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Diffusions and Random Shadows in Negatively Curved Manifolds

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Let M be a d-dimensional complete simply-connected negatively-curved manifold. There is a natural notion of Hausdorff dimension for its boundary at infinity. This is shown to provide a notion of global curvature or average rate of growth in two probabilistic senses: First, on surfaces ($d=2$), it is twice the critical drift separating transience from recurrence for Brownian motion with constant-length radial drift. Equivalently, it is twice the critical β for the existence of a Green function for the operator $\Delta/2-\beta\partial_r$. Second, for any d, it is the critical intensity for almost sure coverage of the boundary by random shadows cast by balls, appropriately scaled, produced from a constant-intensity Poisson point process.

1. Introduction

Let M be a d-dimensional complete simply-connected nonpositively-curved manifold. Such manifolds are called *Cartan-Hadamard*, or *CH* manifolds after the theorem of Cartan and Hadamard that says that the exponential map is a diffeomorphism at every point of the manifold. In particular, every pair of points is joined by a unique geodesic. Of course, trees share this property. Indeed, trees and CH manifolds play analogous roles in some parts of mathematics such as representation theory. Here, we shall present two analogues of probabilistic theorems known for trees (Lyons 1990). One should think of greater branching in a tree as corresponding to more negative curvature of a manifold.

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Fix a point, denoted 0, in M and consider first Brownian motion on M with radial drift $-\beta\partial_r$ towards 0, i.e., the diffusion[1] BM_β whose generator is $\Delta/2-\beta\partial_r$. (This diffusion seems first to have been considered by Kendall (1974), see especially pp. 381–400, where M was the euclidean plane.) As is well known and rederived in Section 5, this diffusion is reversible and corresponds to the scalar conductivity $e^{-2\beta r}$ on M. Since typically M grows exponentially, ordinary Brownian motion ($\beta=0$) is transient. However, there is also typically some β for which BM_β is recurrent. The critical value of β separating the transience and recurrence regimes is denoted $\beta_c(M)$ and may be infinite. Equivalently, for $\beta<\beta_c(M)$, there is a Green function for $\Delta/2-\beta\partial_r$, while for $\beta>\beta_c(M)$, there is no Green function. The quantity $\beta_c(M)$ represents a balance of the growth of M, but is more subtle than the exponential growth rate of volume since how the manifold is "put together" matters. In fact, we shall show that, at least for CH surfaces whose sectional curvatures are bounded below, $\beta_c(M)$ is equal to half the Hausdorff dimension of the ideal boundary $\partial_\infty M$ of M with respect to a certain "metric" (Theorem 5.7).

The *ideal boundary*, or boundary at infinity, of M is the set of geodesic rays emanating from 0. It is denoted $\partial_\infty M$. This can, of course, be identified with the unit tangent sphere at 0. If one wishes to eliminate the dependence on 0, one can take for $\partial_\infty M$, instead, the set of equivalence classes of geodesic rays from arbitrary starting points, where two rays are equivalent if their distance from each other is bounded. As we review in Section 2, Kaimanovich (1990) and Gromov (1987) introduced a natural "metric" ρ on $\partial_\infty M$ that is analogous to one long used for trees. [The reason for the quotation marks is that ρ may not satisfy the triangle inequality.] It is the Hausdorff dimension of $\partial_\infty M$ with respect to ρ that is referred to above.

This result on $\beta_c(M)$ uses not only the analogous result for trees, but an analogous one for planar graphs proved here. This may be of separate interest and Section 4, which is devoted to it, can be read independently.

In order to see more clearly how $\beta_c(M)$ balances the intrinsic growth of M, write the Laplacian in geodesic spherical coordinates (r,Θ): since in

[1] For nonprobabilists, a diffusion whose generator is the differential operator L is, loosely speaking, a random continuous function X_t taking values in M for t in some interval such that the values of X_t for $t\geqslant t_0$ depend probabilistically on those for $t\leqslant t_0$ only through X_{t_0} and such that for smooth functions f, we have $\lim_{t\to t_0}\mathbf{E}[f(X_t)-f(X_{t_0})\mid X_{t_0}]/(t-t_0)=(Lf)(X_{t_0})$. Brownian motion without drift can also be thought of as the limit as $\varepsilon\to 0$ of the following random walk paths, $Y_t^{(\varepsilon)}$: at time $t=(n+1)\varepsilon$, choose a random point $Y_{(n+1)\varepsilon}^{(\varepsilon)}$ on the geodesic sphere of radius $\sqrt{d\varepsilon}$ about $Y_{n\varepsilon}^{(\varepsilon)}$ and join it to $Y_{n\varepsilon}^{(\varepsilon)}$ by a geodesic to define Y_t for $n\varepsilon\leqslant t\leqslant (n+1)\varepsilon$.

general coordinates, we have $\Delta = \sqrt{g(x)}^{-1} \sum \partial_i \sqrt{g(x)}\, g^{ij}(x)\, \partial_j$, where the metric is g_{ij} with inverse g^{ij} and determinant g, we obtain

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial \log \sqrt{g(t, \Theta)}}{\partial r} \frac{\partial}{\partial r} + L_\Theta,$$

where L_Θ has derivatives in Θ only. Note that in geodesic spherical coordinates, $\sqrt{g(r, \Theta)}$ is the $(d-1)$-dimensional volume element on the geodesic sphere of radius r centered at 0. The term involving $\log \sqrt{g(r, \Theta)}$ gives an outward drift to the radial component of Brownian motion. This is counteracted by the inward radial drift of BM_β. The "global" balancing point is at $\beta_c(M)$.

Our second result on manifolds is an analogue of a percolation result on trees. Let $\beta, r > 0$ and distribute points randomly in M according to a Poisson point process of intensity β times volume measure. Place an open ball, which we will refer to as a *cloud*[2] , of radius r centered at each of these points. Each cloud casts a *shadow* on $\partial_\infty M$, namely the set of rays from 0 that intersect the cloud.[3] If the clouds or their intensity are too large, then a.s. the shadows cover the whole boundary.[4] We are interested in the phase boundary for a.s. coverage. In particular, we show that if γ_r denotes the volume of a euclidean ball of radius r in $\mathbf{R}^{d-1}$ and $\beta_c(M, r)$ denotes the critical intensity for a.s. coverage by clouds of radius r, then $\lim_{r \to 0} \beta_c(M, r)\, \gamma_r = \dim \partial_\infty M$ (Corollary 6.2), at least if the sectional curvatures are bounded away from zero and negative infinity. The determination of $\beta_c(M, r)$ for each r is, at least in the case of surfaces, given by a capacity criterion (Theorem 6.3 and Theorem 6.7). In the case of manifolds of constant negative curvature, one can calculate explicitly $\beta_c(M, r)$ for each r, as was done by Kahane (1990), (1991) for surfaces using somewhat different methods.

By analogy, then, with trees (Lyons 1990), we may think of $\dim \partial_\infty M$ as a global measure of curvature or average rate of growth. The following facts about it will be shown in Section 3. Let K be the sectional curvature. If $-a^2 \geqslant K \geqslant -b^2$, then $(d-1)\, a \leqslant \dim \partial_\infty M \leqslant (d-1)\, b$. The Hausdorff dimension is unchanged if the Riemannian metric is changed on a compact subset of M, at least if the sectional curvatures are bounded away from

[2] Despite Mandelbrot's famous statement, "Clouds are not spheres" (Mandelbrot (1983), p. 1.)

[3] Olbers' paradox in physics is based on a similar model of stars in euclidean space (Harrison 1987). Other early results were found by Chernoff and Daly (1957), who investigated the case of shadows cast on a line in the euclidean plane.

[4] No matter what the size or intensity of the clouds, Fubini's theorem implies that a.s. μ-almost every boundary point is blocked by a cloud for any measure μ on $\partial_\infty M$ fixed in advance.

zero. It is always at most the liminf exponential rate of volume growth, but these two numbers are, in general, unequal. In case M is the universal cover of a compact negatively-curved manifold, $\dim \partial_\infty M$ is equal to the rate of volume growth (which equals the topological entropy of the geodesic flow (Manning 1979)).

