Occupation measure of random walks
and wired spanning forests
in balls of Cayley graphs

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Abstract

We show that for finite-range, symmetric random walks on general transient Cayley graphs, the expected occupation time of any given ball of radius r is $O(r^{5/2})$. We also study the volume-growth property of the wired spanning forests on general Cayley graphs, showing that the expected number of vertices in the component of the identity inside any given ball of radius r is $O(r^{11/2})$.

Résumé

On montre que toute marche aléatoire symétrique à pas bornés sur un graphe de Cayley transitoire satisfait que l'espérance du temps d'occupation d'une boule quelconque de rayon r vaut $O(r^{5/2})$. On étudie aussi la croissance du volume des fôrets recouvrantes câblées dans les graphes de Cayley généraux, en montrant que l'espérance du nombre de sommets appartenant à la composante connexe de l'identité dans une boule quelconque de rayon r vaut $O(r^{11/2})$.

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1 Introduction

Given a transient, symmetric random walk S starting from a vertex o in a Cayley graph $G = (V, E)$, let $L_r := \big|\{t : S_t \in B(o, r)\}\big|$, where $B(o, r)$ is the set of vertices within graph distance r of o. Suppose for the moment that S is simple random walk. If G has polynomial growth of degree D, then Varopoulos' estimate $p_t(o, x) \lesssim t^{-D/2}$ (see, e.g., [CGP01, Corollary 7.3]) yields $\mathbb{E}[L_r] \lesssim r^2$ (see Remark 2.7). Here, $a(t) \lesssim b(t)$ means that $\exists\, c > 0$ such that $a(t) \le c\, b(t)$ for all t. Similarly, Varopoulos' estimate $p_t(o, o) \lesssim e^{-ct^{1/3}}$ for groups of exponential growth (see [CGP01, Corollary 7.4]) yields $\mathbb{E}[L_r] \lesssim r^3$ (see the proof of [BB07, Proposition 2.3]). When the walk escapes at a linear rate, a simple argument (Lemma 2.1) shows that $\mathbb{E}[L_r] \lesssim r$. In particular, the linear bound holds for nonamenable Cayley graphs. We believe that the following quadratic bound holds in general; to the best of our knowledge, this is open.

Conjecture 1.1. *For a symmetric random walk S on a transient Cayley graph G, let L_r be the occupation time of $B(o, r)$ defined as above. Then $\mathbb{E}[L_r] \lesssim r^2$.*

As an example of amenable Cayley graphs of exponential growth where a quadratic bound is easy to establish, consider simple random walk on lamplighter groups over any base group which has polynomial growth or, more generally, any base group known to have quadratic occupation time: we can bound the occupation time of balls in the Cayley graph of the lamplighter group by the occupation time of balls of the projection of simple random walk under the quotient map to the base group. In this paper, although we cannot prove Conjecture 1.1, we establish a general $5/2$-power bound for finite-range, symmetric random walks (i.e., symmetric random walks whose jumps have bounded support).

Theorem 1.2. *Let G be a transient Cayley graph and $V(r) := |B(o, r)|$. Then for every finite-range, symmetric random walk on G,*

$$\mathbb{E}[L_r] \lesssim r^2 \sqrt{\log V(r)}. \tag{1}$$

In particular, $\mathbb{E}[L_r] \lesssim r^{5/2}$.

By comparison, if τ_r denotes the first exit time of $B(o, r)$ of a symmetric random walk starting at o, it is known that

$$E[\tau_r] \lesssim r^2$$

for all Cayley graphs. (See Theorem 2.2 for a proof.)

Let $G = (V, E)$ be an infinite graph. The wired spanning forest measure on G is defined as the infinite-volume limit of the wired spanning tree measures on a sequence of finite subgraphs exhausting G: Let $V_1 \subset V_2 \subset \cdots$ be finite subsets of V whose induced subgraphs G_n are connected with $\bigcup_{n=1}^{\infty} V_n = V$. Let μ_n^F be the uniform spanning tree measure on G_n. Then as a probability measure on edge configurations, μ_n^F restricted to any finite subset of E converges. This defines a unique probability measure μ^F on 2^E, which we call the *free spanning forest*. Another way of taking limits of spanning trees is as follows. Suppose G_n are defined as above. Let G_n^W be obtained from G by identifying all the vertices outside G_n to one new vertex and μ_n^W be the uniform spanning tree measure on G_n^W. Then μ_n^W also has a limit μ^W, which we call the *wired spanning forest*. These results are due to [Pem91]. The free and wired spanning forests are the same if G is of polynomial growth or, more generally, amenable [Pem91, BLPS01]. They can be different, such as on the Cayley graph of a free group. See [BLPS01, LP16] for more details.

