ORDINAL RANKINGS ON MEASURES
ANNIHILATING THIN SETS

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ABSTRACT. We assign a countable ordinal number to each probability measure which annihilates all H-sets. The descriptive-set theoretic structure of this assignment allows us to show that this class of measures is coanalytic non-Borel. In addition, it allows us to quantify the failure of Rajchman's conjecture. Similar results are obtained for measures annihilating Dirichlet sets.

A closet subset E of the unit circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is called an H-set if there exists a sequence $\{n_k\}$ of positive integers tending to ∞ and an interval (i.e., a nonempty open arc) $I \subseteq \mathbf{T}$ such that for all k and all $x \in E$, $n_k x \notin I$. These sets play a fundamental role as examples of sets of uniqueness for trigonometric series [KL; Z, Chapters IX, XII]. A (Borel) probability measure μ on $\mathbf{T}$ is called a *Rajchman measure* if $\hat{\mu}(n) \to 0$ as $|n| \to \infty$, where $\hat{\mu}(n) = \int_T e(-nx)\,d\mu(x)$, $e(x) = e^{2\pi i x}$. We denote by R the class of such measures. These measures have also been very important to the study of sets of uniqueness. In particular, every Rajchman measure annihilates every set of uniqueness, hence every H-set. After establishing these relationships [R1, R2], Rajchman conjectured that, in fact, the only measures which annihilate all H-sets are those in R. This, however, is false [L1, L2, L3, L5]. Here, we shall quantify how distant Rajchman's conjecture is from the truth.

Given a class $\mathscr{C}$ of closed subsets of $\mathbf{T}$, denote by $\mathscr{C}^\perp$ the class of probability measures on $\mathbf{T}$ which annihilate all sets in $\mathscr{C}$: $\mu \in \mathscr{C}^\perp \Leftrightarrow \forall E \in \mathscr{C}\ (\mu(E) = 0)$. Thus $R \subsetneqq H^\perp$, where H denotes the class of H-sets. Denote by PROB($\mathbf{T}$) the compact, metrizable space of (Borel) probability measures on $\mathbf{T}$ with the weak* topology. It is easy to check that R is a Borel, in fact $\mathbf{\Pi}_3^0$ (i.e., $F_{\sigma\delta}$), subset of this space. We establish in §3 that $H^\perp$ is a $\mathbf{\Pi}_1^1$ (i.e., coanalytic) but not Borel subset of PROB($\mathbf{T}$). This is the first example of a natural class of measures of such complexity known to the authors and it highlights the distinction between R and $H^\perp$.

Our method of proof actually provides quite a bit of further information on the relationship between R and $H^\perp$. In §1, we assign to each $\mu \in H^\perp$ a countable ordinal number $h(\mu)$ which measures in some sense the complexity of the verification that μ annihilates all H-sets. We show that h has certain definability properties, namely, it is a $\mathbf{\Pi}_1^1$-rank (see [KL]). In §3, using the techniques developed in [L3 and L4], we prove that the rank h is unbounded in ω_1, the first uncountable ordinal;

that is, $\forall \alpha < \omega_1 \; \exists \mu \in H^\perp \; (h(\mu) \geq \alpha)$. It follows from the Boundedness Theorem for $\mathbf{\Pi}_1^1$-ranks (see [**KL**, V.1]) that $H^\perp$ is not Borel.

Denote by $H_\alpha^\perp$ the class of all $\mu \in H^\perp$ with $h(\mu) \leq \alpha$. Thus $H_1^\perp \subseteq H_2^\perp \subseteq \cdots \subseteq H_\alpha^\perp \subseteq \cdots \subseteq H_\beta^\perp \ldots \; (\alpha < \beta < \omega_1)$, $H^\perp = \bigcup_{\alpha < \omega_1} H_\alpha^\perp$ and $H_\alpha^\perp \subsetneq H^\perp$ for each $\alpha < \omega_1$, so that $\{H_\alpha^\perp\}$ is a stratification of $H^\perp$ in a hierarchy of increasing complexity. The "simplest" measures in $H^\perp$ are those in $H_1^\perp$. In §2, we prove that these include the Rajchman measures: $R \subseteq H_1^\perp$. This bound gives quantitative meaning to the statement that R is only a small part of $H^\perp$. We also show that another canonical class of measures in $H^\perp$, the so-called quasisymmetric measures, belong to $H_\omega^\perp$. We do not know if they belong to $H_1^\perp$. An interesting consequence of these upper bounds is that the techniques in [**L3**] for proving measures to be in $H^\perp$ are demonstrated to be more powerful in a quantitative sense than the traditional techniques used (and amplified) in §2.

We conclude this paper by establishing, in §4, analogous results for the class $D^\perp$ of probability measures annihilating all Dirichlet sets. (Recall that a closed set $E \subseteq \mathbf{T}$ is a *Dirichlet set* [**LP**] if there is a sequence of positive integers $\{n_k\}$ tending to ∞ such that $\sup_{x \in E} \|n_k x\| \to 0$, where $\|x\| = \text{dist}(x, \mathbf{Z})$.) $D^\perp$ has been studied before under the name $\mathscr{L}_I$ (see [**HMP**, pp. 212–215, 242–247]).

NOTE. After completion of this manuscript, we learned that B. Host, A. Louveau, and F. Parreau established several months earlier than us that $D^\perp$ is not Borel; they used the characterization of $D^\perp$ given on p. 243 of [**HMP**]. Their work is unpublished.

1. A $\mathbf{\Pi}_1^1$-rank on $H^\perp$. Let $K(\mathbf{T})$ be the space of closed subsets of $\mathbf{T}$ with the standard Hausdorff metric. This is a compact space. As a subset of $K(\mathbf{T})$, it is verified in [**KL**, IV.2.7] that H is $\mathbf{\Sigma}_3^0$ (i.e., $G_{\delta\sigma}$).

We next compute the following upper bound for the complexity of $H^\perp$.