2. The Ideal Boundary

Write $\mathrm{dist}(\bullet, \bullet)$ for the distance function on M. Denote the ball of radius r about x by $B_r(x)$ and the sphere of radius r about 0 by S_r. We call a set *r-separated* if distinct points in the set are at distance at least r from each other. For a point $x \in M$, denote the ray from 0 passing through x by $\vec{x}$. For a ray ξ, let $\xi(r)$ denote the point on ξ at distance r from 0. Kaimanovich (1990) introduced the following function ρ on $\partial_\infty M$:

$$\rho(\xi, \eta) := e^{-t} \qquad (\xi \neq \eta),$$

where t is the number such that $\mathrm{dist}(\xi(t), \eta(t)) = 1$. This gives the usual euclidean topology to $\partial_\infty M$ (identified with the unit tangent sphere at 0). If the sectional curvatures are bounded away from zero, then there is some $\varepsilon > 0$ such that ρ^ε is a metric (Kaimanovich (1990), Prop. 1.2) and then (Kaimanovich (1990), Prop. 1.4) it is equivalent to a metric introduced by Gromov (1987), §§ 7.2.K, 7.2.L, 7.2.M. Although ρ may not be a metric, we still use it to define Hausdorff dimension on $\partial_\infty M$:

$$\dim \partial_\infty M := \inf \left\{ \alpha; \inf \left\{ \sum_i r_i^\alpha ; \partial_\infty M = \bigcup_i B_{r_i}(\xi_i) \right\} = 0 \right\},$$

where

$$B_{r_i}(\xi_i) := \{ \eta \in \partial_\infty M; \rho(\xi_i, \eta) < r_i \}.$$

Let (X, d) be a metric space. Denote by $\mathsf{Prob}(X)$ the set of Borel probability measures on X. Define the *capacitary dimension* of X to be

$$\operatorname{cap\,dim} X := \sup \left\{ \alpha \geqslant 0; \exists \mu \in \mathsf{Prob}(X) \iint \frac{d\mu(x)\, d\mu(y)}{d(x, y)^\alpha} < \infty \right\}.$$

We shall need the Frostman-type result that $\operatorname{cap\,dim} \partial_\infty M = \dim \partial_\infty M$ in Section 6. See Howroyd (1995) for a simple proof of the following.

Theorem 2.1. *If X is a compact metric space, then* $\dim X = \operatorname{cap\,dim} X$.

If some power ρ^ε is a metric (such as when the sectional curvatures are bounded away from zero), then we may apply Theorem 2.1 to $(\partial_\infty M, \rho^\varepsilon)$

in order to derive the same conclusion for $(\partial_\infty M, \rho)$. However, ρ may not have any power which is a metric. Most of our arguments are unaffected by this, but Theorem 2.1 enters in a crucial way. In order to avoid placing extra curvature assumptions in our results, we introduce the following notion of dimension.

Given a compact Hausdorff space X with a function $\rho\colon X \times X \to [0, \infty]$, write

$$B_r(x) := \{y \in X;\ \rho(x, y) < r\}.$$

Suppose that the sets $B_r(x)$ $(r > 0, x \in X)$ are open and $\overline{B_r(x)} \subseteq B_s(x)$ for $r < s$. Define the *weighted Hausdorff dimension* of X by

$$\operatorname{wdim} X := \inf \left\{\alpha;\ \inf \left\{\sum_i c_i r_i^\alpha;\ \mathbf{1}_X \leqslant \sum_i c_i \mathbf{1}_{B_{r_i}(x_i)}\right\} = 0\right\}.$$

In general, it is clear that $\operatorname{wdim} X \leqslant \dim X$. Define $\operatorname{cap\,dim} X$ as before with ρ in place of d. Then Howroyd's (1995) proof shows that

$$\operatorname{cap\,dim} X = \operatorname{wdim} X. \tag{2.1}$$

3. Dimension Comparisons

We now establish some properties of $\dim \partial_\infty M$ that we shall not need in the sequel but help in understanding its significance. First, it is independent of base point used—at least, if the sectional curvatures are bounded away from zero, since then we have uniform equivalence of the functions ρ based at different points (Kaimanovich (1990), Prop. 1.2). (Here, we identify $\partial_\infty M$ not with a unit tangent sphere but with equivalence classes of geodesics.) Also, $\dim \partial_\infty M$ is unchanged when the Riemannian metric is changed in a compact region of M—again, at least if the sectional curvatures are bounded away from zero. For in that case, Kaimanovich (1990), Prop. 1.4, shows that $-\log \rho(\xi, \eta)$ differs by a bounded amount from the distance between 0 and the (unique) geodesic $[\xi, \eta]$ asymptotic to ξ and η. If ξ and η are sufficiently close, then $[\xi, \eta]$ avoids the region where the metric is changed and, hence, its distance from 0 differs by a bounded amount in the two metrics. This again gives uniform equivalence of the functions ρ for the two metrics.

Proposition 3.1. *If M is a CH manifold with sectional curvatures satisfying $-a^2 \geqslant K \geqslant -b^2$, then $(d-1)\,a \leqslant \dim \partial_\infty M \leqslant (d-1)\,b$.*

Proof. We may assume that $a > 0$ or $b < \infty$ in the proofs of the corresponding inequalities. Let M_a and M_b be the d-dimensional CH manifolds of constant curvatures a and b. Fix points $x_a \in M_a$ and $x_b \in M_b$. Identify x_a and x_b with $0 \in M$ and choose isometric identifications of the tangent spaces $T_{x_a}M_a$, $T_{x_b}M_b$ with T_0M. In particular, this identifies the unit tangent spheres and, so, the ideal boundaries, $\partial_\infty M$, $\partial_\infty M_a$, $\partial_\infty M_b$. Choose $\xi, \eta \in \partial_\infty M$ and let $t := -\log \rho(\xi, \eta)$. Then a corollary of the Rauch comparison theorem (Cheeger and Ebin (1975), Cor. 1.30) implies that $\operatorname{dist}_{M_a}(\xi(t), \eta(t)) \leqslant 1 \leqslant \operatorname{dist}_{M_b}(\xi(t), \eta(t))$, whence $\rho_{M_a}(\xi, \eta) \leqslant \rho(\xi, \eta) \leqslant \rho_{M_b}(\xi, \eta)$. Now the hyperbolic sine rule implies that $\rho_{M_a}(\xi\eta) \sim c_a \theta(\xi, \eta)^{1/a}$ for some constant c_a as the angle $\theta(\xi, \eta)$ between ξ and η tends to 0, and likewise for M_b. This proves the result. ▮

Let $N(r)$ be the minimum number of "balls" $B_r(\xi_i)$ of "radius" r needed to cover $\partial_\infty M$. The Minkowski dimension of $\partial_\infty M$ is

$$\operatorname{Mdim} \partial_\infty M := \liminf_{r \to 0} -\log N(r)/\log r.$$

The (liminf exponential) growth rate of volume in M is

$$\operatorname{loggr} M := \liminf_{t \to \infty} \frac{1}{t} \log \operatorname{Vol} B_t(0).$$

We always have

$$\dim \partial_\infty M \leqslant \operatorname{Mdim} \partial_\infty M \leqslant \operatorname{loggr} M.$$

The first inequality follows from the definitions. For the second, let $A(t)$ be a maximal set of points in S_t that are at distance at least 1 from each other and let $N'(t)$ be the size of $A(t)$. Set $r := e^{-t}$. Then $\partial_\infty M = \bigcup_{x \in A(t)} B_r(\vec{x})$, whence $N(r) \leqslant N'(t)$. On the other hand, the open balls of radius 1/2 centered at the points in $A(t)$ are disjoint, lie within $B_{t+1/2}(0)$, and have volume at least that of a euclidean ball of the same radius and dimension. Hence there is some constant c such that $N'(t) \leqslant c \operatorname{Vol} B_{t+1/2}(0)$. Therefore, $N(r) \leqslant c \operatorname{Vol} B_{t+1/2}(0)$. Letting $t \to \infty$, we conclude that $\operatorname{Mdim} \partial_\infty M \leqslant \operatorname{loggr} M$.

