distinct infinite components in $E \setminus \{[x, y]\}$. Call this event $A_{x,y}$. We have $\text{FRC}(p, q)[[x, y] \in \omega \mid A_{x,y}] = p/[p + (1 - p)q] \neq p = \text{WRC}(p, q)[[x, y] \in \omega \mid A_{x,y}]$, which contradicts $\text{FRC}(p, q) = \text{WRC}(p, q)$.

Since there cannot be more than one infinite component, the above relations give $\text{FPt}(\beta) = \text{WPt}(\beta)$. ■

It seems reasonable to suppose that Conjectures 3.6 and 3.8 extend to random cluster models, so that for each $q \geq 1$, there would be three phases on nonamenable transitive graphs with one end. In the case of the graph formed by the product of a regular tree of sufficiently high degree and \mathbb{Z}^d , this follows from Newman and Wu (1990).

It is well known that $\text{RC}(p, q)$ is stochastically increasing in p for each fixed q , where $\text{RC}(p, q)$ denotes either $\text{FRC}(p, q)$ or $\text{WRC}(p, q)$. Therefore, the set of p for which $\theta(\text{RC}(p, q)) = 0$ is an interval for each q . The same holds for the sets of p for which the number of infinite components is ∞ or 1 by the following partial analogue of Theorem 3.5:

PROPOSITION 5.2. *Let G be a transitive unimodular graph. Given $q \geq 1$ and $p_1 < p_2 < 1$, if there is a unique infinite component $\text{RC}(p_1, q)$ -a.s. on G , then there is a unique infinite component $\text{RC}(p_2, q)$ -a.s.*

Proof. Given $\omega \in 2^E$ and $e \in E$, write ω_e for the restriction of ω to $E \setminus \{e\}$. A bond percolation process (\mathbf{P}, ω) on G is **insertion tolerant** if $\mathbf{P}[e \in \omega \mid \omega_e] > 0$ a.s. for all $e \in E$. Theorem 3.7(ii) has the following extension: If \mathbf{P} is any invariant ergodic percolation process on G that is insertion tolerant, then there is a unique infinite component \mathbf{P} -a.s. if $\inf_x \mathbf{P}[o \leftrightarrow x] > 0$ (Lyons and Schramm 2000). The converse holds as well when the percolation process has positive correlations, since then a unique infinite component implies that

$$\mathbf{P}[o \leftrightarrow x] \geq \mathbf{P}[|K(o)| = \infty] \mathbf{P}[|K(x)| = \infty] = \mathbf{P}[|K(o)| = \infty]^2.$$

We have already noted that $\text{RC}(p, q)$ is invariant and has positive correlations. It is easy to see that $\text{RC}(p, q)$ is insertion tolerant, and ergodicity is proved by Borgs and Chayes (1996) (for FRC) and by Biskup, Borgs, Chayes, and Kotecký (2000) (for both measures). Therefore, we may apply this extension of Theorem 3.7(ii) and its converse to $\text{RC}(p_1, q)$ and $\text{RC}(p_2, q)$. ■

The ergodicity needed in this proof has itself a simple proof that seems to have been overlooked. In fact, RC has a trivial tail σ -field on every graph, not merely on transitive graphs. To see this, let B be any increasing cylinder event and let A be any tail event. Let G be exhausted by the finite subgraphs G_n . Suppose that n is large enough that B depends on the edges in G_n only. Given $M > n$, approximate A by a cylinder event C depending only on edges in $G_M \setminus G_n$. Let D denote the event that all edges in $G_M \setminus G_n$ are closed. Then

$$\text{RC}^{G_n}(B) = \text{RC}^{G_M}(B \mid D) \leq \text{RC}^{G_M}(B \mid C),$$

because RC has positive correlations. Letting $M \rightarrow \infty$ shows that $\text{RC}^{G_n}(B) \leq \text{FRC}(B \mid C)$, whence $\text{RC}^{G_n}(B) \leq \text{FRC}(B \mid A)$. Now let $n \rightarrow \infty$ to conclude that $\text{FRC}(B) \leq \text{FRC}(B \mid A)$. Since the same holds with $\neg A$ in place of A , this inequality is, in fact, an equality. That is, A is independent of every increasing cylinder event, whence of every cylinder event, whence of every event. In other words, A is trivial. To prove tail triviality for WRC, we use the same proof with G_n^* in place of G_n , with D being the event that all edges in $G_M^* \setminus G_n^*$ are open, and with reversed inequalities. (A similar proof appears for different measures in Benjamini, Lyons, Peres, and Schramm (2000).)

There are four possible phases in Potts models that are often investigated, i.e., four types of behavior for different values of β . We shall say that a Potts measure at inverse temperature β is **extreme** if, as an element of the convex set of all Potts measures on G at inverse temperature β , it is extreme. The four phases are:

- (I) there is a unique Potts measure (equivalently, $\text{Pt}_k(\beta)$ does not depend on k);
- (II) the free Potts measure is extreme and there are other Potts measures;
- (III) the free Potts measure is not extreme, nor equal to the wired Potts measure;
- (IV) the free Potts measure is equal to the wired Potts measure and there are other Potts measures.

Newman and Wu (1990) showed the existence of three phases, namely, (I), (II) \cup (III), and (IV), each containing an interval of parameter values of positive length, for the q -state Potts model on the graph formed by the product of a regular tree and \mathbb{Z}^d , provided that the tree has sufficiently high degree depending on q . Schonmann (2000) extended this to show that for any q , if G is a transitive graph of degree d with $\iota_E(G)/d$ sufficiently close to 1 and with $p_u(G) < 1$, then there are these same three phases in the q -state Potts model. Wu (1996) showed similar results for a graph which is not transitive but is similar to the hyperbolic plane.