PROPOSITION 1.1. *The set $H^\perp$ is $\mathbf{\Pi}_1^1$ (i.e., coanalytic) in the space $\text{PROB}(\mathbf{T})$.*

PROOF. Since H is Borel and

$$\mu \in H^\perp \Leftrightarrow \forall E \in K(\mathbf{T}) \quad [E \in H \Rightarrow \mu(E) = 0]$$

for $\mu \in \text{PROB}(\mathbf{T})$, it is enough to show that the relation

$$\{(\mu, E) : \mu \in \text{PROB}(\mathbf{T}) \,\&\, E \in K(\mathbf{T}) \,\&\, \mu(E) = 0\}$$

is Borel (in fact, G_δ) in the space $\text{PROB}(\mathbf{T}) \times K(\mathbf{T})$. Now, if $\{V_n\}$ is an open basis for the topology of $\mathbf{T}$ which is closed under finite unions, then for $\mu \in \text{PROB}(\mathbf{T})$ and $E \in K(\mathbf{T})$,

$$\mu(E) = 0 \Leftrightarrow \forall m \exists n \left(E \subseteq V_n \,\&\, \mu(\mathbf{T} \backslash \overline{V}_n) > \frac{m-1}{m} \right).$$

Since $\{E : E \subseteq V\}$ is open in $K(\mathbf{T})$ for each open $V \subseteq \mathbf{T}$, while $\{\mu : \mu(V) > a\}$ is open in $\text{PROB}(\mathbf{T})$ for each $a \in \mathbf{R}$ and V open in $\mathbf{T}$, the result follows. $\square$

Our main result is that $H^\perp$ is not Borel, thus determining exactly the complexity of $H^\perp$.

We will use below notation concerning finite sequences and trees as in [**KL**, IV.1]. We denote by $\mathbf{N}^*$ the set of positive integers. To define our rank $h : H^\perp \to \omega_1$, we

associate first to each $\mu \in \mathrm{PROB}(\mathbf{T})$, each I which is a rational interval in $\mathbf{T}$, and each $\varepsilon \in \mathbf{Q}^+ = \{x \in \mathbf{Q} : x > 0\}$, a tree $T_\mu^{I,\varepsilon}$ on $\mathrm{Seq}\, \mathbf{N}^*$ as follows:

$$T_\mu^{I,\varepsilon} = \{(n_0, n_1, \ldots, n_k) : \forall i \le k(n_i \in \mathbf{N}^*) \& \forall i < k(n_{i+1}/n_i \ge 2) \&$$
$$\mu(\{x \in \mathbf{T} : \forall i \le k(n_i x \notin I)\}) \ge \varepsilon\}.$$

Notice that if $I \supseteq I'$ and $\varepsilon \ge \varepsilon'$, then $T_\mu^{I,\varepsilon} \subseteq T_\mu^{I',\varepsilon'}$.

REMARK. The condition "$n_{i+1}/n_i \ge 2$", instead of the expected one, "$n_{i+1} > n_i$", has the effect of thinning down the trees for normalization purposes. For example, in Theorem 2.4 below, we obtain that $h(\mu) = 1$ for Rajchman measures, rather than $h(\mu) \le \omega$.

PROPOSITION 1.2. *Let $\mu \in \mathrm{PROB}(\mathbf{T})$. Then $\mu \in H^\perp \Leftrightarrow \forall I\ \forall \varepsilon (T_\mu^{I,\varepsilon}$ is well founded).*

PROOF. If for some I, ε, $T_\mu^{I,\varepsilon}$ has an infinite branch $n_0, n_1, n_2, \ldots$, then $\mu(E) \ge \varepsilon > 0$ for $E = \{x \in \mathbf{T} : \forall i(n_i x \notin I)\}$. As $E \in H$, we have $\mu \notin H^\perp$. Conversely, if $\mu \notin H^\perp$, let $\varepsilon \in \mathbf{Q}^+$, $E \in H$ be such that $\mu(E) \ge \varepsilon$ and let $0 < n_0 < n_1 < \cdots$ and I be such that $n_k x \notin I$ for all $k \in \mathbf{N}$, $x \in E$. By going to a subsequence, we can assume that $n_{i+1}/n_i \ge 2$ for all i. Then $(n_0, n_1, \ldots, n_k) \in T_\mu^{I,\varepsilon}$ for all k, i.e., $T_\mu^{I,\varepsilon}$ is not well founded. $\square$

For each well-founded tree T, we define its height, $\mathrm{ht}(T)$, as follows. First, for each $s \in T$, define its height in T, $\mathrm{ht}(s, T)$, by

$\mathrm{ht}(s, T) = 0$ if s is terminal in T, i.e., has no proper extension in T,

$\mathrm{ht}(s, T) = \sup\{\mathrm{ht}(s^\wedge(y), T) + 1 : s^\wedge(y) \in T\}$

$\qquad = \sup\{\mathrm{ht}(t, T) + 1 : t \in T, t \supsetneq s\}$ otherwise.

Then let $\mathrm{ht}(T) = \mathrm{ht}(\varnothing, T)$. (This definition is slightly different from that used in [**KL**, V.1], where one defines $\mathrm{ht}(s, T) = 1$ if s is terminal in T.)

For each $\mu \in H^\perp$, define the rank function $h'(\mu) = \sup\{\mathrm{ht}(T_\mu^{I,\varepsilon}) + 1 : I$ a rational interval and $\varepsilon \in \mathbf{Q}^+\}$. Clearly $h' : H^\perp \to \omega_1$. We will show first that $h'(\mu)$ is always a limit ordinal, thus of the form $\omega \cdot \alpha$ for some $\alpha < \omega_1$. We will then define $h(\mu)$ by $h'(\mu) = \omega \cdot h(\mu)$.

LEMMA 1.3. *For each $\mu \in H^\perp$, $h'(\mu)$ is a limit ordinal.*

PROOF. First notice that $h'(\mu) \ge \omega$ for all $\mu \in H^\perp$. Indeed, given any $0 < n_0 < n_1 < \cdots < n_k$ with $n_{i+1}/n_i \ge 2$, one can easily find I and ε such that $(n_0, \ldots, n_k) \in T_\mu^{I,\varepsilon}$. To complete the proof, it is enough to show that if $\mu \in H^\perp$, $h'(\mu) > \omega \cdot \alpha$ $(\alpha \ge 1)$, and $N \in \mathbf{N}^*$, then $h'(\mu) \ge \omega \cdot \alpha + N$. Since $h'(\mu) > \omega \cdot \alpha$, there are I and ε with $\mathrm{ht}(T_\mu^{I,\varepsilon}) \ge \omega \cdot \alpha$. Fix next a sequence $0 < n_0 < \cdots < n_{N-1}$ with $n_{i+1}/n_i \ge 2$ and a rational interval $I' \subseteq I$ such that $\mu(\{x : \forall i \le N - 1\ (n_i x \notin I')\}) \ge 1 - \varepsilon/2$.