If the sectional curvatures are bounded below, then $\operatorname{Mdim} \partial_\infty M = \operatorname{loggr} M$. For then a simple modification of the preceding argument shows that the surface area $|S_t|$ of S_t is at most $cN(r)$ for some constant c, whence $\operatorname{Vol} B_t(0) = \int_0^t |S_u| \, du < t\,|S_t| \leqslant ctN(r)$, from which the inequality $\operatorname{loggr} M \leqslant \operatorname{Mdim} \partial_\infty M$ follows.

For surfaces, at least, no such curvature assumption is in fact needed for the equality $\operatorname{Mdim} \partial_\infty M = \operatorname{loggr} M$. For if $\partial_\infty M = \bigcup_{i=1}^{N(r)} B_r(\xi_i)$, then let $t := -\log r - 1/2$. For any $\xi \in B_r(\xi_i)$, the distance from 0 to the geodesic joining $\xi_i(-\log r)$ to $\xi(-\log r)$ is at least t. By Lemma 5.5, it follows that $|S_t| < 2N(r)$. The argument is now completed as before.

Call M *rotationally symmetric about* 0 if the rotations of $T_0 M$ induce isometries of M.

PROPOSITION 3.2. *If M is rotationally symmetric about* 0, *then* $\dim \partial_\infty M = \operatorname{loggr} M$.

Proof. We first show that $\dim \partial_\infty M = \operatorname{Mdim} \partial_\infty M$. Let $\theta(\xi, \eta)$ be the angle between ξ and η. Write $F(\varepsilon) := \rho(\xi, \eta)$ when $\theta(\xi, \eta) = \varepsilon$. Choose $\alpha > \dim \partial_\infty M$. We claim that there are arbitrarily small $\varepsilon > 0$ for which $\varepsilon^{d-1} > F(\varepsilon)^\alpha$. For if not, we would have that every cover of $\partial_\infty M$ by balls B_{r_i} with radii sufficiently small would satisfy

$$\sum_i r_i^\alpha \geqslant \sum_i F^{-1}(r_i)^{d-1},$$

yet the latter sum is bounded below because the $(d-1)$-dimensional Hausdorff measure of the euclidean sphere in $\mathbf{R}^d$ is positive.

It follows that $\operatorname{Mdim} \partial_\infty M$ is at most $\alpha/(d-1)$ times the Minkowski dimension of the euclidean sphere in $\mathbf{R}^d$, which is $d-1$. That is, $\operatorname{Mdim} \partial_\infty M \leqslant \alpha$. Since this is true for every $\alpha > \dim \partial_\infty M$, we get $\dim \partial_\infty M = \operatorname{Mdim} \partial_\infty M$.

Now we show that $\operatorname{Mdim} \partial_\infty M = \operatorname{loggr} M$. As in the preceding discussion of surfaces, if $\partial_\infty M = \bigcup_{i=1}^{N(r)} B_r(\xi_i)$, then let $t := -\log r - 1/2$. The intersection of S_t with the rays in $B_r(\xi_i)$ is contained in a ball on S_t of radius less than 1 (measured on S_t) by Lemma 5.5. By spherical symmetry, the intrinsic curvature of S_t is constant, whence it is positive. Therefore, such a ball has volume less than that of a euclidean ball of radius 1 and dimension $d-1$. Hence $|S_t| < \gamma_1 N(r)$, from which the conclusion follows. ∎

As we mentioned in the introduction, if M is the universal cover of a compact negatively-curved manifold, then $\dim \partial_\infty M = \operatorname{loggr} M$. To see this, we need only establish that $\dim \partial_\infty M \geqslant \operatorname{loggr} M$. But the Patterson–Sullivan measure has Hausdorff dimension equal to $\operatorname{loggr} M$ (Kaimanovich (1990), §3.5, or Coornaert (1993)).

An example where $\dim \partial_\infty M < \operatorname{Mdim} \partial_\infty M$ follows. Let (R, θ) be polar coordinates on $\mathbf{R}^2$. Let R_k be a sequence such that $R_{k+1}/R_k \to \infty$ rapidly and $\varepsilon(R) > 0$ a function tending to 0 rapidly as $R \to \infty$. Let $0 < a < b < \infty$. Let $K(R, \theta)$ be a smooth function with values in $[-b^2, -a^2]$ such that

$$K(R, \theta) = \begin{cases} -a^2 & \text{if } R_{2k} \leqslant R \leqslant R_{2k+1} \text{ for some } k \text{ and } |\theta| \leqslant \pi/2 - \varepsilon(R) \\ & \quad \text{or } R_{2k+1} + 1 \leqslant R \leqslant R_{2k+2} - 1 \\ & \quad \text{for some } k \text{ and } |\theta - \pi| \leqslant \pi/2 - \varepsilon(R); \\ -b^2 & \text{if } R_{2k} \leqslant R \leqslant R_{2k+1} \text{ for some } k \text{ and } |\theta - \pi| \leqslant \pi/2 - \varepsilon(R) \\ & \quad \text{or } R_{2k+1} + 1 \leqslant R \leqslant R_{2k+2} - 1 \\ & \quad \text{for some } k \text{ and } |\theta| \leqslant \pi/2 - \varepsilon(R). \end{cases}$$

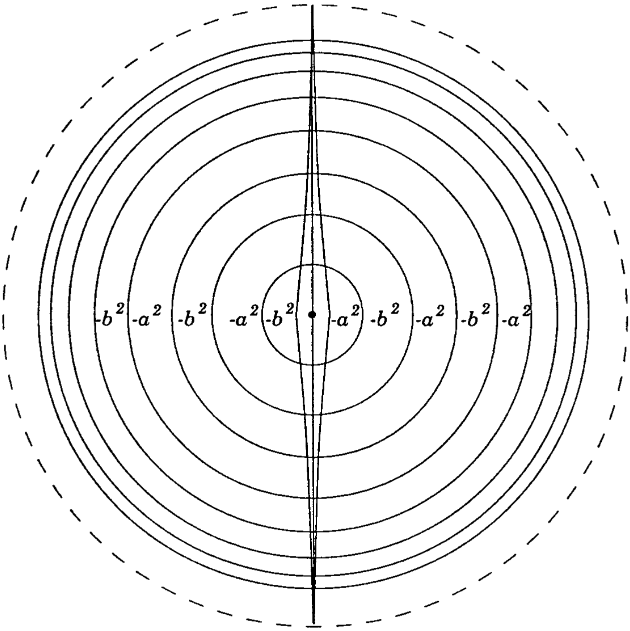

FIG. 3.1. Schematic representation of a surface and its ideal boundary with unequal Hausdorff and Minkowski dimensions.

See Fig. 3.1. Define $f(R, \theta)$ by $f_{RR}(R, \theta) + K(R, \theta) f(R, \theta) = 0$, $f(0, \theta) = 0$, and $f_R(0, \theta) = 1$. Let M be $\mathbf{R}^2$ with the metric $dR^2 + f(R, \theta)^2 \, d\theta^2$. Then M is a CH surface with curvature equal to $K(R, \theta)$, $\dim \partial_\infty M = a^2$, and $\mathrm{Mdim}\, \partial_\infty M = b^2/(2b - a)$. The easiest way to prove the dimension statements is to use Proposition 5.6 below; the tree T_M there is similar to an example on p. 936 of Lyons (1990).