There are some partial results for other graphs. For natural Cayley graphs of co-compact Fuchsian groups, an uncountable number of mutually singular Potts measures were constructed by Series and Sinaï (1990).

In the following results, we use “interval” to mean interval of positive length.

THEOREM 5.3. (JONASSON 1999) *Let G be a nonamenable regular graph and $q \geq 2$ be an integer. Then for all sufficiently large q , there is an interval of p for which $\text{FRC}(p, q) = \text{WRC}(p, q)$ and there is an interval of p for which $\text{FRC}(p, q) \neq \text{WRC}(p, q)$.*

[In fact, $\text{FRC}(p, q) = \text{WRC}(p, q)$ holds for small p and all q since both measures are dominated by \mathbf{P}_p . It would be interesting to show that $\text{FRC}(p, q) = \text{WRC}(p, q)$ can occur for large p if, say, G has one end.]

As a consequence of this and Proposition 5.1(i), we obtain:

THEOREM 5.4. (JONASSON 1999) *Let G be a nonamenable regular graph and $q \geq 2$ be an integer. For all sufficiently large q , there is an interval of β for which $\text{FPt}(\beta) = \text{WPt}(\beta)$ and there is an interval of β for which $\text{FPt}(\beta) \neq \text{WPt}(\beta)$.*

Both of the above theorems fail when G is amenable and transitive (Jonasson 1999).

The last result we mention also gives a characterization of amenability among transitive graphs, but it involves an external field. To define the Ising model with external field h on a finite graph G , modify the energy $H(\omega)$ to be

$$H(\omega) := 2 \sum_{e \in E} I_\omega(e) + 2h \sum_{x \in V} \mathbf{1}_{\{\omega(x) \neq 1\}}.$$

Here, we take $q = 2$. The corresponding probability measure on $\{-1, 1\}^V$ proportional to $e^{-\beta H(\omega)}$ is denoted $\text{Ising}^G(\beta, h)$. For an infinite graph G , two limits over exhaustions G_n are particularly important, namely, $\text{Ising}_\pm^G(\beta, h)$, the weak* limits of $\text{Ising}^{G_n}(\beta, h)$ conditional on $\omega|_{\partial_V^{\text{int}} G_n}$ to be a constant, ± 1 , respectively.

THEOREM 5.5. (JONASSON AND STEIF 1999) *If G is a nonamenable graph of bounded degree, then for some β , there is an interval of h for which $\text{Ising}_+^G(\beta, h) = \text{Ising}_-^G(\beta, h)$ and there is an interval of h for which $\text{Ising}_+^G(\beta, h) \neq \text{Ising}_-^G(\beta, h)$.*

As Jonasson and Steif (1999) show, this is not true for any amenable transitive graph.

§6. Percolation on Trees.

As we have already mentioned, if T is a regular tree, then Bernoulli percolation produces a cluster $K(o)$ that is a Galton-Watson tree (except for the first generation), so its analysis is easy and well known. In fact, it is not hard to find the critical value for percolation on Galton-Watson trees:

PROPOSITION 6.1. (LYONS 1990) *Let T be the family tree of a Galton-Watson process with mean $m > 1$. Then $p_c(T) = 1/m$ a.s. given nonextinction.*

In the proof, as well as below, we write T^x for the descendant subtree of T from x , i.e., the tree formed from all $y \in T$ such that the path from o to y contains x .

Proof. Consider Bernoulli(p) percolation on T . We claim that $K(o)$ has the law (not conditioned on T) of another Galton-Watson tree having mean mp : Let L be a random

variable whose distribution is the offspring law for T and let Y_i represent i.i.d. $\text{Bin}(1, p)$ random variables that are also independent of T . Then

$$\mathbf{E}\left[\sum_{i=1}^L Y_i\right] = \mathbf{E}\left[\mathbf{E}\left[\sum_{i=1}^L Y_i \mid L\right]\right] = \mathbf{E}\left[\sum_{i=1}^L \mathbf{E}[Y_i]\right] = \mathbf{E}\left[\sum_{i=1}^L p\right] = pm.$$

Hence $K(o)$ is finite a.s. if $mp \leq 1$. Since $\mathbf{E}\left[\mathbf{P}[|K(o)| < \infty \mid T]\right] = \mathbf{P}[|K(o)| < \infty]$, this means that for almost every Galton-Watson tree T , the component of its root is finite a.s. if $mp \leq 1$. In other words, $p_c(T) \geq 1/m$ a.s. given nonextinction. Similarly, the component of the root is infinite w.p.p. if $mp > 1$, whence $\forall p > 1/m \quad p_c(T) \leq p$ w.p.p. It remains to show that $\mathbf{P}[T \text{ is infinite and } p_c(T) \leq p] = 1 - q$, the probability of nonextinction, for $p > 1/m$. However, it is easy to see that the event $\{T \text{ is finite or } p_c(T) > p\}$ is inherited in the sense that if T has this property, then so does T^x for each child x of the root of T . It follows (e.g., see Lyons (2001)) that $\mathbf{P}[p_c(T) > p] \in \{q, 1\}$. We have already seen that it is not equal to 1. ■

Results for percolation on more general trees depend on the following notions. Define a **cutset** of a tree T to be a collection Π of vertices whose removal from T would leave o in a finite component. Lyons (1990) defined the **branching number** of T to be

$$\text{br}(T) := \inf \left\{ \lambda > 1; \inf_{\Pi} \sum_{x \in \Pi} \lambda^{-|x|} = 0 \right\},$$

where the infimum is over cutsets Π . This is related to the Hausdorff dimension of the boundary of T : The **boundary** of T , denoted ∂T , is the set of infinite paths from o that do not backtrack. We put a metric on ∂T by letting the distance between ξ and η be e^{-n} if the number of edges common to ξ and η is n . Then $\text{br}(T) = e^{\dim \partial T}$; Furstenberg (1970) was the first to consider $\dim \partial T$. If T is spherically symmetric (about o), meaning that $\deg x$ is a function only of $|x|$ for $x \in T$, then $\text{br}(T) = \text{gr}(T)$, while in general, we have $\text{br}(T) \leq \text{gr}(T)$.