We claim that if $S = \{(n)^\wedge t \in T_\mu^{I,\varepsilon} : n \ge 2n_{N-1}\} \cup \{\varnothing\}$, then $\mathrm{ht}(S) \ge \omega \cdot \alpha$. This is because for some $s \in T_\mu^{I,\varepsilon}$, we have $\mathrm{ht}(s, T_\mu^{I,\varepsilon}) = \omega \cdot \alpha$ and thus $\sup\{\mathrm{ht}(s^\wedge(n), T_\mu^{I,\varepsilon}) + 1\} = \omega \cdot \alpha$. Since the supremum is not attained and $s^\wedge(n)^\wedge t \in T_\mu^{I,\varepsilon} \Rightarrow (n)^\wedge t \in T_\mu^{I,\varepsilon}$, it follows that $\mathrm{ht}(S) \ge \omega \cdot \alpha$.

Now let

$$T = \{(n_0, \ldots, n_{N-1})^\wedge s : s \in S\}.$$

Clearly $\mathrm{ht}(T) \geq \omega \cdot \alpha + N$. We claim that $T \subseteq T_\mu^{I',\varepsilon/2}$, thus $\mathrm{ht}(T_\mu^{I,\varepsilon}) \geq \omega \cdot \alpha + N$ and we are done. Evidently, every element of a sequence $(n_0, \ldots, n_{N-1})^\wedge s \in S$ is at least twice as big as its predecessor. Also, we have

$$\mu(\{x: n_0 x \notin I' \& \cdots \& n_{N-1} x \notin I')\} \geq 1 - \varepsilon/2$$

and, if $(n_0, \ldots, n_{N-1}, n_N, \ldots, n_{N+m}) \in T$, then

$$\mu(\{x: n_N x \notin I' \& \cdots \& n_{N+m} x \notin I'\}$$
$$\geq \mu(\{x: n_N x \notin I \& \cdots \& n_{N+m} x \notin I\}) \geq \varepsilon.$$

Therefore

$$\mu(\{x: n_0 x \notin I' \& \cdots \& n_{N+m} x \notin I'\}) \geq \varepsilon/2$$

and our proof is complete. $\square$

As mentioned before, we may now define $h(\mu)$ for $\mu \in H^\perp$ by $h'(\mu) = \omega \cdot h(\mu)$. The following fact establishes the basic definability properties of this rank.

PROPOSITIN 1.4. *The rank* $h: H^\perp \to \omega_1$ *is a* $\mathbf{\Pi}_1^1$-*rank on the* $\mathbf{\Pi}_1^1$ *set* $H^\perp$.

PROOF. Recall that if X is a Polish space, $P \subseteq X$ is a $\mathbf{\Pi}_1^1$ set in X and $\varphi: P \to \omega_1$ is a rank on P, then we say φ is a $\mathbf{\Pi}_1^1$-rank if, letting $\varphi(x) = \omega_1$ for all $x \notin P$, we have that the relations

$$x \leq_\varphi^* y \Leftrightarrow x \in P \& \varphi(x) \leq \varphi(y),$$
$$x <_\varphi^* y \Leftrightarrow x \in P \& \varphi(x) < \varphi(y) \qquad (\Leftrightarrow \varphi(x) < \varphi(y))$$

are $\mathbf{\Pi}_1^1$ (as subsets of X^2).

A typical example of a $\mathbf{\Pi}_1^1$-rank is the following (see [**KL**, V.1]): Identifying trees on $\mathrm{Seq}\,\mathbf{N}^*$ with their characteristic functions, we can view them as members of the space $2^{\mathrm{Seq}\,\mathbf{N}^*}$ ($= \{0,1\}^{\mathrm{Seq}\,\mathbf{N}^*}$). Let $WF \subseteq 2^{\mathrm{Seq}\,\mathbf{N}^*}$ be the set of well-founded trees on $\mathrm{Seq}\,\mathbf{N}^*$. Then WF is $\mathbf{\Pi}_1^1$. Moreover, the rank $T \mapsto \mathrm{ht}(T)$ is a $\mathbf{\Pi}_1^1$-rank on WF.

Going back to h now, notice that for $\mu, \nu \in H^\perp$, $h(\mu) \leq h(\nu) \Leftrightarrow h'(\mu) \leq h'(\nu)$, so it is enough to show that h' is a $\mathbf{\Pi}_1^1$-rank on $H^\perp$. Again let $h'(\mu) = \omega_1$ if $\mu \notin H^\perp$. Then

$$\mu \leq_{h'}^* \nu \Leftrightarrow \mu \in H^\perp \& h'(\mu) \leq h'(\nu)$$
$$\Leftrightarrow \mu \in H^\perp \& \forall I, \varepsilon \, \exists J, \delta \, [\mathrm{ht}(T_\mu^{I,\varepsilon}) \leq \mathrm{ht}(T_\nu^{J,\delta})]$$
$$\Leftrightarrow \mu \in H^\perp \& \forall I, \varepsilon \, \exists J, \delta \, [T_\mu^{I,\varepsilon} \in WF \& \mathrm{ht}(T_\mu^{I,\varepsilon}) \leq \mathrm{ht}(T_\nu^{J,\delta})]$$

and

$$\mu <_{h'}^* \nu \Leftrightarrow \mu \in H^\perp \& h'(\mu) < h'(\nu)$$
$$\Leftrightarrow \mu \in H^\perp \& \exists J, \delta \, \forall I, \varepsilon \, [T_\mu^{I,\varepsilon} \in WF \& \mathrm{ht}(T_\mu^{I,\varepsilon}) < \mathrm{ht}(T_\nu^{J,\delta})].$$

Since $\mathbf{\Pi}_1^1$ sets are closed under countable intersections and unions as well as Borel pre-images and ht is a $\mathbf{\Pi}_1^1$-rank on WF, it is enough to show that for each *fixed* pair I, ε, the map $\mu \mapsto T_\mu^{I,\varepsilon}$ is a Borel map from $\mathrm{PROB}(\mathbf{T})$ into $2^{\mathrm{Seq}\,\mathbf{N}^*}$. Since $2^{\mathrm{Seq}\,\mathbf{N}^*}$ has the product topology, this reduces to showing that for each fixed $s = (n_0, \ldots, n_k)$ with $2n_i \leq n_{i+1}$, the set

$$\{\mu \in \mathrm{PROB}(\mathbf{T}): s \in T_\mu^{I,\varepsilon}\} = \{\mu \in \mathrm{PROB}(\mathbf{T}): \mu(F) \geq \varepsilon\},$$

where $F = \bigcap_{i=0}^{k}\{x \colon n_i x \notin I\}$, is a Borel set. Since F is closed, let $F = \bigcap_n V_n$, V_n decreasing and open in $\mathbf{T}$. Then

$$\mu(F) \geq \varepsilon \Leftrightarrow \forall \varepsilon' < \varepsilon \; \forall n \; (\mu(V_n) > \varepsilon'),$$

so actually this set is a G_δ and we are done. $\square$

2. Measures of low rank. Let $A = \{f \in C(\mathbf{T}) \colon \hat{f} \in l^1(\mathbf{Z})\}$ with $\|f\|_A = \|\hat{f}\|_{l^1}$. For $f \in A$, $\mu \in R$, and $\varepsilon > 0$, define

$$N(f,\varepsilon) = \min\left\{ N \geq 0 \colon \sum_{|n|>N} |\hat{f}(n)| < \varepsilon \right\}$$

and

$$N'(\mu,\varepsilon) = \min\{N \geq 0 \colon |n| > N \Rightarrow |\hat{\mu}(n)| < \varepsilon\}.$$

We write $(T_m f)(x) = f(mx)$. Thus, $\|T_m f\|_A = \|f\|_A$ and $N(T_m f, \varepsilon) = mN(f, \varepsilon)$ for $m \geq 1$.