4. Planar Graphs

Let G be an infinite connected locally-finite graph and let 0 be a vertex in G. The distance between points in G is the usual graph distance; in particular, the distance from v to 0 is written $|v|$ and the distance from 0 to the closest endpoint of an edge e is written $|e|$. Denote the sphere of radius n by $S_n := \{v;\ |v| = n\}$. Its cardinality is written $|S_n|$. Let Π_n be the set of edges joining S_{n-1} to S_n. We call $\mathrm{gr}(G) := \liminf_{n \to \infty} |\Pi_n|^{1/n}$ the *growth* of G.

Given $\lambda \geqslant 1$, we define a nearest-neighbor random walk on G denoted RW_λ as follows. Let $\deg v$ be the degree of a vertex $v \in G$ and let $\deg^- v$

stand for the number of edges connecting v to $S_{|v|-1}$. Then the transition probability from v to an adjacent vertex w is

$$p(v, w) := \begin{cases} \lambda/(\deg v + (\lambda - 1) \deg^{-} v) & \text{if} \quad w \in S_{|v|-1} \\ 1/(\deg v + (\lambda - 1) \deg^{-} v) & \text{otherwise.} \end{cases}$$

That is, from any vertex v, each edge connecting v to a vertex closer to 0 is λ times more likely to be taken than any other edge incident to v. (For $\lambda = 1$, this is simple random walk.) The walk RW_λ is reversible since it corresponds to the electrical network on G with conductance $\lambda^{-|e|}$ associated to the edge e (see, e.g., Doyle and Snell (1984)). Such random walks are sometimes referred to as "homesick", with λ being the measure of homesickness. They have been studied before in the context of trees (Berretti and Sokal (1985), Krug (1985), Lawler and Sokal (1988), Lyons (1990), and Lyons, Pemantle, and Peres (1996)) and Cayley graphs (Lyons 1995).

Define $\lambda_c(G) := \inf\{\lambda; \mathsf{RW}_\lambda \text{ is recurrent}\}$. In case G is spherically symmetric about 0, it is easy to show that $\lambda_c(G) = \operatorname{gr}(G)$. In general, however, this is not the case, even for trees. Instead, the following notion is more important for deciding the type of RW_λ.

Call a collection Π of edges a *cutset* in G if the removal of Π from G leaves 0 in a finite connected component. Set

$$\operatorname{br}(G) := \inf\left\{\lambda; \inf_{\Pi} \sum_{e \in \Pi} \lambda^{-|e|} = 0\right\}.$$

It is easy to see that $\operatorname{br}(G)$ does not depend on the choice of distinguished vertex, 0. Since Π_n are particular cutsets, we always have that $\operatorname{br}(G) \leqslant \operatorname{gr}(G)$. It was shown in Lyons (1990) that when G is a tree, then $\lambda_c(G) = \operatorname{br}(G)$. We call a subtree T of G rooted at 0 *geodesic* if for every vertex $v \in T$, the distance from v to 0 is the same in T as in G. Note that for such trees, $\operatorname{br}(T) \leqslant \operatorname{br}(G)$ since any cutset of G restricts to one in T. A subtree of G is called *spanning* if it includes every vertex of G. It follows from Lyons (1995) that when G is a Cayley graph of a finitely generated group, there is a geodesic spanning tree T of G with $\operatorname{br}(T) = \operatorname{gr}(G)$, whence we may conclude that $\operatorname{br}(G) = \operatorname{gr}(G)$. However, this equality is not generally valid even for trees.

For any graph, we have $\lambda_c(G) \leqslant \operatorname{br}(G)$ since if $\lambda > \operatorname{br}(G)$, the Nash–Williams criterion (Nash–Williams (1959), Griffeath and Liggett (1982), T. Lyons (1983)) shows that RW_λ is recurrent. For many planar graphs, we have also the converse:

THEOREM 4.1. *Let G be an infinite connected planar graph of bounded degree that can be embedded in the plane in such a way that only finitely* *many vertices are embedded in any bounded region. Assume that G has a geodesic spanning tree with no leaves. Then* $\lambda_c(G) = \mathrm{br}(G)$.

LEMMA 4.2. *Let G be an infinite connected planar graph of bounded degree that can be embedded in the plane in such a way that only finitely many vertices are embedded in any bounded region. Suppose that T is a geodesic spanning tree of G with no leaves. Then* $\mathrm{br}(T) = \mathrm{br}(G)$.

Proof. It suffices to show that for $\lambda > \mathrm{br}(T)$, we have $\lambda \geqslant \mathrm{br}(G)$. Given a cutset Π of T, we shall define a cutset Π^* of G whose corresponding cutset sum is not much larger than that of Π. Embed G in the plane in the manner assumed possible. We may assume then that 0 is at the origin of the plane and that all vertices in S_n are on the circle of radius n in the plane. Now every vertex $v \in T$ has a descendant subtree $T^v \subseteq T$. For $n \geqslant |v|$, this subtree cuts off an arc of the circle; in the clockwise order of $T^v \cap S_n$, there is a least element $\underline{v}_n$ and a greatest element $\bar{v}_n$. Each edge in Π has two endpoints; collect the ones farther from 0 in a set W. Define Π^* to be the collection of edges incident to the set of vertices

$$W^* := \{\underline{v}_n, \bar{v}_n;\, v \in W, n \geqslant |v|\}.$$

We claim that Π^* is a cutset of G. For if $0 = x_1, x_2, \ldots$ is a path in G with an infinite number of distinct vertices, let x_k be the first vertex belonging to T^v for some $v \in W$. Planarity implies that $x_k \in W^*$, whence the path intersects Π^*, as desired.

Now let c be the maximum degree of vertices in G. We have

$$\begin{aligned} \sum_{e \in \Pi^*} \lambda^{-|e|} &\leqslant c \sum_{v \in W^*} \lambda^{-|v|+1} \leqslant c \sum_{v \in W} \sum_{n \geqslant |v|} 2\lambda^{-n+1} \\ &= \frac{2c\lambda}{\lambda - 1} \sum_{v \in W} \lambda^{-|v|+1} = \frac{2c\lambda}{\lambda - 1} \sum_{e \in \Pi} \lambda^{-|e|}. \end{aligned}$$

Now the desired conclusion is evident. ∎

Proof of Theorem 4.1. We need only show that $\lambda_c(G) \geqslant \mathrm{br}(G)$. Let $\lambda < \mathrm{br}(G)$ and let T be a geodesic spanning tree of G. By the lemma, $\lambda < \mathrm{br}(T)$, whence by the result of Lyons (1990) quoted above, RW_λ is transient on T. Therefore, RW_λ is also transient on G. ∎

5. DIFFUSIONS ON MANIFOLDS

In order to apply the results of the preceding section to manifolds, we require a result of Kanai (1986) showing that diffusions can be "approximated" by random walks on a graph that is a discrete approximation of the manifold.

Let M be a complete Riemannian manifold. Given a function $\sigma(x)$ which is Borel-measurable, locally bounded and locally bounded below, called the (*scalar*) *conductivity*, we associate the diffusion whose generator is $(2\sigma(x)\sqrt{g(x)})^{-1}\sum \partial_i \sigma(x)\sqrt{g(x)}\, g^{ij}(x)\,\partial_j$ in coordinates, where the metric is g_{ij} with inverse g^{ij} and determinant g. In coordinate-free notation, this is $(1/2)\,\Delta + (1/2)\,\nabla \log \sigma$. In other words, the diffusion is Brownian motion with drift of half the gradient of the log of the conductivity, as is well known. In particular, BM_β corresponds to the conductivity $e^{-2\beta r}$. The main result of Ichihara (1978) [see also the exposition by Durrett (1986), p. 75; Fukushima (1980), Theorem 1.5.1, and Fukushima (1985); or Grigor'yan (1985)] gives the following test for transience.

THEOREM 5.1. *On a complete Riemannian manifold, the diffusion corresponding to the scalar conductivity $\sigma(x)$ is transient iff*

$$\inf\left\{\int |\nabla u(x)|^2\,\sigma(x)\,dx;\ u \in C_0^\infty(M),\ u \restriction B_1(0) \equiv 1\right\} > 0,$$

where dx is the volume form.