The following theorem was first proved (in different but equivalent language) by Hawkes (1981) for trees T with bounded degree and by Lyons (1990) in general:

THEOREM 6.2. *If T is any tree, then $p_c(T) = 1/\text{br}(T)$.*

From this and Proposition 6.1, we find that $\text{br}(T) = m$ a.s. for Galton-Watson trees with mean m ; this was first shown (in the language of Hausdorff dimension) by Hawkes (1981).

The issue of uniqueness of infinite clusters on trees was settled in folklore, but appeared in print for the first time by Peres and Steif (1998).

PROPOSITION 6.3. *For any tree T and $p < 1$, the number of infinite clusters on T is \mathbf{P}_p -a.s. 0 or \mathbf{P}_p -a.s. ∞ .*

Similarly, one can describe the number of ends of the clusters for percolation on trees:

THEOREM 6.4. (PEMANTLE AND PERES 1995) *If T is any tree and $0 < p < 1$, then \mathbf{P}_p -a.s. either $K(o)$ is finite or $K(o)$ has infinitely many ends.*

In order to determine the behavior of percolation at the critical value, we need to introduce the notion of capacity. Let μ be a probability measure on ∂T . For $p < 1$, we define the p -energy of μ as

$$\mathcal{E}_p(\mu) := \iint p^{-|\xi_1 \wedge \xi_2|} d\mu(\xi_1) d\mu(\xi_2),$$

where $\xi_1 \wedge \xi_2$ denotes the vertex in $\xi_1 \cap \xi_2$ that is furthest from o . (If $\xi = \eta$, we interpret $p^{-|\xi_1 \wedge \xi_2|} := \infty$.) The p -capacity of ∂T , denoted $\text{cap}_{(p)}(\partial T)$, is the reciprocal of the minimum $\mathcal{E}_p(\mu)$ over all probability measures μ on ∂T . If T is spherically symmetric, then

$$\text{cap}_{(p)}(\partial T) = \left(1 + (1-p) \sum_{n=1}^{\infty} \frac{1}{p^n |T_n|} \right)^{-1},$$

where T_n denotes the set of vertices x with $|x| = n$.

The second part of the following theorem was shown by Fan (1989, 1990) when T has bounded degree, and the full theorem by Lyons (1992) in general:

THEOREM 6.5. *If T is any tree with root o and $0 < p < 1$, then*

$$\text{cap}_{(p)}(\partial T) \leq \theta_o(p) \leq 2 \text{cap}_{(p)}(\partial T).$$

In particular, the probability of an infinite cluster is positive iff $\text{cap}_{(p)}(\partial T) > 0$.

Using this result, it is easy to construct trees T for which $\theta(p_c(T)) > 0$ or for which $\theta(p)$ is discontinuous at other p . Similarly, nothing like Theorems 3.13 or 3.12 hold for general trees.

An extension and sharpening of Theorem 6.5 is known for arbitrary survival parameters. Given any survival parameters $p(e)$ on the edges e of T , we define the corresponding energy as

$$\mathcal{E}(\mu) := \iint \mathbf{P}[o \leftrightarrow \xi_1 \wedge \xi_2]^{-1} d\mu(\xi_1) d\mu(\xi_2)$$

and define capacity as before but using this energy. The following theorem was proved by Lyons (1992), with the sharpening provided by the second inequality due to Marchal (1998):

THEOREM 6.6. *If T is any tree and with any survival parameters $p(\bullet)$ and corresponding capacity $\kappa := \text{cap}(\partial T)$, we have*

$$\kappa \leq \mathbf{P}[o \leftrightarrow \infty] \leq 1 - e^{-2\kappa/(1-\kappa)} \leq 2\kappa.$$

Häggström, Peres, and Steif (1997) introduced a version of (bond) percolation on graphs that evolves in time. Given $p \in (0, 1)$, the set of open edges evolves so that at any fixed time $t \geq 0$, the distribution of this set is \mathbf{P}_p . Let the initial distribution at time 0 be given by \mathbf{P}_p , and let each edge change its status (open or closed) according to a continuous-time, stationary 2-state Markov chain, independently of all other edges. Each edge flips (changes its value) at rate p when closed and rate $1 - p$ when open. Let Ψ_p denote the probability measure for this Markov process, called **dynamical percolation** with parameter p . This process is most interesting for $p = p_c(G)$ because of the following general result:

THEOREM 6.7. (HÄGGSTRÖM, PERES, AND STEIF 1997) *For any graph G , if $p > p_c(G)$, then Ψ_p -a.s. there is an infinite cluster for every time t , while if $p < p_c(G)$, then Ψ_p -a.s. there is an infinite cluster for no time t .*

On trees, one can decide what happens at criticality by means of a capacity condition (that we express for comparison via percolation instead of capacity):

THEOREM 6.8. (HÄGGSTRÖM, PERES, AND STEIF 1997) *Let T be a tree and $0 < p < 1$. Write \mathbf{P}^* for the probability measure of percolation on T that independently retains each edge joining T_{n-1} to T_n with probability $p + p/n$. Then there is Ψ_p -a.s. some time $t > 0$ at which there is an infinite cluster on T iff \mathbf{P}^* -a.s. there is an infinite cluster on T . If T is spherically symmetric, then this is equivalent to*

$$\sum_{n=1}^{\infty} \frac{1}{np^n |T_n|} < \infty.$$

No reasonable necessary and sufficient condition is known so that with $\Psi_{p_c(T)}$ -probability 1, there exists an infinite cluster for *all* times $t > 0$. However, Peres and Steif (1998) have shown that when $p > p_c(T)$, there are infinitely many infinite clusters for all times t simultaneously Ψ_p -a.s. This follows from the proof of Proposition 6.3 together with Theorem 6.7.