On Cayley graphs, the wired spanning forest (WSF) has a single component if the graph has at most quartic growth; otherwise, there are infinitely many components in the WSF [Pem91]. In the latter case, the geometry of the WSF has intriguing behaviors. Let $\mathcal{T}_o$ be the component containing o in the WSF of G. For Cayley graphs with polynomial growth of order at least quartic, $\mathbb{E}\big[|\mathcal{T}_o \cap B(o, r)|\big] \asymp r^4$, whereas nonamenable Cayley graphs satisfy $\mathbb{E}\big[|\mathcal{T}_o \cap B(o, r)|\big] \asymp r^2$ [BLPS01, Section 13]. Here, $a(t) \asymp b(t)$ means $a(t) \lesssim b(t)$ and $b(t) \lesssim a(t)$. In [BKPS04], the authors provided a detailed analysis of the geometry of the WSF on $\mathbb{Z}^d$ $(d \geq 5)$. Among other results, they showed that the tree components have "stochastic dimension" 4. In this paper, we extend the investigation of the volume-growth property of the WSF to general Cayley graphs (Theorems 1.3 and 1.4).

Using a similar method as we use to prove Theorem 1.2, we show the following upper bound:

Theorem 1.3. *Let G be a Cayley graph and $V(r) = |B(o, r)|$. Then*

$$\mathbb{E}\big[|\mathcal{T}_o \cap B(o, r)|\big] \lesssim r^4 \log^{3/2} V(r). \tag{2}$$

In particular, $\mathbb{E}\big[|\mathcal{T}_o \cap B(o, r)|\big] \lesssim r^{11/2}$.

Let $C(o, r)$ be the connected component of $\mathcal{T}_o \cap B(o, r)$ containing o. This provides another way to measure the growth of the WSF. We show the following upper bound in terms of the exit time τ_r for random walk:

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Theorem 1.4. *Given a Cayley graph G of superpolynomial growth, let $C(o,r)$ be defined as above. Then there exists r_0 such that*

$$\mathbb{E}\big[|C(o,r)|\big] \leq 4\,\mathbb{E}^2[\tau_{6r}] \quad for \quad r > r_0\,. \tag{3}$$

Remark 1.5. As will be clear from our proof of (3), the constants involved are not optimal.

For Cayley graphs of polynomial growth, we have $\mathbb{E}\big[|C(o,r)|\big] \leq \mathbb{E}\big[|\mathcal{T}_o \cap B(o,r)|\big] \lesssim r^4$. Since $\mathbb{E}[\tau_r] \lesssim r^2$ for all Cayley graphs, Theorem 1.4 implies that $\mathbb{E}\big[|C(o,r)|\big] \lesssim r^4$ in general. We believe that $\mathbb{E}\big[|\mathcal{T}_o \cap B(o,r)|\big] \asymp \mathbb{E}\big[|C(o,r)|\big]$ and hence $\mathbb{E}\big[|\mathcal{T}_o \cap B(o,r)|\big] \lesssim r^4$ for general Cayley graphs.

2 Occupation measure of random walks

2.1 Preliminaries

The only random walks $S = (S_0, S_1, \ldots)$ on groups that we consider are those where for all $t \geq 1$, the random variables $S_{t-1}^{-1}S_t$ are independent and identically distributed. Such a random walk is called *symmetric* if for all g, we have $\mathbb{P}\big[S_0^{-1}S_1 = g\big] = \mathbb{P}\big[S_0^{-1}S_1 = g^{-1}\big]$. We usually choose S_0 to be the identity, o.

Suppose Γ is a group generated by a finite subset X, i.e., every element in Γ can be written as a product of elements in $X \cup X^{-1}$. The Cayley graph G associated to (Γ, X) is the unoriented graph with vertices Γ and edges $\big\{[g, gx] : g \in \Gamma,\, x \in X\big\}$. Every Cayley graph is a connected, vertex-transitive graph.