A graph G is called an ε-*net* of M if the vertices of G form a maximal ε-separated subset of M and edges join distinct vertices iff their distance in M is at most 3ε. When a conductivity σ is given on M, we assign conductances C to the edges of G by

$$C(v, w) := \int_{B_\varepsilon(v)} \sigma(x)\,dx + \int_{B_\varepsilon(w)} \sigma(x)\,dx. \tag{5.1}$$

Recall that the associated nearest-neighbor random walk on G has transition probabilities proportional to the conductances. In general, for a graph with conductances such that for every vertex, the sum of the conductances of the incident edges is finite, so that a random walk can be associated to it, we have the following well-known criterion for transience (see, e.g., Woess (1994), Theorem 4.8). Here, $dF(e)$ denotes the difference of F at the endpoints of e.

THEOREM 5.2. *A network G with conductances C is transient iff*

$$\inf\left\{\sum_{e\in G} dF(e)^2\,C(e);\ F \text{ has finite support and } F\restriction A \equiv 1\right\} > 0$$

for (some or) any finite set of vertices $A \subset G$.

An evident modification of the proof of Theorem 2 of Kanai (1986) shows the following. Recall that M is said to have *bounded geometry* if its Ricci curvature is bounded below and the injectivity radius is positive. We shall say that σ is *ε-slowly varying* if

$$\sup\{\sigma(x)/\sigma(y);\, \mathrm{dist}(x, y) \leqslant \varepsilon\} < \infty.$$

Note that Ricci curvature being bounded below implies that nets have bounded degree (Kanai (1985), Lemma 2.3).

THEOREM 5.3. *Suppose that M is a complete Riemannian manifold of bounded geometry, that ε is at most half the injectivity radius of M, that σ is an ε-slowly varying Borel-measurable conductivity on M, and that G is an ε-net in M. Then the associated diffusion on M is transient iff the associated random walk on G is transient.*

Remark. The condition that σ be ε-slowly varying can be weakened to the following: There is a constant $c < \infty$ such that if $\mathrm{dist}(v, w) \leqslant 3\varepsilon$, then

$$\left(\int_{B_\varepsilon(v)} \sigma(x)\, dx + \int_{B_\varepsilon(w)} \sigma(x)\, dx\right)\left(\int_{B_{4\varepsilon}(v)} \sigma(x)^{-1}\, dx + \int_{B_{4\varepsilon}(w)} \sigma(x)^{-1}\, dx\right) \leqslant c.$$

Other methods can be used to weaken this still further.

Given two networks G and G' with conductances C and C', we say that a map ϕ from the vertices of G to those of G' is *bounded* if there is a constant $c < \infty$ and a map Φ defined on the edges of G such that

(i) for every edge $(v, w) \in G$, $\Phi(v, w)$ is a path of edges from $\phi(v)$ to $\phi(w)$ with

$$\sum_{e' \in \Phi(v, w)} C'(e')^{-1} \leqslant cC(v, w)^{-1};$$

(ii) for every edge $e' \in G'$, there are no more than c edges in G whose image under Φ contains e'.

(One should think of the resistances C^{-1} as lengths of edges.) We call two networks *roughly equivalent* if there are bounded maps in both directions. A straightforward modification of the proof of Corollary 7 of Kanai (1986) shows:

THEOREM 5.4. *If G and G' are roughly equivalent networks, then G is transient iff G' is transient.*

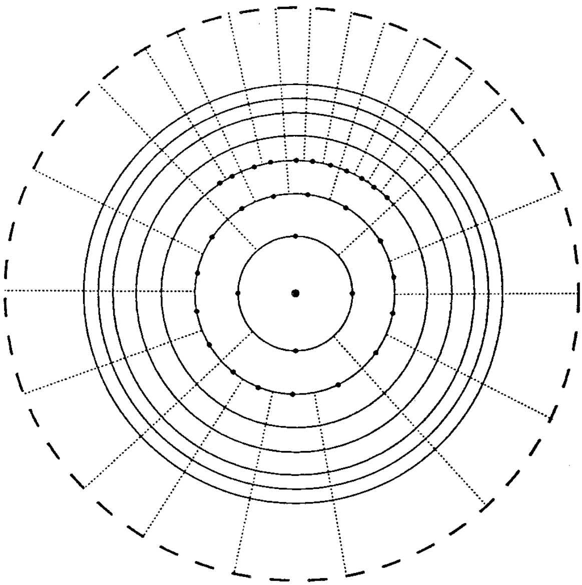

FIG. 5.1. The sets T_n and τ_n.

We next approximate $\partial_\infty M$ by the boundary of a tree, T_M. We have been able to do this only for surfaces. Let M be a CH surface. We shall define inductively finite sets $T_n \subset S_n \subset M$ $(n \geqslant 0)$ and $\tau_n \subset \partial_\infty M$ $(n \geqslant 1)$ in such a way that

(i) for each $n \geqslant 1$, $\tau_n \subset \tau_{n+1}$;

(ii) for each $n \geqslant 1$, between each consecutive pair of rays in τ_n, there is exactly one point in T_n that is at distance at least 1/4 and at most 1 from the intersections of the rays with S_n.

See Fig. 5.1. If the point of T_n in (ii) is v, write $\mathsf{Cone}(v)$ for the set of points between the rays in τ_n nearest to v. Also, set $\mathsf{Cone}(0) := \partial_\infty M$.

The construction can be done as follows. Let $T_0 := \{0\}$, of course, and let T_1 be a maximal 1-separated subset of S_1. Choose points in S_1 that are equidistant from consecutive pairs of points in T_1 and let τ_1 be the rays passing through them. Then (ii) holds for $n = 1$ by maximality and the triangle inequality.

Now suppose that T_n and τ_n have been defined. For $v \in T_n$, choose a subset of the arc $\mathsf{Cone}(v) \cap S_{n+1}$ that, with the addition of the endpoints of the arc, is a maximal 1/2-separated subset. (Do not include the endpoints in the subset itself.) Again, bisect consecutive pairs of such points by rays. The set of all such points on S_{n+1} forms T_{n+1} and the rays, together with

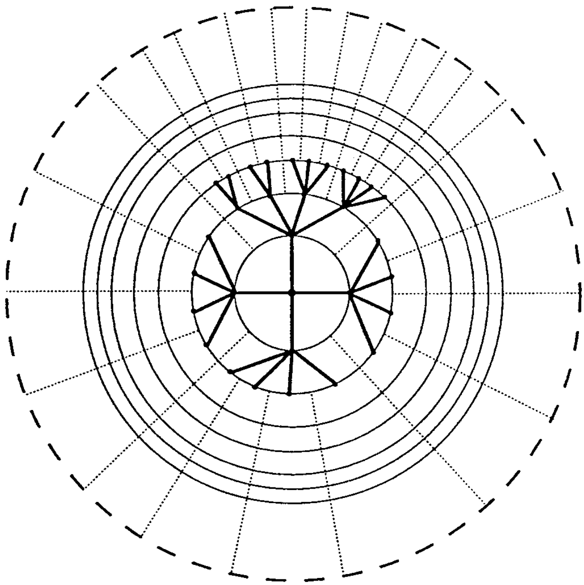

FIG. 5.2. The tree T_M.