LEMMA 2.1. *For $f, g \in A$ and $\varepsilon > 0$,*

$$N(fg, \varepsilon) \leq N\left(f, \frac{\varepsilon}{2\|g\|_A}\right) + N\left(g, \frac{\varepsilon}{2\|f\|_A}\right).$$

PROOF. Denote the terms on the right by N_1 and N_2. If $|k + l| > N_1 + N_2$, then $|k| > N_1$ or $|l| > N_2$. Therefore

$$\sum_{|n|>N_1+N_2} |\widehat{fg}(n)| = \sum_{|n|>N_1+N_2} \left| \sum_{k+l=n} \hat{f}(k)\hat{g}(l) \right|$$

$$\leq \sum_{\substack{|k|>N_1 \\ l\in\mathbf{Z}}} |\hat{f}(k)\hat{g}(l)| + \sum_{\substack{|l|>N_2 \\ k\in\mathbf{Z}}} |\hat{f}(k)\hat{g}(l)| < \varepsilon. \quad \square$$

LEMMA 2.2. *For $f, g \in A$, $\mu \in R$, and*

$$n > 2\max\left(N\left(f, \frac{\varepsilon}{2\|g\|_A}\right), N'\left(\mu, \frac{\varepsilon}{2\|f\|_A\|g\|_A}\right) \right),$$

we have

$$\left| \int f \cdot T_n g \, d\mu - \int f \, d\mu \cdot \hat{g}(0) \right| < \varepsilon.$$

PROOF. Indeed, we have

$$\left| \int f \cdot T_n g \, d\mu - \int f \, d\mu \cdot \hat{g}(0) \right| = \left| \sum_{r\neq 0} \hat{g}(-r) \sum_l \hat{f}(-l)\hat{\mu}(nr+l) \right|$$

$$\leq \sum_{r\neq 0} |\hat{g}(-r)| \left\{ \sum_{|l|\leq n/2} |\hat{f}(-l)\hat{\mu}(nr+l)| + \sum_{|l|>n/2} |\hat{f}(-l)\hat{\mu}(nr+l)| \right\}$$

$$< \varepsilon$$

since if $|l| \leq n/2$ and $r \neq 0$, $|nr + l| \geq n/2$. $\square$

LEMMA 2.3. *If $\mu \in R$, $f \in A$, $|\hat{f}(0)| \leq 1$, $\|f\|_A \geq 1$, $K \geq 2$, $m_1, m_2, \ldots, m_K \in$ N*,*

$$\frac{m_{j+1}}{m_j} \geq 2 \max \left(KN \left(f, \frac{\varepsilon}{2^{K-1} K \|f\|_A^{K-1}} \right), N' \left(\mu, \frac{\varepsilon}{2K \|f\|_A^K} \right) \right)$$

for $1 \leq j < K$, then

$$\left| \int \prod_{j=1}^K T_{m_j} f \, d\mu - \hat{f}(0)^K \right| < \varepsilon.$$

PROOF. Let $F_k = \prod_{j=1}^k T_{m_j} f$. We shall show that

$$(2.1) \qquad \left| \int F_{k+1} \, d\mu - \int F_k \, d\mu \cdot \hat{f}(0) \right| < \frac{\varepsilon}{K}$$

for $0 \leq k < K$, where $F_0 \equiv 1$. Multiplying both sides by $|\hat{f}(0)|^{K-k-1}$ and adding over k yields the desired inequality. Now (2.1) will follow if

$$m_{k+1} \geq 2 \max \left(N \left(F_k, \frac{\varepsilon}{2K \|f\|_A} \right), N' \left(\mu, \frac{\varepsilon}{2K \|f\|_A^{k+1}} \right) \right).$$

That m_{k+1} is at least twice the second term follows immediately from the hypothesis. Thus, it suffices to show that

$$m_k K N \left(f, \frac{\varepsilon}{2^{K-1} K \|f\|_A^{K-1}} \right) \geq N \left(F_k, \frac{\varepsilon}{2K \|f\|_A} \right).$$

By repeated application of Lemma 2.1, we obtain

$$N \left(F_k, \frac{\varepsilon}{2K \|f\|_A} \right) \leq N \left(F_{k-1}, \frac{\varepsilon}{4K \|f\|_A^2} \right) + N \left(T_{m_k} f, \frac{\varepsilon}{4K \|f\|_A \cdot \|F_{k-1}\|_A} \right)$$

$$\leq \cdots \leq N \left(T_{m_1} f, \frac{\varepsilon}{2^k K \|f\|_A^k} \right) + N \left(T_{m_2} f, \frac{\varepsilon}{2^k K \|f\|_A^k} \right)$$

$$+ N \left(T_{m_3} f, \frac{\varepsilon}{2^{k-1} K \|f\|_A^k} \right) + \cdots + N \left(T_{m_k} f, \frac{\varepsilon}{4K \|f\|_A^k} \right)$$

$$\leq k m_k N \left(f, \frac{\varepsilon}{2^k K \|f\|_A^k} \right).$$

This implies the desired inequality. $\square$

THEOREM 2.4. *If $\mu \in R$, then $h(\mu) = 1$; i.e., $R \subseteq H_1^\perp$.*

PROOF. We shall show that $h'(\mu) \leq \omega$. Let I be a rational interval and $\varepsilon \in \mathbf{Q}^+$. Choose $f \in A$ so that $0 \leq f \leq 1$, $f = 1$ on I^c, and $\hat{f}(0) < 1$. Then $\|f\|_A \geq 1$ and $\mathbf{1}_{I^c} \leq f$. Choose K so that $\hat{f}(0)^K < \varepsilon/2$ and choose

$$L \geq 2K \log_2 \max \left(2KN \left(f, \frac{\varepsilon}{2^K K \|f\|_A^{K-1}} \right), 2N' \left(\mu, \frac{\varepsilon}{4K \|f\|_A^K} \right) \right).$$