For a Cayley graph G, a vertex $o \in G$, and $r > 0$, let d_G denote the graph distance in G and $B(o,r) := \{v \in G : d_G(o,v) \leq r\}$. We call $V(r) := |B(o,r)|$ the *volume function* of G. Due to Gromov's theorem [Gro81], it is well known that either $V(r) \asymp r^D$ for some $D \in \mathbb{N}$ or $\lim_{r\to\infty} V(r)/r^D = \infty$ for all $D \in \mathbb{N}$. In the former case, we say that G has *polynomial growth of degree D*. In the latter case, we say that G has *superpolynomial growth*. These properties are independent of the choice of the generating set X of G.

Given a Cayley graph G with $d := |X \cup X^{-1}|$, lazy simple random walk on G is the Markov chain $S = (S_t)_{t=0}^\infty$ on Γ with transition probabilities $p(g, gx) = 1/(2d)$ for $x \in X \cup X^{-1}$ and $p(g, g) = 1/2$. We assume that the identity is not an element of X.

The following facts concerning the occupation time L_r and the escape time τ_r are not needed for the rest of the paper. We record them for completeness.

Lemma 2.1. *Suppose S_t is a random walk on a Cayley graph G such that $\liminf_{t \to \infty} d_G(o, S_t)/t > 0$ a.s. Then $\mathbb{E}[L_r] \lesssim r$.*

Proof. We may choose $\epsilon > 0$ and $t_0 < \infty$ so that

$$\mathbb{P}_o[\forall t \geq t_0 \quad d_G(o, S_t) > \epsilon t] > 1/2 \,.$$

Let $s(r) := \max\{2r/\epsilon, t_0\}$. Then for every t, we have

$$\mathbb{P}\big[\forall m \geq s(r) \quad S_{t+m} \notin B(o, r) \mid S_t \in B(o, r)\big] > 1/2 \,,$$

so $\mathbb{E}[L_r] < 2s(r)$. □

Note that if Γ is a nonamenable group, then the hypothesis of Lemma 2.1 holds: [Kes59a, Kes59b] showed that there is some $\rho < 1$ such that for all $x \in \Gamma$ and all $t \in \mathbb{N}$, we have $p_t(o, x) \leq \rho^t$. The result then follows from a Borel–Cantelli argument.

The following argument was noted by Anna Erschler (personal communication, 2005).

Theorem 2.2. $\mathbb{E}[\tau_r] \lesssim r^2$ *for symmetric random walks on Cayley graphs.*

Proof. Because of the linear bound on nonamenable Cayley graphs even for occupation time (Lemma 2.1) and of the stochastic domination of τ_r by L_r, it remains to show this bound on escape time when G is amenable. Furthermore, we may assume that the support of the random walk generates the group Γ, as otherwise we take the subgroup it generates. Let W be a finite subset of the support of S_1 such that W generates Γ. Because distances in any Cayley graph of G differ from those in the Cayley graph generated by W by a bounded factor, we may assume that G is in fact the Cayley graph determined by W. We may also assume that the support of S_1 is contained in $B(o, 2r)$ since if not, we may replace all jumps outside that ball by staying in place; the new random walk leaves $B(o, r)$ no earlier than

the original random walk does. By [Mok95, KS97], there is a harmonic, equivariant, Hilbert-space valued, nonconstant function ϕ on V (also see [LP13, Theorem 3.1] for an explicit construction). Here, "equivariant" means with respect to some affine isometric action of the group on the Hilbert space. Let $c := \mathbb{E}\big[\|\phi(S_1) - \phi(o)\|^2\big] > 0$. Let $p_* := \min\{p(o, x) : x \in W\}$. Then $\|\phi(x) - \phi(y)\|^2 \leq c/p_*$ when x and y are neighbors in G, whence $\|\phi(x) - \phi(y)\| \leq \sqrt{c/p_*} \cdot d_G(x, y)$ for all vertices x, y of G. In particular, $\|\phi(x) - \phi(o)\| \leq 3r\sqrt{c/p_*}$ for $x \in B(o, 3r)$. Since ϕ is harmonic, the sequence of random variables $\|\phi(S_n) - \phi(o)\|^2 - cn$ forms a martingale, thus the optional-stopping theorem gives $\mathbb{E}\|\phi(S_{\tau_r}) - \phi(o)\|^2 = c\,\mathbb{E}[\tau_r]$. Since the support of S_1 is within $B(o, 2r)$ and τ_r is the exit time of $B(o, r)$, the triangle inequality gives $S_{\tau_r} \in B(o, 3r)$. Therefore

$$\mathbb{E}[\tau_r] \leq \big(3r\sqrt{c/p^*}\big)^2 \cdot c^{-1} = 9r^2/p_* \,. \qquad \square$$

When the random walk has bounded jumps, a stronger result on the distribution of τ_r follows from the main result of [LPS14].