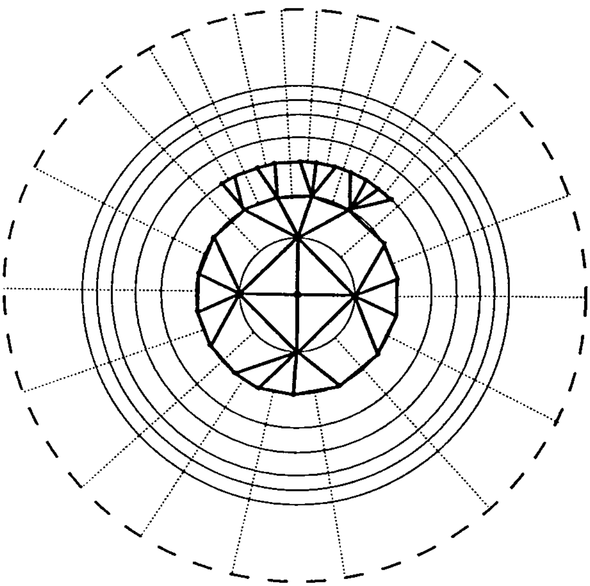

FIG. 5.3. The graph G_M.

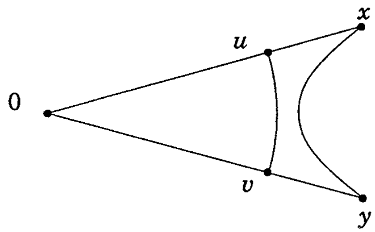

FIGURE 5.4.

τ_n, form the set τ_{n+1}. Note that the points of T_{n+1} closest to the endpoints of $\mathsf{Cone}(v) \cap S_{n+1}$ are at distance less than 1 from them since otherwise the triangle inequality and maximality would give a contradiction. Thus, (ii) holds for $n+1$.

Next, form a tree T_M as in Fig. 5.2 by connecting each point $v \in T_{n+1}$ to the point $w \in T_n$ such that $v \in \mathsf{Cone}(w)$. Finally, form a planar graph G_M as in Fig. 5.3 by connecting consecutive points in T_n to each other.

LEMMA 5.5. *Let M be a CH surface. Given two points $x, y \neq 0$, let t be at most the distance from 0 to the geodesic joining x to y and let $u := \vec{x}(t)$ and $v := \vec{y}(t)$, as in Fig. 5.4. Then the length of the arc on S_t from u to v is less than* $\operatorname{dist}(x, y)$.

Proof. Use geodesic polar coordinates $dr^2 + f(r, \theta)^2\, d\theta^2$ on M. Then f is a (convex) increasing function of r for each fixed θ because the curvature is nonpositive: $f_{rr}/f = -K$ with the initial conditions $f(0, \theta) = 0$ and $f_r(0, \theta) = 1$. The result follows immediately on comparing integrals for arclengths. ∎

PROPOSITION 5.6. *If M is a CH surface, then* $\dim \partial_\infty M = \log \operatorname{br} T_M$.

Proof. Let Π be a cutset in T_M. From the definition of T_M, it follows that for every $\xi \in \partial_\infty M$, there is some $v \in \Pi$ such that $\operatorname{dist}(v, \xi(|v|)) \leqslant 1$, i.e., $\rho(\vec{v}, \xi) \leqslant e^{-|v|}$. Therefore, $\partial_\infty M = \bigcup_{v \in \Pi} \overline{B_{e^{-|v|}}(\vec{v})}$. This gives immediately that $\dim \partial_\infty M \leqslant \log \operatorname{br} T_M$.

For the converse direction, let $B_i := B_{r_i}(\xi_i)$ $(1 \leqslant i \leqslant k)$ form a cover of $\partial_\infty M$. Write B_i also as the arc in $\partial_\infty M$ from ξ_i' to ξ_i''. Thus,

$$\operatorname{dist}(\xi_i'(-\log r_i), \xi_i(-\log r_i)) = \operatorname{dist}(\xi_i(-\log r_i), \xi_i''(-\log r_i)) = 1.$$

We may assume that all r_i are at most e^{-2}. For each i, let $n_i := \lfloor -\log r_i - 1/2 \rfloor$. Note that the distances from 0 to the geodesics joining

$\xi_i(-\log r_i)$ to $\xi_i'(-\log r_i)$ and $\xi_i''(-\log r_i)$ are at least n_i. Thus, by Lemma 5.5, the length of the circular arc $I_i \subset S_{n_i}$ from $\xi_i'(n_i)$ to $\xi_i''(n_i)$ is less than 2. Let Π_i be the set of points in T_{n_i} that are either in I_i or adjacent to I_i. Since the length of I_i is less than 2, there are at most 5 points in Π_i. Let $\Pi := \bigcup_{1 \leqslant i \leqslant k} \Pi_i$. Since every ray lies in some B_i, every ray lies in $\mathsf{Cone}(v)$ for some $v \in \Pi_i$. Therefore, Π is a cutset of T_M. Since for any $\alpha > 0$,

$$\sum_{v \in \Pi} e^{-\alpha |v|} < 5e^{3\alpha/2} \sum_{1 \leqslant i \leqslant k} r_i^{\alpha},$$

we obtain that $\dim \partial_\infty M \geqslant \log \operatorname{br} T_M$. ∎

Remark. Similar reasoning shows that $\operatorname{wdim} \partial_\infty M = \log \operatorname{br} T_M$, so that $\operatorname{wdim} \partial_\infty M = \dim \partial_\infty M$.

We may now put together the pieces to prove our main theorem on diffusions.

THEOREM 5.7. *If M is a CH surface with curvature bounded below, then $\beta_c(M) = (1/2) \dim \partial_\infty M$.*

Proof. Let G be a net in M. Fix $\beta > 0$ and set $\lambda := e^{2\beta}$. Give G_M the conductances $\lambda^{-|e|}$ described in Section 4 and G those described above in (5.1). It is easy to see that G_M and G are roughly equivalent. By Theorem 5.3, BM_β is transient iff G is transient. By Theorem 5.4, this holds iff G_M is transient.

Since G_M has bounded degree, Theorem 4.1 and Lemma 4.2 imply that $\lambda_c(G_M) = \operatorname{br}(T_M)$. Therefore $\beta_c(M) = (1/2) \log \lambda_c(G_M) = (1/2) \log \operatorname{br} T_M$. By Proposition 5.6, this is the same as asserted. ∎

Remark. The same reasoning shows that for any CH manifold M with sectional curvatures bounded below, $\beta_c(M) = (1/2) \log \lambda_c(G) \leqslant (1/2) \log \operatorname{br} G$ for every net G in M.

Remark. If M is rotationally-symmetric CH manifold, then $\beta_c(M) = (1/2) \operatorname{loggr} M$ without any assumption on the curvature: Let X_t be the diffusion BM_β. Then $\operatorname{dist}(0, X_t)$ is the diffusion on $\mathbf{R}^+$ corresponding to the conductivity $\sigma(r) := e^{-2\beta r} |S_r|$ (compare Durrett (1986) and Doyle (1988)). This is transient iff

$$\int_1^\infty \sigma(r)^{-1} \, dr = \int_1^\infty e^{2\beta r} |S_r|^{-1} \, dr < \infty.$$

Hence $\beta_c(M) = (1/2) \liminf_{r \to \infty} \log |S_r|/r = (1/2) \operatorname{loggr} M$.

Remark. Since the generator of the diffusion BM_β is symmetric with respect to the measure $\mu := \sigma(x)\,dx$, we have that BM_β is reversible with respect to μ and hence μ is a stationary measure for BM_β. Thus, BM_β is positive recurrent iff μ is finite. In particular, we see that if β is larger than half the limsup volume growth rate on any CH manifold, then BM_β is positive recurrent.

6. Random Shadows

Let $\gamma_r := (\sqrt{\pi}\, r)^{d-1}/\Gamma((d+1)/2)$ be the volume of a ball of radius r in $\mathbf{R}^{d-1}$. Recall that clouds are open balls centered at the points of a Poisson point process of intensity β times volume measure, which means that if A and B are disjoint regions in M, then the centers in A and B are independent and A has no centers with probability $\exp\{-\beta \operatorname{Vol} A\}$.