§7. The Ising Model on Trees.

We resume the notation of Section 5. The Ising model on trees was first studied by Kurata, Kikuchi, and Watari (1953), who showed that if T is regular of degree $b + 1$, then its critical β equals $\coth^{-1} b$, meaning that there is a unique Ising measure for $\beta > \coth^{-1} b$, but not for $\beta < \coth^{-1} b$ (see also Preston (1974)). In other words, this is the boundary of phase (I) in the phase divisions given in Section 5. This calculation was extended to all trees by Lyons (1989):

THEOREM 7.1. *If T is any tree, then its critical β equals $\coth^{-1} \text{br}(T)$.*

Bleher, Ruiz, and Zagrebnev (1995) showed that the critical β for the free Ising model on a regular tree T of degree $b + 1$ equals $\coth^{-1} \sqrt{b}$. This means that the free Ising measure is extreme for $\beta < \coth^{-1} \sqrt{b}$, but not for $\coth^{-1} \sqrt{b} < \beta < \coth^{-1} b$. This is the boundary of phase (II). A simpler proof was given by Ioffe (1996a). The result was extended to all trees by Evans, Kenyon, Peres, and Schulman (2000):

THEOREM 7.2. *If T is any tree, then the critical value of β for the free Ising model equals $\coth^{-1} \sqrt{\text{br}(T)}$.*

Theorem 7.7 of Georgii (1988) shows that an Ising measure is extremal iff it has a trivial tail, and Lemma 4.2 of Evans, Kenyon, Peres, and Schulman (2000) or Lemma 2 of Ioffe (1996b) shows that, for the free Ising measure, this is equivalent to independence of $\omega(o)$ from the tail. Thus, another interpretation of Theorem 7.2 involves asymptotic reconstruction of $\omega(o)$ given $\omega(x)$ for all $x \in T_n$ as $n \rightarrow \infty$.

We next discuss Edwards-Anderson spin glasses. For a graph G , let $J(e)$ be independent uniform ± 1 -valued random variables indexed by the edges $e \in E$. If G is finite, define the energy of a configuration $\omega \in \{1, -1\}^V$ to be

$$H(\omega) := 2 \sum_{e \in E} J(e) I_\omega(e). \quad (7.1)$$

The corresponding probability measure at inverse temperature β on $\{1, -1\}^V$ is the one proportional to $e^{-\beta H(\omega)}$, denoted $\text{SpGl}^G(\beta)$. Note that this measure depends on the values of J . We call \mathbf{P} a **spin glass measure** on an infinite graph G at inverse temperature β and with interactions $J(e)$ if \mathbf{P} is a Markov random field and for all finite $V' \subset V$ and all $\omega' \in \{1, \dots, q\}^{V'}$,

$$\mathbf{P} \left[\omega|_{V'} = \omega' \mid \omega|_{\partial_V^{\text{int}} G'} = \partial\omega' \right] = \text{SpGl}^{G'}(\beta) \left[\omega = \omega' \mid \omega|_{\partial_V^{\text{int}} G'} = \partial\omega' \right].$$

Define

$$\beta_c^{SG}(G) := \sup\{\beta \geq 0; \text{ for a.e. } J(\bullet) \text{ and for every } \beta' \in [0, \beta] \\ \text{there is a unique spin glass measure on } G \\ \text{at inverse temperature } \beta' \text{ and with interactions } J(\bullet)\}.$$

See Newman (1997) for background.

If T is a regular tree of degree $b + 1$, then Chayes, Chayes, Sethna, and Thouless (1986) showed that $\beta_c^{SG}(T) = \coth^{-1} \sqrt{b}$. On trees, the spin glass model is equivalent via a gauge transformation to having random independent boundary conditions in the Ising model. Under this transformation, let \mathbf{P}_β denote the limiting Ising measure. The phase transition defining β_c^{SG} is equivalent to \mathbf{P}_β going from not being a.s. extreme to being a.s. extreme as β passes $\coth^{-1} \sqrt{b}$. This calculation was extended to all trees by Pemantle and Peres (2000):

THEOREM 7.3. *If T is any tree, then $\beta_c^{SG} = \coth^{-1} \sqrt{\text{br}(T)}$. Furthermore, there is a.s. more than one spin glass measure on T for every $\beta > \beta_c^{SG}$.*

The critical cases in each of the above three theorems can be decided based on a capacity criterion, although the capacity for Theorem 7.1 is not the usual double-integral type, but a triple-integral type. These capacity criteria hold for varying interaction strengths $J(e)$ as well. (The interaction strengths affect the Hamiltonian as in (7.1).) In order to state these criteria, we shall use the following notation: For a vertex $x \in T$ and an edge or vertex $a \in T$, write $a \leq x$ if a is on the path from o to x . If $x \in \partial T$, then $a \leq x$ will mean $a \in x$. Let

$$C(x) := \prod_{e \leq x} \tanh(J(e)\beta)$$

and

$$k(x) := \sum_{o \neq y \leq x} C(y)^{-2}.$$

THEOREM 7.4. (PEMANTLE AND PERES 2000) *Let T be any tree without leaves (except possibly at o). Let $0 < \inf_{e \in E} J(e) \leq \sup_{e \in E} J(e) < \infty$.*