2.2   Proof of Theorem 1.2

There are three main ingredients in our proof of Theorem 1.2. The first ingredient is a bound for the return probability of lazy random walks using the volume function $V(r)$, which is obtained in [LO17] by spectral embedding:

Lemma 2.3. *Given a vertex-transitive graph G, let $p_m(o, o) := \mathbb{P}[S_m = o]$ be the return probability of a lazy, finite-range, symmetric random walk, S. Let V be the volume function defined as above. Then there exist constants $c \in (0, 1)$ and $c' < \infty$ such that*

$$\forall m \in \mathbb{N}^+ \qquad p_m(o, o) \leq c'm \int_0^1 \frac{e^{-\lambda m}}{V\big(c/\sqrt{\lambda}\big)} \, d\lambda \,. \qquad (4)$$

Proof. Combine Lemma 3.5 and Theorem 6.1 in [LO17]. $\qquad \square$

The second ingredient is immediate from the main result of [LP13] in the amenable case and Lemma 2.1 in the nonamenable case:

Lemma 2.4. *Given a vertex-transitive graph G, let $p_m(o, o) := \mathbb{P}[S_m = o]$ be the return probability of a lazy, finite-range, symmetric random walk, S. Then there exists a constant $c < \infty$ such that*

$$\forall r, n \in \mathbb{N}^+ \qquad \sum_{m=0}^{n} \mathbb{P}\big[S_m \in B(o, r)\big] \leq cr\sqrt{n}\,.$$

The third ingredient is an important growth property of the volume function of Cayley graphs of superpolynomial growth, established in [BGT12]:

Lemma 2.5. *Suppose G is a Cayley graph of superpolynomial growth. Then for all $k \in \mathbb{N}$, there exists $c_k > 0$ such that*

$$\text{for all } a \geq 1 \text{ and } r \geq 1, \qquad \frac{V(ar)}{V(r)} \geq c_k a^k\,. \tag{5}$$

Proof. This is an immediate consequence of [BGT12, Corollary 11.2]. □

Corollary 2.6. *Suppose G is a Cayley graph of superpolynomial growth of a group, Γ. Let S be a lazy, finite-range, symmetric random walk on G whose support generates Γ. Write $p_m(x, y) := \mathbb{P}_x[S_m = y]$. Then there is a constant $c > 0$ such that for all $k \in \mathbb{N}^+$, there is some $c'' > 0$ (depending on k) such that for all $r, m \in \mathbb{N}^+$ and all $x, y \in \Gamma$,*

$$p_m(x, y) \leq c''\big(m^{-k/2}r^k/V(r) + e^{-c^2 m/r^2}\big)\,. \tag{6}$$

Proof. Choose c as in (4). From the preceding two lemmas, we have

$$
\begin{aligned}
p_m(x, y) \leq p_m(o, o) &\lesssim m \int_0^1 \frac{e^{-\lambda m}}{V\big(c/\sqrt{\lambda}\big)} \, d\lambda \\
&= m \int_0^{c^2/r^2} \frac{e^{-\lambda m}}{V\big(c/\sqrt{\lambda}\big)} \, d\lambda + m \int_{c^2/r^2}^1 \frac{e^{-\lambda m}}{V\big(c/\sqrt{\lambda}\big)} \, d\lambda \\
&\lesssim \frac{m}{V(r)} \int_0^{c^2/r^2} \lambda^{k/2} r^k e^{-\lambda m} \, d\lambda + m \int_{c^2/r^2}^1 e^{-\lambda m} \, d\lambda \\
&\lesssim m^{-k/2} r^k/V(r) + e^{-c^2 m/r^2}\,,
\end{aligned}
$$

where in the last line, we use the change of variable $u := m\lambda$. The implied constants depend on k. This proves (6). □

Proof of Theorem 1.2. We may clearly assume that the support of the walk generates the group, as otherwise we simply take the subgroup it generates together with a Cayley graph of the subgroup. We may also assume that S is lazy, i.e., $p_1(o, o) \geq 1/2$. We wish to show that

$$\mathbb{E}[L_r] = \sum_{m=0}^{\infty} \mathbb{P}[S_m \in B(o, r)] \lesssim r^2 \sqrt{\log V(r)}. \tag{7}$$