Theorem 6.1. *Let M be a CH manifold with sectional curvatures bounded below. Consider random clouds with intensity β and radius r.*

(i) *If $\beta\gamma_r > \operatorname{wdim} \partial_\infty M$, then the shadows of the clouds cover the boundary a.s.*

(ii) *For all $\varepsilon > 0$, there is some $r_0 > 0$ such that if $r \leqslant r_0$ and $\beta\gamma_r < \operatorname{wdim} \partial_\infty M - \varepsilon$, then with positive probability, the shadows do not cover the boundary.*

Recall that $\beta_c(M, r)$ is the critical intensity of clouds of radius r.

Corollary 6.2. *If M is a CH manifold with sectional curvatures bounded below, then $\lim_{r\to 0} \beta_c(M, r)\, \gamma_r = \operatorname{wdim} \partial_\infty M$.*

In fact, we shall prove the following sharper form of part (ii) of the theorem. Given $\xi, \eta \in \partial_\infty M$ and $n \geqslant 1$, let $p_n(\xi, \eta)$ be the probability that both $\xi(n)$ and $\eta(n)$ are visible i.e., that the geodesics from 0 to these points avoid every cloud. Write $p_n(\xi)$ for $p_n(\xi, \xi)$. Set

$$\kappa_n(\xi, \eta) := \frac{p_n(\xi, \eta)}{p_n(\xi)\, p_n(\eta)}.$$

For a set $A \subset M$, let $\mathsf{Nbd}_r(A)$ be the set of points at distance less than r from A. Then

$$\begin{aligned}\kappa_n(\xi, \eta) &= \frac{\exp\{-\beta \operatorname{Vol} \mathsf{Nbd}_r((\xi \cup \eta) \cap B_n(0))\}}{\exp\{-\beta \operatorname{Vol} \mathsf{Nbd}_r(\xi \cap B_n(0))\}\, \exp\{-\beta \operatorname{Vol} \mathsf{Nbd}_r(\eta \cap B_n(0))\}}\\ &= \exp\{\beta \operatorname{Vol} \mathsf{Nbd}_r(\xi \cap B_n(0)) \cap \mathsf{Nbd}_r(\eta \cap B_n(0))\}.\end{aligned}$$

Write

$$\kappa^{(\beta)}(\xi,\eta) := \lim_{n\to\infty} \kappa_n(\xi,\eta) = \exp\{\beta \operatorname{Vol} \mathsf{Nbd}_r(\xi) \cap \mathsf{Nbd}_r(\eta)\}. \tag{6.1}$$

Given a kernel $\kappa \geqslant 0$, the κ-*energy* of a measure $\mu \in \mathsf{Prob}(\partial_\infty M)$ is

$$\int\int \kappa(\xi,\eta)\, d\mu(\xi)\, d\mu(\eta)$$

and the corresponding *capacity* $\operatorname{cap}_\kappa(\partial_\infty M)$ of $\partial_\infty M$ is the reciprocal of the infimum of κ-energies.

THEOREM 6.3. *Let M be any CH manifold and $\beta > 0$. The probability that some ray is visible is at least* $\operatorname{cap}_{\kappa^{(\beta)}}(\partial_\infty M)$.

Proof. If $\operatorname{cap}_{\kappa^{(\beta)}}(\partial_\infty M) > 0$, then there is a probability measure μ on $\partial_\infty M$ of finite $\kappa^{(\beta)}$-energy, $\mathscr{E}$. Let $I_n(\xi)$ be the indicator of the event that $\xi(n)$ is visible. Use weak integrals to define the random variables

$$X_n := \int_{\partial_\infty M} I_n(\xi)\, p_n(\xi)^{-1}\, d\mu(\xi).$$

Then $\mathbf{E}[X_n] = 1$ and $\mathbf{E}[X_n^2] = \int\int \kappa_n(\xi,\eta)\, d\mu(\xi)\, d\mu(\eta) \leqslant \mathscr{E}$. By the Cauchy–Buniakowski–Schwarz inequality, it follows that

$$\mathbf{P}[X_n > 0] \geqslant \mathbf{E}[X_n]^2 / \mathbf{E}[X_n^2] \geqslant 1/\mathscr{E}.$$

Thus the set of points $\xi \in \partial_\infty M$ such that $\xi(n)$ is visible is compact and, with probability at least $1/\mathscr{E}$, nonempty. Since these sets are nested as n varies, it follows that some point of $\partial_\infty M$ is visible also with probability at least $1/\mathscr{E}$. Taking the infimum of energies $\mathscr{E}$ gives the result. ∎

Now, if t is such that $\operatorname{dist}(\xi(t), \eta(t)) = 2r$, then

$$\mathsf{Nbd}_r(\xi) \cap \mathsf{Nbd}_r(\eta) \subset \mathsf{Nbd}_r(\xi) \cap B_{t+r}(0). \tag{6.2}$$

Define the *tube* $\operatorname{Tube}_r^s(\xi)$ of a ray ξ for $r, s > 0$ to be the set of points in M from which there issues a geodesic of length less than r meeting ξ

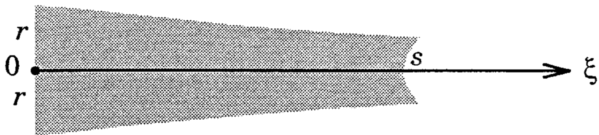

FIG. 6.1. A tube.

orthogonally at a point $\xi(u)$ for some $u \in [0, s]$; see Fig. 6.1. Then if the sectional curvatures are bounded below, $\mathrm{Vol}\,\mathsf{Tube}_r^s(\xi)/(\gamma_r s) \to 1$ as $r \to 0$ uniformly in $\xi \in \partial_\infty M$ and $s > 0$. Combining this with (6.1) and (6.2), we obtain the following.

LEMMA 6.4. *Let M be a CH manifold with sectional curvatures bounded below. There is a function $\varepsilon(r)$ such that $\lim_{r \to 0} \varepsilon(r) = 0$ and for all $\beta > 0$ and all $\xi, \eta \in \partial_\infty M$,*

$$\kappa^{(\beta)}(\xi, \eta) \leqslant \rho(\xi, \eta)^{-\beta\gamma_r(1+\varepsilon(r))}.$$

In light of (2.1), this implies part (ii) of Theorem 6.1.

For part (i), we use the following (see, e.g., Gray (1990), Cor. 8.6(ii), p. 163).

LEMMA 6.5. *Let M be a CH manifold. For every $\xi \in \partial_\infty M$, $r, s > 0$, we have* $\mathrm{Vol}\,\mathsf{Tube}_r^s(\xi) \geqslant s\gamma_r$.

Part (i) is easier to prove when the sectional curvatures are bounded away from zero, in which case we may use the following well-known fact; see, e.g., Kaimanovich (1990), Prop. 1.1.

LEMMA 6.6. *Let $a, D > 0$. There is a constant c such that for any CH manifold M with sectional curvatures at most $-a^2$, any rays $\xi, \eta \in \partial_\infty M$, and any $t > 0$, if $\mathrm{dist}(\xi(t), \eta(t)) \leqslant D$, then for all $t' \in [0, t]$,*

$$\mathrm{dist}(\xi(t'), \eta(t')) \leqslant e^{c(t'-t)}\,\mathrm{dist}(\xi(t), \eta(t)).$$

Thus, if M is a CH manifold with sectional curvatures bounded away from zero, let $\beta\gamma_r > \mathrm{wdim}\,\partial_\infty M = \dim \partial_\infty M$. Then for any $\varepsilon > 0$, there is a cover of $\partial_\infty M$ by balls $B_{r_i}(\xi_i)$ with $\sum_i r_i^{\beta\gamma_r} < \varepsilon$. Lemma 6.6 implies that there is some $\delta > 0$ such that the entire ball $B_{r_i}(\xi_i)$ lies in the shadow of every cloud centered in $\mathsf{Nbd}_r(\xi_i) \cap B_{-\log r_i - \delta}(0)$, whence the probability that some point of $B_{r_i}(\xi_i)$ is visible is at most

$$\exp\{-\beta\,\mathrm{Vol}\,\mathsf{Nbd}_r(\xi_i) \cap B_{-\log r_i - \delta}(0)\} \leqslant c r_i^{\beta\gamma_r},$$

for some constant c. Thus, the probability that some point of $\partial_\infty M$ is visible is at most $c\varepsilon$. Since this holds for each $\varepsilon > 0$, we find that a.s. no ray is visible.