- (i) *There is a unique Ising measure at inverse temperature β iff there is a probability measure μ on ∂T such that*

$$\iiint k(\xi_1 \wedge \xi_2 \wedge \xi_3) d\mu(\xi_1) d\mu(\xi_2) d\mu(\xi_3) < \infty.$$

If $J(e) \equiv J$ and T is spherically symmetric, then this is equivalent to

$$\sum_{n \geq 1} \frac{1}{[\tanh(J\beta)]^{2n} |T_n|^2} < \infty.$$

- (ii) *The free Ising measure at inverse temperature β is extreme iff \mathbf{P}_β is a.s. extreme iff there is a probability measure μ on ∂T such that*

$$\iint k(\xi_1 \wedge \xi_2) d\mu(\xi_1) d\mu(\xi_2) < \infty.$$

If $J(e) \equiv J$ and T is spherically symmetric, then this is equivalent to

$$\sum_{n \geq 1} \frac{1}{[\tanh(J\beta)]^{2n} |T_n|} < \infty.$$

Yet another phase transition of the Ising model concerns the **magnetic susceptibility**, $\lim_{n \rightarrow \infty} \text{Var} \left(\sum_{x \in B(o, n)} \omega(x) \right) / |B(o, n)|$, where Var is variance with respect to the free Ising measure. Matsuda (1974) and Falk (1975) showed that the magnetic susceptibility becomes infinite when β passes the critical value $\coth^{-1} \sqrt{b}$ if T is a regular tree of degree $b + 1$.

We turn now to models other than the Ising model. Pemantle and Steif (1999) have shown that the location of a phase transition for the q -state Potts model on trees with $q \geq 3$ depends on subtle aspects of the structure of the tree, and most certainly not on $\text{br}(T)$. However, the location of a robust phase transition still depends only on $\text{br}(T)$. Here, we are using the following notion. Given a cutset Π of T , let $\Pi(o)$ denote the component of o in $(T \setminus \Pi) \cup \Pi$. For $\epsilon > 0$, let $s_\beta(\Pi, \epsilon)$ denote the distribution of $\omega(o)$ with respect to the Potts measure on $\Pi(o)$ with inverse temperature β , interaction strengths

$$J(e) := \begin{cases} \epsilon & \text{if } e \text{ has an endpoint in } \Pi, \\ 1 & \text{if not,} \end{cases} \tag{7.2}$$

and conditioned on $\omega \upharpoonright \partial_V^{\text{int}} \Pi(o) \equiv 1$. Then the critical value for a **robust phase transition** is defined to be

$$\sup \left\{ \beta; \forall \epsilon > 0 \inf_{\Pi} \|s_\beta(\Pi, \epsilon) - 1/q\|_\infty > 0 \right\},$$

where the infimum is over all cutsets Π . [Instead of considering arbitrarily small boundary interactions strengths ϵ , one could instead keep the interaction strengths constant and use high temperatures at the boundaries Π .]

THEOREM 7.5. (PEMANTLE AND STEIF 1999) *If T is any tree with bounded degrees and $q \geq 2$, the critical value of β for a robust phase transition in the q -state Potts model on T is the unique value of β satisfying*

$$\frac{e^\beta + (q - 1)e^{-\beta}}{e^\beta - e^{-\beta}} = \text{br}(T).$$

In particular, the location for a robust phase transition in the Ising model is the same as for the usual phase transition.

Lastly, we consider some continuous models on trees. Let S^d denote the d -dimensional unit sphere in \mathbb{R}^{d+1} . Given a finite graph G and interaction strengths $J(e)$ for $e \in E$, define the energy of $\omega \in (S^d)^V$ as

$$H(\omega) := \sum_{e \in E} H_\omega(e),$$

where for any edge $e = [x, y]$, we write $H_\omega(e) := -J(e)\omega(x) \cdot \omega(y)$. The **d -dimensional spherical measure** on G at inverse temperature β is the probability measure proportional to $e^{-\beta H(\omega)} \mathbf{P}(\omega)$, where \mathbf{P} is the product measure on $(S^d)^V$ with marginals on each coordinate equal to Lebesgue measure (i.e., normalized surface measure) on S^d . When $d = 1$, this is called the “rotor” measure; when $d = 2$, it is called the “Heisenberg” measure.

For a tree T and $\epsilon > 0$, let $s_\beta(\Pi, \epsilon)$ denote the density of $\omega(o)$ with respect to the d -dimensional spherical measure on $\Pi(o)$ with inverse temperature β , interaction strengths as in (7.2), and conditioned on $\omega|_{\partial V^{\text{int}}\Pi(o)} \equiv \hat{1}$, where $\hat{1}$ denotes any fixed element of S^d . The critical value for a **robust phase transition** is defined to be

$$\sup \left\{ \beta; \forall \epsilon > 0 \inf_{\Pi} \|s_\beta(\Pi, \epsilon) - 1\|_\infty > 0 \right\}.$$

THEOREM 7.6. (PEMANTLE AND STEIF 1999) *If T is any tree with bounded degrees and $d \geq 1$, the critical value of β for a robust phase transition in the d -dimensional spherical model on T is the unique value of β satisfying*

$$\frac{\int_{-1}^1 e^{\beta r} (1 - r^2)^{d/2-1} dr}{\int_{-1}^1 r e^{\beta r} (1 - r^2)^{d/2-1} dr} = \text{br}(T).$$

§8. The Contact Process on Trees.