Since the result is known for groups of polynomial growth, we assume G is of superpolynomial growth. Write $\varphi(m)$ for the right-hand side of (6). Then $\forall m \in \mathbb{N}$ and $r > 0$,

$$\mathbb{P}[S_m \in B(o, r)] \leq \varphi(m)V(r).$$

Set $\alpha := c^{-2}$, where c is as defined in (6). Put

$$\Sigma_r^{(1)} := \sum_{m=0}^{\lfloor \alpha r^2 \log V(r) \rfloor} \mathbb{P}[S_m \in B(o, r)]$$

and

$$\Sigma_r^{(2)} := \sum_{m > \alpha r^2 \log V(r)} \varphi(m)V(r).$$

By Lemma 2.4,
$$\Sigma_r^{(1)} \lesssim r^2 \sqrt{\log V(r)}.$$

Since
$$\sum_{m=0}^{\infty} \mathbb{P}[S_m \in B(o, r)] \lesssim \Sigma_r^{(1)} + \Sigma_r^{(2)},$$

to prove (7), it suffices to show that $\Sigma_r^{(2)} \lesssim r^2$. Choose $k > 2$ with Corollary 2.6 in mind. Now

$$\sum_{m > \alpha r^2 \log V(r)} m^{-k/2} r^k \lesssim \left(r^2 \log V(r)\right)^{-k/2+1} r^k \lesssim r^2. \tag{8}$$

On the other hand,

$$\sum_{m > \alpha r^2 \log V(r)} V(r)e^{-c^2 m/r^2} \lesssim V(r)r^2 e^{-\alpha c^2 \log V(r)} = r^2. \tag{9}$$

Therefore, $\Sigma_r^{(2)} \lesssim r^2$, as claimed. $\qquad \square$

Remark 2.7. If G has polynomial growth, then we can separate the sum in (7) at αr^2 instead of at $\alpha r^2 \log V(r)$. The same argument as above combined with the bounds $V(r) \asymp r^D$ and $p_{2m}(o, o) \asymp m^{-D/2}$ then gives a proof of the quadratic bound on occupation time; one does not need Lemma 2.4, but only the trivial bound that every probability is at most 1.

3   Volume growth of the WSF

Given a finite path $\mathcal{P} = \langle v_0, v_1, \ldots, v_n \rangle$ in a graph G, we define the forward loop erasure of $\mathcal{P}$ (denoted by $\mathbf{LE}[\mathcal{P}]$) by erasing cycles in $\mathcal{P}$ chronologically. More precisely, $\mathbf{LE}[\mathcal{P}]$ is defined inductively as follows. The first vertex u_0 of $\mathbf{LE}[\mathcal{P}]$ is the vertex v_0 of $\mathcal{P}$. Supposing that u_j has been set, let k be the last index such that $v_k = u_j$. Set $u_{j+1} := v_{k+1}$ if $k < n$; otherwise, let $\mathbf{LE}[\mathcal{P}] := \langle u_0, \ldots, u_j \rangle$. If S is a simple random walk on a Cayley graph G, then $\mathbf{LE}[S]$ is called the *loop-erased random walk (LERW)*. There is no trouble defining the forward loop erasure of S a.s. if G is transient. For recurrent Cayley graphs of quadratic growth, loop-erased random walk can be defined by taking a limit (see [Law13, BLPS01]). We omit the details, because we focus exclusively on transient graphs in the rest of the paper.

In [Wil96], Wilson discovered an algorithm for sampling uniform spanning trees on finite graphs using loop-erased random walk. In [BLPS01], Wilson's algorithm was adapted to sample the WSF on transient graphs: Order the vertex set V as $V = (v_1, v_2, \ldots)$. Set $\mathcal{T}_0 := \emptyset$. Inductively, for each $n = 1, 2, \ldots$, run an independent simple random walk starting at v_n. Stop the walk when it hits $\mathcal{T}_{n-1}$ if it does; otherwise, let it run indefinitely. Denote the resulting path by $\mathcal{P}_n$, and set $\mathcal{T}_n := \mathcal{T}_{n-1} \cup \mathbf{LE}[\mathcal{P}_n]$. According to [BLPS01, Theorem 5.1] no matter the ordering of V, the resulting forest is always distributed as the WSF on G. This method of generating the WSF is called *Wilson's method rooted at infinity*.