To eliminate the assumption that the sectional curvatures are bounded away from zero, we need to modify this argument a bit and use the assumption that the sectional curvatures are bounded below. Again, let

$\beta\gamma_r > \operatorname{wdim} \partial_\infty M$. For sufficiently small $\varepsilon > 0$, we have $\beta\gamma_{r-\varepsilon} > \operatorname{wdim} \partial_\infty M$. Take a collection of balls $B_{r_i}(\xi_i)$ and constants c_i with

$$\mathbf{1}_{\partial_\infty M} \leqslant \sum_i c_i \mathbf{1}_{B_{r_i}(\xi_i)} \tag{6.3}$$

and

$$\sum_i c_i r_i^{\beta\gamma_{r-\varepsilon}} < \varepsilon. \tag{6.4}$$

Since the sectional curvatures are bounded below, there is a constant c such that for each ξ_i, there are $\xi_{i,j}$ $(1 \leqslant j \leqslant c)$ with the property that if $\xi \in B_{r_i}(\xi_i)$, then for some j, $\operatorname{dist}(\xi(-\log r_i), \xi_{i,j}(-\log r_i)) < \varepsilon$. Thus, if $\xi \in B_{r_i}(\xi_i)$ is visible, then for some j, there is no cloud centered in $\mathsf{Tube}_{r-\varepsilon}^{-\log r_i}(\xi_{i,j})$, whence the probability of some point in $B_{r_i}(\xi_i)$ being visible is at most

$$\sum_j \exp\{-\beta \operatorname{Vol} \mathsf{Tube}_{r-\varepsilon}^{-\log r_i}(\xi_{i,j})\} \leqslant \sum_j \exp\{\beta \log r_i \gamma_{r-\varepsilon}\} = c r_i^{\beta\gamma_{r-\varepsilon}}. \tag{6.5}$$

For $A \subset \partial_\infty M$, let $J(A)$ be the indicator of the event that some point of A is visible. By (6.3), we have

$$J(\partial_\infty M) \leqslant \sum_i c_i J(B_{r_i}(x_i)),$$

whence

$$\mathbf{E}[J(\partial_\infty M)] \leqslant \sum_i c_i \mathbf{E}[J(B_{r_i}(x_i))].$$

Combining this with (6.5), we have that the probability that some point of $\partial_\infty M$ is visible is at most

$$\sum_i c_i c r_i^{\beta\gamma_{r-\varepsilon}} < c\varepsilon$$

by (6.4). Since this holds for all small ε, it follows that a.s. no ray is visible. This completes the proof of Theorem 6.1.

Remark. Since part (i) of Theorem 6.1 holds for CH manifolds with either sectional curvatures bounded away from zero or sectional curvatures bounded below, it seems rather likely that no special assumptions on the curvatures should be needed.

It would be very interesting to know whether a converse of Theorem 6.3 holds. One does hold for surfaces:

THEOREM 6.7. *Let M be a CH surface and $\beta > 0$. The probability that some ray is visible is at most* $4 \operatorname{cap}_{\kappa^{(\beta)}}(\partial_\infty M)$.

Proof. Assume that the probability, p, that some ray is visible is positive. Choose geodesic polar coordinates (R, θ) on M and consider the semicircles of $\partial_\infty M$ going counterclockwise from a starting angle θ. Some such semicircle has probability at least $p/2$ of containing a visible ray. We may assume that it is the semicircle I from 0 to π.

Fix $n \geqslant 1$ and let ξ^* be the ray in I with least angle such that $\xi(n)$ is visible if there is such a ray. Let v_n be its distribution, a subprobability measure on $I \subset \partial_\infty M$. For $\xi \in I$, let $A(\xi)$ be the event that $\xi(n)$ is visible. Write $\xi \preccurlyeq \eta$ to mean that the angle of ξ is at most the angle of η. The importance of I being a semicircle is that for $\xi, \eta \in I$ and $\xi \preccurlyeq \eta$,

$$\mathbf{P}[A(\eta) \mid A(\xi)] = \mathbf{P}[A(\eta) \mid \xi^* = \xi].$$

Thus, we have

$$\begin{aligned}
\int_{\xi \preccurlyeq \eta} \kappa_n(\xi, \eta)\, dv_n(\xi) &= \frac{1}{\mathbf{P}[A(\eta)]} \int_{\xi \preccurlyeq \eta} \mathbf{P}[A(\eta) \mid A(\xi)]\, dv_n(\xi) \\
&= \frac{1}{\mathbf{P}[A(\eta)]} \int_{\xi \preccurlyeq \eta} \mathbf{P}[A(\eta) \mid \xi^* = \xi]\, dv_n(\xi) \\
&= \frac{1}{\mathbf{P}[A(\eta)]} \int_I \mathbf{P}[A(\eta) \mid \xi^* = \xi]\, dv_n(\xi) = 1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int\!\!\int \kappa_n(\xi, \eta)\, dv_n(\xi)\, dv_n(\eta) \\
&\qquad \leqslant 2 \int_{\eta \in I} \int_{\xi \preccurlyeq \eta} \kappa_n(\xi, \eta)\, dv_n(\xi)\, dv_n(\eta) = 2 \int_I dv_n(\eta) = 2 v_n(I).
\end{aligned}$$

Consequently, the κ_n-energy of the probability measure $v_n/(v_n(I)$ is at most $2/v_n(I)$ and so $\operatorname{cap}_{\kappa_n}(\partial_\infty M) \geqslant v_n(I)/2$.

Now let n tend to infinity. Then $v_n(I)$ tends to the probability that some ray in I is visible, which is at least $p/2$, while standard arguments show that

$$\lim_{n \to \infty} \operatorname{cap}_{\kappa_n}(\partial_\infty M) \leqslant \operatorname{cap}_{\kappa^{(\beta)}}(\partial_\infty M). \quad \blacksquare$$

Remark. In case the sectional curvatures are bounded away from zero, then by Lemma 6.6 and Lemma 6.5, there is a constant c such that $\kappa^{(\beta)}(\xi, \eta) \geqslant c\rho(\xi, \eta)^{-\beta\gamma_r}$. Thus, Theorem 6.7 implies part (i) of Theorem 6.1

for CH surfaces whose sectional curvatures are bounded away from zero. Of course, a better lower bound on $\kappa^{(\beta)}$ is available in terms of the upper bound on the sectional curvatures. For example, in case of constant curvature $-a^2<0$, there are constants c, c' so that

$$c\rho(\xi,\eta)^{-\beta\Gamma_r(a)} \leqslant \kappa^{(\beta)}(\xi,\eta) \leqslant c'\rho(\xi,\eta)^{-\beta\Gamma_r(a)},$$

where $\Gamma_r(a)$ is the volume of a tube of radius r and length 1. Now the general formula for $\Gamma_r(a)$ in d dimensions is $\gamma_1(\sinh ar/a)^{d-1}$ (Gray and Vanhecke (1982), Theorem 6.3). Thus, for $d=2$, $\partial_\infty M$ is covered a.s. iff $\beta(e^{ar}-e^{-ar})\geqslant a^2$, as shown by Kahane (1990), (1991). In higher dimensions ($d>2$), the Hausdorff dimension argument extends in a similar fashion to give that $\beta_c(M,r)=(d-1)\,a/\Gamma_r(a)$, but this does not determine whether coverage obtains at criticality.

QUESTION. *Is there a capacity criterion on $\partial_\infty M$ that determines transience of* BM_β, *as there is for trees* (*Lyons* 1990)?

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