We claim that $\operatorname{ht}(T_\mu^{I,\varepsilon}) < L$. Indeed, if $(m_0', \ldots, m_{L-1}') \in T_\mu^{I,\varepsilon}$, then there is a subsequence $\{m_j\}_{j=1}^K$ of $\{m_i'\}_{i=0}^{L-1}$ such that

$$\frac{m_{j+1}}{m_j} \geq 2 \max \left(KN \left(f, \frac{\varepsilon}{2^K K \|f\|_A^{K-1}} \right), N' \left(\mu, \frac{\varepsilon}{4K \|f\|_A^K} \right) \right).$$

Thus

$$\mu\{x: \forall i < L(m_i' x \notin I)\} \leq \int \prod_{j=1}^{K} \mathbf{1}_{I^c}(m_j x)\, d\mu(x)$$

$$\leq \int \prod_{1}^{K} f(m_j x)\, d\mu(x) < \hat{f}(0)^K + \frac{\varepsilon}{2} < \varepsilon,$$

a contradiction. $\square$

Let us say that a probability measure μ on $\mathbf{T}$ is *quasisymmetric*, or $\mu \in QS$, if for some C, whenever I_1 and I_2 are adjacent intervals on $\mathbf{T}$ of the same length,

$$\mu I_1 \leq C \cdot \mu I_2.$$

(These measures are related to quasiconformal mappings; see [**BA**].) Note that every $\mu \in QS$ is continuous. Examples include Riesz products (for background, see [**K**, p. 107])

$$\mu = \prod_{k \geq 1}(1 + \mathrm{Re}\{\alpha_k e(n_k x)\})$$

with $\overline{\lim}\,|\alpha_k| < 1$, $n_{k+1}/n_k \geq 3$, and either $n_k | n_{k+1}$ or $n_{k+1}/n_k \to \infty$; we omit the proof. We do not know whether $QS \subseteq H_1^\perp$, but we now show that $QS \subseteq H_\omega^\perp$.

THEOREM 2.5. $QS \subseteq H_\omega^\perp$.

PROOF. Let $\mu \in QS$, $I \subseteq \mathbf{T}$ be a rational interval, and $\varepsilon \in \mathbf{Q}^+$. Denote $T_n^{-1} I = \{x: nx \in I\}$. Then $T_n^{-1} I$ consists of n equally spaced intervals separated by the n intervals of $T_n^{-1} I^c$; the ratio of the lengths of adjacent intervals is $|I|/|I^c|$ and therefore the ratio of their μ-measures is bounded by a constant, C', independent of n. It follows that if A is a finite union of intervals, then

$$\mu(A \cap T_n^{-1} I^c) \leq 2C' \mu(A \cap T_n^{-1} I)$$

for all large n. Let K be such that $(2C'/(1+2C'))^K < \varepsilon$. We claim that $\mathrm{ht}(T_\mu^{I,\varepsilon}) < \omega \cdot K$, which will complete the proof.

Indeed, if $\mathrm{ht}(T_\mu^{I,\varepsilon}) \geq \omega \cdot K$, then there exist arbitrarily large m such that $\mathrm{ht}((m), T_\mu^{I,\varepsilon}) > \omega \cdot (K-1)$. (Note that $s^\wedge t^\wedge u^\wedge v \in T_\mu^{I,\varepsilon} \Rightarrow t^\wedge v \in T_\mu^{I,\varepsilon}$.) Choose m_0 such that

$$\mathrm{ht}((m_0), T_\mu^{I,\varepsilon}) > \omega \cdot (K-1)$$

and

$$\mu(T_{m_0}^{-1} I^c) \leq 2C' \mu(T_{m_0}^{-1} I).$$

In a similar fashion, we may find inductively m_k $(1 \leq k \leq K-1)$ such that

$$\mathrm{ht}((m_0, m_1, \ldots, m_k), T_\mu^{I,\varepsilon}) > \omega \cdot (K-1-k)$$

and

(2.2) $$\mu(E_{k-1} \cap T_{m_k}^{-1} I^c) \leq 2C' \mu(E_{k-1} \cap T_{m_k}^{-1} I),$$

where

$$E_k = \bigcap_{i=0}^{k} T_{m_i}^{-1} I^c.$$

Now (2.2) implies that

$$\mu E_k \le 2C'(\mu E_{k-1} - \mu E_k),$$

or

$$\mu E_k \le \frac{2C'}{1 + 2C'}\mu E_{k-1}.$$

Therefore

$$\mu E_{K-1} \le \left(\frac{2C'}{1 + 2C'}\right)^K < \varepsilon,$$

which contradicts the fact that $(m_0, m_1, \ldots, m_{K-1}) \in T_\mu^{I,\varepsilon}$. $\square$

3. Measures of large rank. For each $\alpha < \omega_1$, we shall construct a canonical tree T_α of height α. Let $\mathscr{F}$ be the set of strictly increasing maps $f: \mathbf{N}^* \to \mathbf{N}^*$. Any $f \in \mathscr{F}$ induces a map f^* on trees in the obvious way. Given $\alpha < \omega_1$, $p \in {]0, 1[}$ and $f \in \mathscr{F}$, we shall construct a measure $\nu_{\alpha,p,f} \in H^\perp$ with h'-rank at least α. This shows that $H^\perp$ is not Borel.

Given a tree T, let

$$(T) = \{n: \exists s \ (n \in s \in T)\},$$

where, if $s = (n_0, n_1, \ldots, n_k)$, we write $n \in s \Leftrightarrow \exists i \le k \ (n = n_i)$, and let $(T)^c = \mathbf{N}^* \backslash (T)$. Define $f_n(x) = x^{2^{n+1}} \cdot 2^{2^n}$; note that if x and y are integral powers of 2 and $n \ne m$, then $f_n(x) \ne f_m(y)$. For each limit ordinal $\alpha < \omega_1$, choose a bijection $g_\alpha: \alpha \to \mathbf{N}$.

We construct our canonical trees T_α by induction, beginnning with $T_0 = \{\varnothing\}$. It will turn out that (T_α) consists entirely of powers of 2. If T_γ have been constructed for $\gamma < \alpha$, let

$$T_\alpha = \begin{cases} \{(2^2)^\wedge s: s \in f_0^* T_\beta\} \cup \{\varnothing\} & \text{if } \alpha = \beta + 1, \\ \bigcup_{\beta < \alpha} f_{g_\alpha(\beta)}^* T_\beta & \text{if } \alpha \text{ is a limit.} \end{cases}$$

It is clear that $\mathrm{ht}(T_\alpha) = \alpha$.