In fact, the theory of wired spanning forests extends to general networks, i.e., general reversible random walks; see [BLPS01] or [LP16] for details. Thus, we will prove the following extension of Theorem 1.3:

Theorem 3.1. *Let G be a Cayley graph of a group Γ and $V(r) := |B(o, r)|$. Consider the WSF on Γ corresponding to a finite-range symmetric random walk S whose support generates Γ. Then*

$$\mathbb{E}\big[|\mathcal{T}_o \cap B(o, r)|\big] \lesssim r^4 \log^{3/2} V(r). \tag{10}$$

In particular, $\mathbb{E}\big[|\mathcal{T}_o \cap B(o,r)|\big] \lesssim r^{11/2}$.

Proof. The polynomial-growth case is known when the WSF is generated by simple random walk; the proof of its extension to finite-range symmetric random walks will be clear following Remark 2.7. Thus, we assume G has superpolynomial growth. We may further assume that S is lazy, since adding laziness simply produces loops in the random walk paths, which are then erased.

Let $\{S^v\}_{v \in G}$ be a family of independent random walks with the same increment distribution as S but such that S^v starts from v. Let $\mathbb{P}_v$ be the law of S^v. By Wilson's algorithm rooted at infinity,

$$\mathbb{P}[x \in \mathcal{T}_o] \leq \mathbb{P}[\exists y \in G \ \ \exists m \geq k \geq 0 \ \ \ S^o(k) = S^x(m-k) = y]$$

$$\leq \sum_{y \in G} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \mathbb{P}_o[S_k = y]\, \mathbb{P}_x[S_{m-k} = y]\,. \tag{11}$$

By reversibility and the Markov property,

$$\sum_{y \in G} \mathbb{P}_o[S_k = y]\, \mathbb{P}_x[S_{m-k} = y] = \mathbb{P}_o[S_m = x]\,.$$

Combined with (11), this leads to

$$\mathbb{P}[x \in \mathcal{T}_o] \leq \sum_{m=0}^{\infty} (m+1)\, \mathbb{P}_o[S_m = x]\,.$$

Summing over $x \in B(o,r)$, we arrive at

$$\mathbb{E}\big[|\mathcal{T}_o \cap B(o,r)|\big] \leq \sum_{m=0}^{\infty} (m+1)\, \mathbb{P}_o\big[S_m \in B(o,r)\big]\,.$$

Decomposing this last sum similarly to the proof of Theorem 1.2, we have

$$\sum_{m=0}^{\infty} (m+1)\, \mathbb{P}[S_m \in B(o,r)] \lesssim \Sigma_r^{(3)} + \Sigma_r^{(4)}\,,$$

where

$$\Sigma_r^{(3)} := \sum_{m=0}^{\lfloor \alpha r^2 \log V(r) \rfloor} (m+1)\, \mathbb{P}_o\big[S_m \in B(o,r)\big]\,,$$

$$\Sigma_r^{(4)} := \sum_{m > \alpha r^2 \log V(r)} V(r)(m+1)\varphi(m)\,,$$

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and φ is the right-hand side of (6). Using a very similar argument as in Theorem 1.2, by choosing $k > 4$ and $\alpha := 2c^{-2}$, we obtain

$$\Sigma_r^{(3)} \lesssim r^4 \log^{3/2} V(r) \quad \text{and} \quad \Sigma_r^{(4)} \lesssim r^4,$$

thus concluding the proof. $\qquad\square$

To prove Theorem 1.4, we first record an elementary fact concerning simple random walk on Cayley graphs.

Lemma 3.2. *Let G be a Cayley graph of superpolynomial growth and S be a simple random walk starting from $o \in G$. For a vertex $x \in G$, let $|x|$ denote the graph distance from x to o. Then for every $D > 0$ there exists a positive constant c_D such that*

$$\mathbb{P}_o[S \text{ hits } x] \leq \frac{c_D}{|x|^D}. \tag{12}$$

Proof. Indeed, by Lemma 2.3, for example,

$$\mathbb{P}_o[S \text{ hits } x] \leq \sum_{n \geq |x|} p_n(o, x) \lesssim \sum_{n \geq |x|} n^{-D-1} \lesssim |x|^{-D}. \qquad\square$$

Proof of Theorem 1.4. Suppose the WSF is generated via Wilson's algorithm by first sampling a simple random walk S from o and then sampling simple random walks from other vertices in a certain order. Let $\mathrm{Ray}_o := \mathbf{LE}[S]$ be the infinite ray emanating from o in the WSF, $\mathrm{Ray}(o, r) := \mathrm{Ray}_o \cap C(o, r)$, and $N_r := |\mathrm{Ray}(o, r)|$. We first claim that $\mathbb{E}[N_r] \leq 2\,\mathbb{E}[\tau_{3r}]$ for r large enough.