The **contact process** with parameter λ on a graph G is a continuous-time Markov chain ξ_t on 2^V . The subset $\xi_t \subseteq V$ is called the set of **infected** (or **occupied**) sites at time t , while $V \setminus \xi_t$ is the set of **healthy** (or **vacant**) sites. Infected sites wait an exponential time with parameter 1 and then become healthy, while a healthy site becomes infected at a rate equal to λ times the number of its infected neighbors. The measure \mathbf{P}_λ^A is the measure of the above Markov chain when the initial state is $\xi_0 = A$. The contact process is said to go **extinct** if $\mathbf{P}_\lambda^o[\forall t \xi_t \neq \emptyset] = 0$. Otherwise, it **survives**. We make the further distinction that it **survives strongly** (or **survives locally** or is **recurrent**) if

$\mathbf{P}_\lambda^o[\forall T \exists t > T \ o \in \xi_t] > 0$, while it **survives weakly** (or **globally**) if it survives but it does not survive strongly. It is easy to couple two copies of this Markov chain with different parameter values so that the infected sites corresponding to the larger value always contain the infected sites corresponding to the smaller value. Thus, we may define

$$\lambda_1 := \lambda_1(G) := \sup\{\lambda; \mathbf{P}_\lambda^o \text{ goes extinct}\} = \inf\{\lambda; \mathbf{P}_\lambda^o \text{ survives}\}.$$

We also define

$$\lambda_2 := \lambda_2(G) := \inf\{\lambda; \mathbf{P}_\lambda^o \text{ survives strongly}\}.$$

Thus, for any graph, we have $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$.

It is well known and easy to show that $\lambda_1 > 1/d$ on any graph whose degrees are bounded above by d : just dominate the size of the infection started from a single site by a continuous-time branching process with mean λd . However, with rather small tails in the offspring distribution, one can get $\lambda_1 = \lambda_2 = 0$ a.s. on Galton-Watson trees (Pemantle 1992).

It is significantly more difficult to study contact processes on trees than any of the models on trees of the preceding sections. (One way to see why this should be true is to observe that the graphical representation of the contact process on a graph G involves partially oriented percolation on $G \times \mathbb{R}^+$.) Although this section is devoted to trees, we shall briefly discuss other graphs at the end of the section.

The first graph for which it was shown that $0 < \lambda_1 < \lambda_2 < \infty$ was a regular tree:

THEOREM 8.1. *If T is a regular tree of degree at least 3, then $0 < \lambda_1(T) < \lambda_2(T) < \infty$.*

This was proved for trees of degree at least 4 by Pemantle (1992), then for trees of degree 3 by Liggett (1996). Stacey (1996) gave a simpler proof of this result that extends to certain other trees.

The following theorem describes the behavior at the critical values:

THEOREM 8.2. *Let T be a regular tree of degree $b + 1 \geq 3$ and consider the contact process on T .*

- (i) *There is extinction at $\lambda_1(T)$.*
- (ii) *There is weak survival at $\lambda_2(T)$.*

Part (i) was shown by Pemantle (1992) for $b \geq 3$ and by Morrow, Schinazi, and Zhang (1994) for $b = 2$. Part (ii) was proved by Zhang 1996.

A basic duality property of contact processes is that for any $A, B \subset G$, we have

$$\mathbf{P}^A[\xi_t \cap B \neq \emptyset] = \mathbf{P}^B[\xi_t \cap A \neq \emptyset]$$

(see Liggett (1985), Theorem 1.7, Chapter VI). When $\xi_0 = G$, the distribution of ξ_t is stochastically decreasing in time, whence it has a limit, $\bar{\nu}$, called the **upper stationary measure**. The **lower stationary measure** is the probability measure δ_\emptyset concentrated on the empty configuration. From duality, it follows that $\bar{\nu} = \delta_\emptyset$ iff the process goes extinct. One says that **complete convergence** holds if for every initial configuration ξ_0 , the distribution of ξ_t converges to a mixture of the lower and upper stationary measures. In particular, when complete convergence holds, there are no stationary measures other than the lower and upper ones.

The argument of Harris (1976) extends to show that if G is transitive, then the only automorphism-invariant extremal stationary measures are the lower and upper ones. However, there may well be others that are not invariant:

THEOREM 8.3. (DURRETT AND SCHINAZI (1995), ZHANG (1996)) *Let T be a regular tree of degree at least 3. The contact process on T for $\lambda \leq \lambda_1(T)$ has only one stationary measure; for $\lambda_1(T) < \lambda \leq \lambda_2(T)$, it has infinitely many extremal stationary measures; and for $\lambda > \lambda_2(T)$, it has only two extremal stationary measures and complete convergence holds.*

A simpler proof of the last part of Theorem 8.3 was given by Salzano and Schonmann (1997, 1998).

Let $u_n(\lambda)$ be the probability that if the contact process on a regular tree starts with one infected site at o , then a given site x at distance n from o will be infected at some time. It is easy to see that $u_{m+n}(\lambda) \geq u_m(\lambda)u_n(\lambda)$, whence

$$\beta(\lambda) := \lim_{n \rightarrow \infty} u_n(\lambda)^{1/n}$$

exists. Of course, $\beta(\lambda) = 1$ when the process survives strongly. Liggett (1997) conjectured that $\beta(\lambda) \leq 1/\sqrt{b}$ when $\lambda \leq \lambda_2(T)$. This was established by Lalley and Sellke (1998) and the equality case was determined by Lalley (1999):

THEOREM 8.4. *If T is a regular tree of degree $b+1 \geq 3$, then $\beta(\lambda) \leq 1/\sqrt{b}$ for $\lambda \leq \lambda_2(T)$, with equality iff $\lambda = \lambda_2(T)$.*

Theorem 8.4 implies Theorem 8.2(ii). Another proof that $\beta(\lambda) < 1/\sqrt{b}$ for $\lambda < \lambda_2(T)$ was given by Salzano and Schonmann (1998). Theorem 8.4 has the following beautiful consequence for the **limit set** of ξ_t , by which we mean the set of boundary points of T each of whose vertices is infected at some time. We use the same metric on ∂T as in Section 6 for defining Hausdorff dimension on ∂T .