Given a set $A \subseteq \mathbf{N}^*$, define the Bernoulli convolution

$$\lambda_A = \mathop{*}_{n \in A} \left[\frac{1}{2}\delta(0) + \frac{1}{2}\delta(2^{-n})\right],$$

where $\delta(x)$ represents the Dirac measure concentrated at x. For $\alpha < \omega_1$, $p \in {]0, 1[}$, $f \in \mathscr{F}$, we construct preliminary measures $\mu_{\alpha,p,f}$ by induction. Let $\mu_{0,p,f} = \delta(0)$ for all p, f. If $\mu_{\gamma,p,f}$ have been constructed for all $\gamma < \alpha$, $p \in {]0, 1[}$, and $f \in \mathscr{F}$, define

$$\mu_{\alpha,p,f} = \sqrt{p}\mu_{\alpha,p,f}^{(0)} + (1 - \sqrt{p})\lambda_{(f \cdot T_\alpha)},$$

where

$$\mu_{\alpha,p,f}^{(0)} = \begin{cases} \mu\langle\beta, \sqrt{p}, f \circ f_0\rangle & \text{if } \alpha = \beta + 1, \\ \mathop{*}_{\beta < \alpha} \mu\langle\beta, \sqrt{p}, f \circ f_{g_\alpha(\beta)}\rangle & \text{if } \alpha \text{ is a limit;} \end{cases}$$

here, and below, we are using the notation $\langle \cdots \rangle$ for subscripts.

Finally, define

$$\nu_{\alpha,p,f} = \lambda_{(f \cdot T_\alpha)^c} * \mu_{\alpha,p,f}.$$

The reader should have in mind the following probabilistic interpretation of $\nu_{\alpha,p,f}$. First define i.i.d. random variables Y_n for $n \ge 1$ which take the values

0, 1 each with probability 1/2. For $A \subseteq \mathbf{N}^*$, let $Y_A = \sum_{n \in A} Y_n 2^{-n}$. Let Z_p be independent random variables for $0 < p < 1$, independent also of $\{Y_n\}$, which take the values 0, 1 with probabilities $\sqrt{p}$ and $1 - \sqrt{p}$, respectively. We define random variables $X_{\alpha,p,f}$ by induction. Let $X_{0,p,f} \equiv 0$. To define $X_{\alpha,p,f}$ given $X_{\gamma,q,g}$ for $\gamma < \alpha$, first let $\{\tilde{X}_{\gamma,q,g} : \gamma < \alpha, q \in]0,1[, g \in \mathscr{F}\}$ be independent copies of $\{X_{\gamma,q,g}\}$ which are also independent of $\{Z_p\}$ and $\{Y_n\}$. Now define

$$X_{\alpha,p,f} = (1 - Z_p)X_{\alpha,p,f}^{(0)} + Z_p Y_{(f^*T_\alpha)},$$

where

$$X_{\alpha,p,f}^{(0)} = \begin{cases} \tilde{X}_{\beta,\sqrt{p},f \circ f_0} & \text{if } \alpha = \beta + 1, \\ \sum_{\beta < \alpha} \tilde{X}\langle \beta, \sqrt{p}, f \circ f_{g_\alpha(\beta)} \rangle & \text{if } \alpha \text{ is a limit.} \end{cases}$$

Finally, define

$$W_{\alpha,p,f} = X_{\alpha,p,f} + Y_{(f^*T_\alpha)^c}.$$

Then the distributions of $Y_A, X_{\alpha,p,f}, X_{\alpha,p,f}^{(0)}$, and $W_{\alpha,p,f}$ are $\lambda_A, \mu_{\alpha,p,f}, \mu_{\alpha,p,f}^{(0)}$ and $\nu_{\alpha,p,f}$, respectively. One may show by induction that $X_{\alpha,p,f}$ can be written in the form

$$X_{\alpha,p,f} = \sum_{n \geq 1} U_n 2^{-n}$$

for some $\{0,1\}$-valued random variables U_n with $U_n \equiv 0$ if $n \notin (f^*T_\alpha)$.

The following proposition shows that $f^*T_\alpha \subseteq T_{\nu_{\alpha,p,f}}^{]1/2,1[,p}$. Combined with Theorem 3.2, this shows that $h'(\nu_{\alpha,p,f}) \geq \alpha$.

PROPOSITION 3.1. $\forall \alpha < \omega_1 \; \forall p \in]0,1[\; \forall f \in \mathscr{F} \; \forall s \in f^*T_\alpha$

(3.1) $$\nu_{\alpha,p,f}(E_s) = \mu_{\alpha,p,f}(E_s) \geq p$$

and

(3.2) $$\mu_{\alpha,p,f}(E_{(f^*T_\alpha)^c}) = 1,$$

where for $A \subseteq \mathbf{N}^*$,

$$E_A = \left\{ x \in \mathbf{T} : \forall n \in A \left(2^{n-1}x \notin [\tfrac{1}{2},1[\right) \right\}.$$

PROOF. We proceed by induction. Condition (3.2) is easily established, so we concentrate on (3.1). It is clear that $\nu_{\alpha,p,f}E_s = \mu_{\alpha,p,f}E_s$. Now (3.1) is trivial for $\alpha = 0$, so suppose it holds for all $\gamma < \alpha$. If $\alpha = \beta + 1$ and $s \in f^*T_\alpha$, then $s = (f(2^2))^\wedge s_0$ for some $s_0 \in (f \circ f_0)^*T_\beta$. Therefore

$$\begin{aligned} \mu_{\alpha,p,f}(E_s) &\geq \sqrt{p}\mu\langle \beta, \sqrt{p}, f \circ f_0\rangle(E_s) \\ &\geq \sqrt{p}\mu\langle \beta, \sqrt{p}, f \circ f_0\rangle(E_{s_0} \cap E\langle((f \circ f_0)^*T_\beta)^c\rangle) \\ &= \sqrt{p}\mu\langle \beta, \sqrt{p}, f \circ f_0\rangle(E_{s_0}) \quad \text{(by (3.2))} \\ &> \sqrt{p} \cdot \sqrt{p} = p \end{aligned}$$

by the inductive hypothesis. On the other hand, if α is a limit and $s \in f^*T_\alpha$, then $s \in (f \circ f_{g_\alpha(\beta)})^*T_\beta$ for some $\beta < \alpha$. Therefore

$$\mu_{\alpha,p,f}E_s \geq \sqrt{p}\mu_{\alpha,p,f}^{(0)}E_s = \sqrt{p}\mu\langle \beta, \sqrt{p}, f \circ f_{g_\alpha(\beta)}\rangle(E_s) \geq p. \quad \square$$

It remains to prove that $\nu_{\alpha,p,f} \in H^\perp$, for which we use the methods of [L3].