To verify this claim, we use the argument illustrated in Figure 1. Let $\rho_0 := 0$ and $\tau_{2r}^0 := \tau_{2r}$. For $i \geq 1$, let

$$\rho_i := \inf\{t : t > \tau_{2r}^{i-1}, \, S_t \in \mathbf{LE}[S(0, \tau_{2r}^{i-1})] \cap B(o, r)\} \tag{13}$$

and

$$\tau_{2r}^i := \inf\{t : t > \rho_i, \, S_t \notin B(o, 2r)\}. \tag{14}$$

Since G has superpolynomial growth, by Lemma 3.2, conditioned on $S[0, \tau_{2r}^{i-1}]$, the probability that S hits a certain point in $B(o, r)$ after τ_{2r}^{i-1} is bounded by cr^{-4}, where c depends only on G. Let L_r be the occupation measure of $B(o, r)$ as defined in Theorem 1.2. Then by conditioning on $\mathbf{LE}[S(0, \tau_{2r}^{i-1})] \cap B(o, r)$ and applying Theorem 1.2, we get

$$\mathbb{P}[\rho_i < \infty \mid \rho_{i-1} < \infty] \leq \mathbb{E}\big[cr^{-4}\,|\mathbf{LE}[S(0, \tau_{2r}^{i-1})] \cap B(o, r)| \,\big|\, \rho_{i-1} < \infty\big]$$
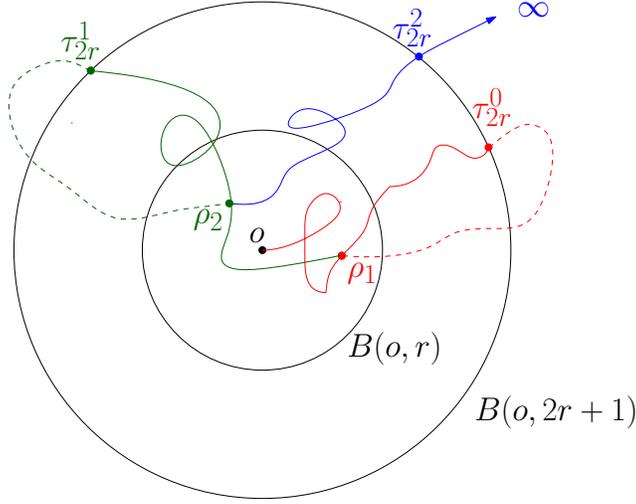$$\leq cr^{-4}\,\mathbb{E}[L_r] \lesssim r^{-1}.$$

Figure 1: Every time S exits $B(o, 2r)$, there is at least $\frac{1}{2}$ chance that afterwards it never visits the vertex set in $B(o, r)$ that is already occupied by S. In this figure, $\xi = 3$. Note that the dashed part does not contribute to $\mathbf{LE}[S] \cap C(o, r)$. Therefore $N_r \leq \sum_{i=0}^{\xi}(\tau_{2r}^i - \rho_i)$.

Therefore we may choose r large enough that

$$\mathbb{P}[\rho_i < \infty \mid \rho_{i-1} < \infty] < 1/2. \tag{15}$$

Fix such an r. We have by the strong Markov property that

$$\mathbb{P}[\tau_{2r}^i - \rho_i > a \mid \rho_i < \infty,\, S_{\rho_i} = x] \leq \mathbb{P}[\tau_{3r} > a] \tag{16}$$

for every $a \geq 0$ and every x. Let $\xi := \inf\{m : \rho_m = \infty\}$. Then by (15) and (16), $\sum_{i=0}^{\xi-1}(\tau_{2r}^i - \rho_i)$ is stochastically dominated by $\sum_{i=0}^{\widetilde{\xi}-1} \tau_{3r}^i$, where $\{\tau_{3r}^i\}_{i \geq 0}$ is a sequence of i.i.d. random variables with the same distribution as τ_{3r} and $\widetilde{\xi}$ is an independent geometric random variable with mean 2.