THEOREM 8.5. (LALLEY AND SELLKE (1998), LALLEY (1999)) *If T is a regular tree of degree $b + 1 \geq 3$, then the contact process on T for $\lambda_1(T) < \lambda \leq \lambda_2(T)$ has a limit set on ∂T whose Hausdorff dimension is at most $\frac{1}{2} \log b$ a.s. on the event of survival, with equality iff $\lambda = \lambda_2(T)$.*

Is it the case that $\lambda_1 = \lambda_2$ on amenable transitive graphs and $\lambda_1 \neq \lambda_2$ on nonamenable transitive graphs? It is known that $\lambda_1 = \lambda_2$ on the usual Cayley graphs of \mathbb{Z}^d (Bezuidenhout and Grimmett 1990).

Salzano and Schonmann (1997) give many results for general graphs. In particular, they prove

THEOREM 8.6. *Let G be a graph of bounded degree.*

- (i) *If $\lambda > \lambda_1(\mathbb{Z})$, then the contact process on G survives and has complete convergence. In particular, $\lambda_2(G) \leq \lambda_1(\mathbb{Z}) < \infty$.*
- (ii) *If G is transitive and $\lambda > \lambda_2(G)$, then there are exactly two extremal stationary measures.*

Finally, Schonmann (2000) has proved the existence of two phase transitions on transitive graphs that are sufficiently nonamenable:

THEOREM 8.7. *If G is a transitive graph of degree d with $\iota_E(G)/d \geq 1/\sqrt{2}$, then $0 < \lambda_1(G) < \lambda_2(G) < \infty$.*

However, Pemantle and Stacey (2000) have exhibited nonamenable trees of bounded degree with $\lambda_1 = \lambda_2$.

§9. Biased Random Walks.

Given $\lambda \geq 1$, we define a nearest-neighbor random walk on G denoted RW_λ as follows. Let $\text{deg}_- x$ stand for the number of edges $[x, y]$ with $|y| = |x| - 1$. Then the transition probability from x to an adjacent vertex y is

$$p(x, y) := \begin{cases} \lambda / (\text{deg } x + (\lambda - 1) \text{deg}_- x) & \text{if } |y| = |x| - 1, \\ 1 / (\text{deg } x + (\lambda - 1) \text{deg}_- x) & \text{otherwise.} \end{cases}$$

That is, from any vertex x , each edge connecting x to a vertex closer to o is λ times more likely to be taken than any other edge incident to x . (For $\lambda = 1$, this is simple random walk.) Such random walks were first studied on trees, by Berretti and Sokal (1985), Krug (1988) and Lawler and Sokal (1988).

These biased random walks are reversible and thus correspond to an electrical network on G (see, e.g., Doyle and Snell (1984), Kemeny, Snell, and Knapp (1976), Chapter IX, Section 10, or Lyons (2001)). The conductances are given by

$$C(x, y) := \lambda^{-(|x| \wedge |y|)},$$

where x and y are adjacent vertices.

THEOREM 9.1. (LYONS 1995) *Let G be a transitive graph. If $\lambda < \text{gr}(G)$, then RW_λ is transient, while if $\lambda > \text{gr}(G)$, then RW_λ is recurrent. Equivalently, if $\lambda < \text{gr}(G)$, then the effective conductance from o to infinity is positive, but not if $\lambda > \text{gr}(G)$.*

One may also consider the **rate of escape** of RW_λ from o when $\lambda < \text{gr}(G)$, i.e., $\lim_{n \rightarrow \infty} |X_n|/n$, where X_n is the location of the random walk at time n . There are Cayley graphs with $\text{gr}(G) > 1$ but that have the surprising property that the rate of escape of simple random walk is 0. One example is the “lamplighter” group denoted G_1 by Kaĭmanovich and Vershik (1983). For this example, Lyons, Pemantle, and Peres (1996) showed that the rate of escape of RW_λ is positive when $1 < \lambda < \text{gr}(G_1)$. This lack of monotonicity of behavior is quite unusual for models on transitive graphs. It might be that for every transitive graph G , the rate of escape of RW_λ is positive as long as $1 < \lambda < \text{gr}(G)$.

The method of proof of Theorem 9.1 uses a corresponding result on trees:

THEOREM 9.2. (LYONS 1990) *Let T be any tree. If $\lambda < \text{br}(T)$, then RW_λ is transient, while if $\lambda > \text{br}(T)$, then RW_λ is recurrent.*

The critical case in Theorem 9.2 is decided by a capacity criterion:

THEOREM 9.3. (LYONS 1990) *Let T be any tree and $\lambda \geq 1$. Then RW_λ is transient iff there is a probability measure μ on ∂T such that*

$$\iint \sum_{n=0}^{|\xi_1 \wedge \xi_2|} \lambda^n d\mu(\xi_1) d\mu(\xi_2) < \infty.$$

If T is spherically symmetric, then this is equivalent to

$$\sum_{n \geq 1} \frac{\lambda^n}{|T_n|} < \infty.$$

Of course, when T is spherically symmetric, this reduces to a random walk on \mathbb{N} and is well known.