THEOREM 3.2. $\forall \alpha < \omega_1$ $\forall p \in]0,1[$ $\forall f \in \mathscr{F}$ $\nu_{\alpha,p,f} \in H^\perp$.

PROOF. For a sequence $\{m_j\} \subseteq \mathbf{Z}$, a measure $\mu \in M(\mathbf{T})$, and measures $\sigma_{x,\mu} \in M(\mathbf{T})$ $(x \in \mathbf{T})$, we shall write $\{m_j\} \sim \sigma_{x,\mu}$ if for all $r \in \mathbf{Z}$,

$$e(-rm_jx) \to \hat{\sigma}_{x,\mu}(r) \quad \text{weak}^* \text{ in } L^\infty(\mu).$$

According to Theorem 13 of [L3], if $m_j \to \infty$ and $\{m_j\} \sim \sigma_{x,\mu}$, then $\operatorname{supp}\sigma_{x,\mu} = \mathbf{T}$ μ-a.e. iff μ annihilates all H-sets based on any subsequence of $\{m_j\}$.

By the method of proof of Theorem 16 of [L3], we see that to show that $\nu_{\alpha,p,f} \in H^\perp$, it is enough to show that

$$\operatorname{supp}\sigma\langle x, \nu_{\alpha,p,f}\rangle = \mathbf{T} \quad \nu_{\alpha,p,f}\text{-a.e.}$$

when $\{2^{n_j-1}x\} \sim \sigma\langle x, \nu_{\alpha,p,f}\rangle$ for some increasing sequence $\{n_j\} \subseteq (f^*T_\alpha)$; in particular, we can take $\alpha \geq \omega$. Furthermore, by taking a subsequence, if necessary, $\{e(-r2^{n_j-1}x)\}$ has weak* limits in $L^\infty(\lambda_{[f^*T_\alpha]^c})$ and $L^\infty(\mu_{\alpha,p,f})$ as well, and

$$\sigma\langle x+y, \nu_{\alpha,p,f}\rangle = \sigma\langle x, \lambda_{(f^*T_\alpha)^c}\rangle * \sigma\langle y, \mu_{\alpha,p,f}\rangle \quad \lambda_{(f^*T_\alpha)^c} \otimes \mu_{\alpha,p,f}\text{-a.e. } [(x,y)]$$

(see [L4]). It is easy to calculate that

$$\sigma\langle x, \lambda_{(f^*T_\alpha)^c}\rangle = \lambda_{\{n:n\geq 2\}} \quad \text{a.e.}$$

Hence it suffices to show that $\exists q_y \in]0,1[$ such that

(3.3) $$\sigma\langle y, \mu_{\alpha,p,f}\rangle = (1-q_y)\delta(0) + q_y\delta(2^{-1}) \quad \text{a.e.}$$

Now

(3.4) $$\operatorname{supp}\mu_{\alpha,p,f} = \left\{ \sum_{n \in (f^*T_\alpha)} a_n 2^{-n} : a_n = 0, 1 \right\},$$

whence (3.3) holds for some $q_y \in [0,1]$. Now $q_y = \frac{1}{2}$ $\lambda_{(f^*T_\alpha)}$-a.e., so it remains to consider $\mu_{\alpha,p,f}^{(0)}$.

We proceed by induction, beginning with the case of a limit ordinal α, which includes the possibility that $\alpha = \omega$. Suppose first that $\{n_j\}$ intersects infinitely many $((f \circ f_{g_\alpha(\beta)})^*T_\beta)$. Then we may assume that $n_j \in ((f \circ f_{g_\alpha(\beta_j)})^*T_{\beta_j})$ with $\{\beta_j\}$ distinct. Now if $q_y \in \{0,1\}$ on a set of positive $\mu_{\alpha,p,f}^{(0)}$-measure, then by Theorem 13 of [L3] quoted above, there is an H-set based on a subsequence of $\{2^{n_j-1}\}$ of positive $\mu_{\alpha,p,f}^{(0)}$-measure. By relabelling, we take this subsequence to be the whole sequence. Furthermore, because of (3.4), we conclude that $\mu_{\alpha,p,f}^{(0)}E^{(a)} > 0$ for either $a = 0, 1$, where $E^{(a)} = \bigcap_{j<\infty} E_j^{(a)}$,

$$E_j^{(a)} = \left\{ x : 2^{n_j-1}x \in \frac{a}{2} + [0, \tfrac{1}{2}[\right\}.$$

But $E_j^{(a)}$ are $\mu_{\alpha,p,f}^{(0)}$-independent since $\{\beta_j\}$ are distinct. Also,

$$\mu_{\alpha,p,f}^{(0)}E_j^{(0)} = \mu\langle \beta_j, \sqrt{p}, f \circ f_{g_\alpha(\beta_j)}\rangle(E_j^{(0)}) \geq \sqrt{p}$$

and

$$\mu_{\alpha,p,f}^{(0)}E_j^{(1)} = \mu\langle \beta_j, \sqrt{p}, f \circ f_{g_\alpha(\beta_j)}\rangle(E_j^{(1)})$$
$$\geq (1-p^{1/4})\lambda\langle((f \circ f_{g_\alpha(\beta_j)})^*T_{\beta_j})\rangle(E_j^{(1)}) = (1-p^{1/4}) \cdot \tfrac{1}{2}.$$

Since $E_j^{(0)}$ and $E_j^{(1)}$ are complementary events, it follows that $\mu_{\alpha,p,f}^{(0)} E^{(a)} = 0$, a contradiction.

Now in the second case, $\{n_j\}$ intersects only finitely many $((f \circ f_{g_\alpha(\beta)})^* T_\beta)$. Here we may then assume that all $n_j \in ((f \circ f_{g_\alpha(\beta_0)})^* T_{\beta_0})$ for some β_0. We have

$$\sigma\langle x + y, \mu_{\alpha,p,f}^{(0)}\rangle = \sigma\langle x, \mu\langle\beta_0, \sqrt{p}, f \circ f_{g_\alpha(\beta_0)}\rangle\rangle * \sigma\left\langle y, \underset{\substack{\beta<\alpha\\\beta\neq\beta_0}}{*} \mu\langle\beta, \sqrt{p}, f \circ f_{g_\alpha(\beta)}\rangle\right\rangle$$

a.e. $[(x,y)]$ and by the inductive hypothesis, the first term on the right equals $q_x'\delta(0) + (1 - q_x')\delta(\frac{1}{2})$ with $0 < q_x' < 1$ a.e. $[x]$. Hence (3.3) holds with $0 < q_x < 1$.