Since $\mathbf{LE}[S] \cap C(o, r)$ is covered by the set $\bigcup_{i=0}^{\xi-1} S[\rho_i, \tau_{2r}^i]$ when $S(0) = o$, we have

$$\mathbb{E}[N_r] \leq \mathbb{E}\sum_{i=0}^{\widetilde{\xi}-1} \tau_{3r}^i = 2\,\mathbb{E}[\tau_{3r}]\,,$$

as claimed.

To bound $|C(o, r)|$, we need to bound the number of vertices in $B(o, r)$ that connect to $\mathrm{Ray}(o, r)$ through the WSF *entirely inside* $B(o, r)$.

For $x, v \in B(o, r)$, write $x \sim_C v$ for the event that $v \in \mathrm{Ray}(o, r)$ and x and v are connected in $C(o, r)$ via a path containing no vertices of $\mathrm{Ray}(o, r)$ other

than v. For all $y \in B(o, r)$, let T_y be the hitting time of y for a simple random walk. Let $\mathbb{P}_y$ be the distribution of a simple random walk S starting from y. Given $\{v_j : 1 \leq j \leq N\} \subset B(o, r)$, write A for the event that $\mathrm{Ray}(o, r) = \{v_j : 1 \leq j \leq N\}$. For all $1 \leq i \leq N$ and $\{v_j : 1 \leq j \leq N\} \subset B(o, r)$,

$$
\begin{aligned}
\mathbb{P}\big[y \sim_C v_i \mid A\big] &= \mathbb{P}_y\big[S \text{ hits } \mathrm{Ray}(o, r) \text{ at } v_i \text{ and } \mathbf{LE}[S(0, T_{v_i})] \subset B(o, r) \mid A\big] \\
&\leq \mathbb{P}_y\big[\mathbf{LE}[S(0, T_{v_i})] \subset B(o, r)\big] = \mathbb{P}_{v_i}\big[\mathbf{LE}[S(0, T_y)] \subset B(o, r)\big],
\end{aligned}
\tag{17}
$$

where the last equality is by reversibility of LERW [Law13, Lemma 7.2.1].

Let $M_v := |\{y \in B(o, r) : y \sim_C v\}|$. Then

$$
\begin{aligned}
\mathbb{E}\big[M_{v_i} \mid A\big] &\leq \sum_{y \in B(o, r)} \mathbb{P}_{v_i}\big[\mathbf{LE}[S(0, T_y)] \subset B(o, r)\big] \\
&= \mathbb{E}_{v_i}\big[|\{y \in B(o, r) : \mathbf{LE}[S(0, T_y)] \subset B(o, r)\}|\big] \\
&\leq \mathbb{E}_o\big[|\{y \in B(o, 2r) : \mathbf{LE}[S(0, T_y)] \subset B(o, 2r)\}|\big].
\end{aligned}
$$

Let τ_{4r}^i, ρ_i, ξ be defined as in (13) but replacing $B(o, r)$ and $B(o, 2r)$ by $B(o, 2r)$ and $B(o, 4r)$, respectively. Then $\{y \in B(o, 2r) : \mathbf{LE}[S(0, T_y)] \subset B(o, 2r)\}$ is covered by the set $\bigcup_{i=0}^{\xi-1} S[\rho_i, \tau_{4r}^i]$ when $S(0) = o$. By the same argument above that proved $\mathbb{E}[N_r] \leq 2\,\mathbb{E}[\tau_{3r}]$, we have

$$
\mathbb{E}_o\big[|\{y \in B(o, 2r) : \mathbf{LE}[S(0, T_y)] \subset B(o, 2r)\}|\big] \leq 2\,\mathbb{E}[\tau_{6r}].
$$

Therefore, writing $\mathrm{Ray}(o, r) = \{v_i : 1 \leq i \leq N_r\}$, we have

$$
\begin{aligned}
\mathbb{E}\big[|C(o, r)|\big] &= \mathbb{E}\Big[\mathbb{E}\big[\sum_{i=1}^{N_r} M_{v_i} \mid \mathrm{Ray}(o, r)\big]\Big] \\
&\leq 2\,\mathbb{E}[\tau_{6r}]\,\mathbb{E}[N_r] \leq 4\,\mathbb{E}[\tau_{6r}]^2. \qquad \square
\end{aligned}
$$

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