An interesting model for which few results are known is that of **edge-reinforced random walk** X_n ($n \geq 0$) on a graph G with parameter λ . We begin with weights on all edges equal to 1. If $X_n = x$, then $[X_n, X_{n+1}]$ is an edge incident to x chosen with probability proportional to the weights at time n of the edges incident to x . The weights of the edges at time $n + 1$ are the same as those at time n except that the weight of $[X_n, X_{n+1}]$ is increased by λ . We call edge-reinforced random walk **recurrent** if it returns to its starting position infinitely often a.s. and **transient** if it returns to its starting position only finitely often a.s. It seems reasonable to suppose that as λ increases, the walk goes from transient to recurrent as long as G is nonamenable. The existence and location of a phase transition was completely solved on trees by Pemantle (1988) for regular and Galton-Watson trees and by Lyons and Pemantle (1992) in general:

THEOREM 9.4. *There is a strictly increasing function $\lambda_E : [1, \infty) \rightarrow [0, \infty)$ with $\lambda_E(1) = 0$ such that if T is any tree, then edge-reinforced random walk on T is transient for $\lambda < \lambda_E(\text{br}(T))$ and is recurrent for $\lambda > \lambda_E(\text{br}(T))$.*

§10. Directions of Current Research.

We outline some of the themes that characterize much research in nonamenable phase transitions and highlight some of the most important open questions.

One contemporary theme in geometry and combinatorial group theory is the investigation of rough-isometry invariants (see, e.g., Gromov (1999, 1993)). Here, a map $\phi : (X, d) \rightarrow (X', d')$ between metric spaces is called a **rough isometry** (or **quasi-isometry**) if there are positive constants a and b such that for all $x, y \in X$,

$$ad(x, y) - a \leq d'(\phi(x), \phi(y)) \leq bd(x, y) + b$$

and such that every point in X' is within distance b of the image of X . Being roughly isometric is an equivalence relation.

In the context of graphs, we use the usual graph distance as the metric on the vertex set. As an example, it is easy to see that different Cayley graphs of the same group are roughly isometric. What properties of our models are invariant under rough isometry? For example, in the context of Bernoulli percolation, is $p_c(G) < p_u(G)$ invariant when G is a transitive graph? If so, Theorem 3.10 would solve Conjecture 3.8 for Cayley graphs. As another example, are critical exponents invariant under rough isometry? (However, they may turn out to be the same for all nonamenable transitive graphs.) Potential-theoretic

rough-isometry invariants are known, but no nontrivial ones are known in percolation theory.

If we specialize from rough isometries to changing generators for a fixed group, we encounter a more refined sort of question having to do with uniform properties: For example, it is easy to see that if Γ is any finitely generated group, then $\inf_S p_c(G(\Gamma, S)) = 0$, where the infimum is over all finite generating sets of Γ and $G(\Gamma, S)$ denotes the Cayley graph of Γ with respect to S . But is $\sup_S p_c(G(\Gamma, S)) < 1$? This would follow for groups of exponential growth from Theorem 3.1 if it were known that $\inf_S \text{gr}(G(\Gamma, S)) > 1$, but this latter is an open question (see Grigorchuk and de la Harpe (1997) for what is known about this growth problem). No nontrivial uniform properties are known at present for, say, all nonamenable groups.

In the other direction, rather than specializing rough isometries, we may enlarge our equivalence classes from roughly isometric to various classes of groups, such as nonamenable, word hyperbolic (see Gromov (1987) or Coornaert and Papadopoulos (1993)), or Kazhdan [although this last is not known to be invariant under rough isometries]. Thus, we may search for characterizations of these classes of groups through Bernoulli percolation or through other models, similar to Theorem 3.10 (in combination with the theorem of Burton and Keane (1989) and Gandolfi, Keane, and Newman (1992)) or Theorems 5.3, 5.4, and 5.5. Characterizations via invariant percolation, such as Theorem 4.3, would also be interesting. As the astute reader will have observed, all of these characterizations are of amenability only. Particularly interesting would be a characterization of hyperbolicity. An important probabilistic characterization of Kazhdan groups, though abstract from our point of view, is given by Glasner and Weiss (1997).

Another geometric theme concerns the appearance of spherical symmetry. Transitive graphs are almost never spherically symmetric, i.e., it is rare for a transitive graph to have the property that if $|x| = |y|$, then there is an automorphism fixing o that carries x to y . This lack of spherical symmetry can manifest itself in probabilistic models. As one clear example, $\tau_p(o, x)$ can decay to 0 as $|x| \rightarrow \infty$ in some directions while not decaying to 0 in other directions (on a given graph); see Lyons and Schramm (2000) for a Cayley graph with this property. What other results show the lack of spherical symmetry? On the other hand, Theorem 9.1 has a conclusion that holds for all spherically symmetric graphs: Here, the lack of spherical symmetry does not affect the critical value of λ . Are there other results where one might expect the lack of spherical symmetry to play a role, yet where it does not? For example, it was suggested in Section 9 that for all transitive graphs, the rate of escape of RW_λ is positive as long as $1 < \lambda < \text{gr}(G)$.

Aside from the geometrically motivated questions above, there are a plethora of purely

probabilistic questions. The possibilities for presence or absence of various phase transitions of random cluster and Potts models are barely understood. The results for contact processes that are known for trees need to be examined for transitive graphs. Except for branching random walks, other interacting particle systems have barely been investigated.

For example, we often lack monotonicity results (such as Proposition 5.2) for processes other than Bernoulli percolation. In fact, some such results are known to fail on quasi-transitive graphs (see Brightwell, Häggström, and Winkler (1999), for example), although there are no known comparable failures on transitive graphs.

Finally, some of the most basic open questions for Bernoulli percolation are: Is $p_c(G) < p_u(G)$ when G is a nonamenable transitive graph? Is $p_u(G) < 1$ when G is a nonamenable transitive graph with one end? Are Theorems 3.7(ii), 3.11, and 3.13 valid in the nonunimodular case? Which transitive graphs have a unique infinite cluster at p_u ? What other types of phase transition are there, such as discontinuities of $\tau_p(o, x)$ as a function of p for fixed x ?

In most situations, planar graphs are much easier to analyze due to the availability of duality. We expect considerably faster progress for planar graphs than for general transitive graphs.

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