Finally, in case $\alpha = \beta + 1$ is a successor ordinal, we have

$$\sigma\langle x, \mu_{\alpha,p,f}\rangle = \sigma\langle x, \mu\langle\beta, \sqrt{p}, f \circ f_0\rangle\rangle \quad \mu_{\alpha,p,f}^{(0)}\text{-a.e.,}$$

whence the desired result follows directly from the inductive hypothesis. $\square$

4. Measures annihilating Dirichlet sets. Every Dirichlet set is evidently an H-set: $D \subseteq H$. Thus $D^\perp \supseteq H^\perp$. It turns out that $D^\perp$ is again $\mathbf{\Pi}_1^1$ non-Borel. The method of proof is similar to that for $H^\perp$, and we shall only outline it here.

First one checks that D is Borel, in fact $\mathbf{\Pi}_2^0$ (i.e., G_δ). Thus, $D^\perp$ is $\mathbf{\Pi}_1^1$. Given $\mu \in \text{PROB}(\mathbf{T})$ and $\varepsilon \in \mathbf{Q}^+$, we define a tree T_μ^ε on $\text{Seq}(\mathbf{N}^* \times \mathbf{Q}^+)$ as follows:

$$T_\mu^\varepsilon = \{((n_0, r_0), (n_1, r_1), \ldots, (n_k, r_k)): \forall i < k(n_{i+1} \geq 2n_i \,\&\, r_{i+1} < r_i)$$
$$\&\mu\{x \in \mathbf{T}: \forall i \leq k(\|n_i x\| \leq r_i)\} \geq \varepsilon\}.$$

Thus, $\mu \in D^\perp \Leftrightarrow \forall \varepsilon \; (T_\mu^\varepsilon$ is well founded). If we define

$$h_D'(\mu) = \sup\{\text{ht}(T_\mu^\varepsilon) + 1: \varepsilon \in \mathbf{Q}^+\},$$

then $h_D'(\mu)$ is a limit ordinal, $\omega \cdot h_D(\mu)$. One proves as before that $h_D: D^\perp \to \omega_1$ is a $\mathbf{\Pi}_1^1$-rank on $D^\perp$. Note that $h \geq h_D$, so that all measures of h-rank one are also of h_D-rank one.

We now construct measures of arbitrarily large h_D-rank. Given $A \subseteq \mathbf{N}^*$, denote

$$A_2 = \{k \in \mathbf{N}: \exists n \in A \; n \leq k < 2n\}.$$

We define $\tau_{\alpha,p,f}$ by induction, using the notation of §3: let $\tau_{0,p,f} = \delta(0)$ and

$$\tau_{\alpha,p,f} = \sqrt{p}\tau_{\alpha,p,f}^{(0)} + (1 - \sqrt{p})\lambda_{(f^* T_\alpha)_2},$$

where

$$\tau_{\alpha,p,f}^{(0)} = \begin{cases} \tau\langle\beta, \sqrt{p}, f \circ f_0\rangle & \text{if } \alpha = \beta + 1, \\ \underset{\beta<\alpha}{*}\tau\langle\beta, \sqrt{p}, f \circ f_{g_\alpha(\beta)}\rangle & \text{if } \alpha \text{ is a limit.} \end{cases}$$

Now set $\eta_{\alpha,p,f} = \lambda_{((f^* T_\alpha)_2)^c} * \tau_{\alpha,p,f}$. We claim that $\eta_{\alpha,p,f} \in D^\perp$ and $h_D'(\eta_{\alpha,p,f}) \geq \alpha$.

One first shows that if $s \in f^* T_\alpha$, then

$$\eta_{\alpha,p,f}\{x \in \mathbf{T}: \forall n \in s \; 2^{n-1}x \in [0, 2^{-n}[\} \geq p.$$

Next, notice that $\mu \in D^\perp$ iff $\sigma_{x,\mu} \neq \delta(0)$ μ-a.e. whenever $\{m_j\} \sim \sigma_{x,\mu}$ (see [**L4**]). As before,

$$\sigma\langle x + y, \eta_{\alpha,p,f}\rangle = \sigma\langle x, \lambda_{(f^* T_\alpha)_2^c}\rangle * \sigma\langle y, \tau_{\alpha,p,f}\rangle \quad \text{a.e. } [(x,y)].$$

Suppose that the left-hand side is $\delta(0)$ on a set of positive measure. Then both members of the right-hand side are Dirac measures on sets of positive measure (since both are probability measures). Also, $\lambda_{(f \cdot T_\alpha)_2^c}$ is monotrochic (see the proof of Proposition 6.4.5, p. 173, of [GM]), whence if $\sigma\langle x, \lambda_{(f \cdot T_\alpha)_2^c}\rangle$ is a Dirac measure on a set of positive $\lambda_{(f \cdot T_\alpha)_2^c}$-measure and we denote $\chi(x) = (\sigma\langle x, \lambda_{(f \cdot T_\alpha)_2^c}\rangle)^\wedge(1)$, then $|\chi(x)| = 1$ $\lambda_{(f \cdot T_\alpha)_2^c}$-a.e. Now there exist subsequences $\{m_j'\}$, $\{m_j''\}$ of $\{m_j\}$ such that $m_j'' > m_j'$ and

$$e(-(m_j'' - m_j')x) \to |\hat{\sigma}_{x,\mu}(1)|^2 \quad \text{weak}^* \text{ in } L^\infty(\mu)$$

for $\mu = \eta_{\alpha,p,f}$, $\lambda_{(f \cdot T_\alpha)_2^c}$, and $\tau_{\alpha,p,f}$ (see Lemme I.5, p. 17, of [HMP]). Thus, if $\{m_j'' - m_j'\} \sim \sigma_{x,\mu}'$ and $\sigma_{x,\mu}$ is a Dirac measure on a set E of positive μ-measure, then $\sigma_{x,\mu}' = \delta(0)$ on almost all of E, where μ is any of the above three measures. Hence we may relabel $m_j'' - m_j'$ as m_j without loss of generality. But now we have $\chi \geq 0$, whence $\chi(x) = 1$ a.e., so that $\hat{\lambda}_{(f \cdot T_\alpha)_2^c}(m_j) \to 1$, which is the same as $m_j = a_j 2^{n_j}$ for some $n_j \in (f^* T_\alpha)$ and $a_j \in \mathbf{Z}$ with $a_j 2^{-n_j} \to 0$. An inductive argument shows that for such $\{m_j\}$, $\sigma\langle y, \tau_{\alpha,p,f}\rangle \neq \delta(0)$ a.e., which contradicts our supposition and completes the proof that $\eta_{\alpha,p,f} \in D^\perp$.

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