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THE LOCAL STRUCTURE OF SOME MEASURE-ALGEBRA HOMOMORPHISMS

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Extending classical theorems, we obtain representations for bounded linear transformations from L-spaces to Banach spaces with a separable predual. In the case of homomorphisms from a convolution measure algebra to a Banach algebra, we obtain a generalization of Šreĭder's representation of the Gelfand spectrum via generalized characters. The homomorphisms from the measure algebra on a LCA group, G, to that on the circle are analyzed in detail. If the torsion subgroup of G is denumerable, one consequence is the following necessary and sufficient condition that a positive finite Borel measure on G be continuous: $\exists \gamma_\alpha \to \infty$ in $\widehat{G}$ such that $\forall n \neq 0$ $\hat{\mu}(\gamma_\alpha^n) \to 0$.

1. Introduction. Given a measurable space X and a (bounded) complex measure μ on X, the Banach space dual of $L^1(\mu)$ is commonly represented as $L^\infty(\mu)$. We shall call M an *L-space* on X if M is a Banach space of complex measures on X (under the measure norm) such that $\nu \ll \mu \in M \Rightarrow \nu \in M$ **[Sc]**. Šreĭder **[Šr]** gave a representation of the dual M^* of M as a space of so-called generalized functions, i.e., families of functions $f_\mu \in L^\infty(\mu)$ satisfying

$$\nu \ll \mu \Rightarrow f_\nu = f_\mu \quad \nu\text{-a.e.}, \tag{1.1}$$

$$\sup_{\mu \in M} \|f_\mu\|_{L^\infty(\mu)} < \infty. \tag{1.2}$$

The representation of M^*, like that of $L^1(\mu)^*$, is by integration:

$$\mu \mapsto \int f_\mu \, d\mu.$$

Now, given two Banach spaces, B_1 and B_2, we denote by $L(B_1, B_2)$ the Banach space of bounded linear transformations from B_1 to B_2. Since $M^* = L(M, \mathbf{C})$, we may ask, in generalizing the above, for a representation of $L(M, B)$, where B is an arbitrary Banach space. Again, the case where $M = L^1(\mu)$ is classical **[DS]**; here, the hypothesis that B has a separable predual is made. In §2, we extend this theorem to general L-spaces M in a manner similar to Šreĭder's representation above. In essence, functions are replaced by

B-valued functions. Our treatment will be entirely self contained, thus giving an apparently new proof of [**DS**, Theorem VI.8.6]. However, another point of view could be adopted. Namely, if we use the Radon-Nikodym theorem to identify $L(\mu) = \{\nu \ll \mu : \nu \text{ bounded}\}$ with $L^1(\mu)$, then we may regard an L-space M as the direct limit $\lim_{\mu \in M} L^1(\mu)$, where M is directed by $\ll$ and for $\nu \ll \mu$, $L^1(\nu)$ is included in $L^1(\mu)$. Now $L(\cdot, B)$ is a functor from the category of Banach spaces to its opposite category and, furthermore, is easily checked to be a left adjoint. Since left adjoints preserve direct limits and inverse limits are dual to direct limits, it follows that $L(M, B)$ is the inverse limit $\lim_{\mu \in M} L(L^1(\mu), B)$, where, for $\nu \ll \mu$, $L(L^1(\mu), B)$ is mapped by restriction to $L(L^1(\nu), B)$. Hence, given a representation of $L(L^1(\mu), B)$ (as in [**DS**]) and a construction of inverse limits, we may obtain a representation of $L(M, B)$. This amounts to the same as our Theorem 2.1.

Now Šreĭder was actually interested in representing ΔM, the multiplicative linear functionals on M, when M was a convolution measure algebra on a locally compact abelian group. He showed that in addition to (1.1) and (1.2), the following property was necessary and sufficient for f_μ to define an element of ΔM:

(1.3) $\quad \forall \mu, \nu \geq 0 \; f_{\mu * \nu}(xy) = f_\mu(x) f_\nu(y) \quad \mu \times \nu\text{-a.e. } [(x, y)].$

We, too, are mainly interested in the subset of homomorphisms $\mathrm{Hom}(M, B) \subseteq L(M, B)$ when B is a Banach algebra. A similar condition to (1.3) is found in Theorem 3.2. In particular, when $M = M(G)$, the complex Borel measures on a locally compact abelian group, G, and $B = M(\mathbf{T})$, $\mathbf{T}$ the circle, $\mathrm{Hom}(M(G), M(\mathbf{T}))$ contains in a natural way $\mathrm{Hom}(G, \mathbf{T}) = \widehat{G}$. The closure of $\widehat{G}$ in a certain weak topology is related to the behavior of Fourier transforms at infinity and contains much information about a measure μ when regarded locally, i.e., when restricted to $L(\mu)$, or, what is the same, when viewed via the Šreĭder representation. For example, this analysis will lead to the following surprising result: if the torsion subgroup of G is denumerable, then a positive measure $\mu \in M(G)$ is continuous iff there is a net $\{\gamma_\alpha\} \subseteq \widehat{G}$ tending to infinity such that for all $n \neq 0$, $\lim_\alpha \hat{\mu}(\gamma_\alpha^n) = 0$. Characterizations of certain other classes of measures are found in §4; these have proved useful in [**KL**] and [**L4**]. Other analyses of the local structure of the closure of $\widehat{G}$ for certain μ can be found in [**L3**], [**L4**], and [**L5**]. The local structure of $\widehat{G}$ is also related to asymptotic distribution; this relationship, described here, has been used in [**KL**] and [**L4**].

The Šreĭder representation, Theorem 3.2, has been given before in **[IgK]** for the case $\mathrm{Hom}(M, M(\mathbf{T}))$, M being an L-subalgebra of $M(\mathbf{T})$, though in slightly different notation. An alternative representation for $\mathrm{Hom}(M, M(G))$, where M is a semisimple commutative convolution measure algebra in the sense of Taylor and G is a compact abelian group, analogous to Taylor's representation of ΔM via a structure semigroup, has been given in **[InK]**.

2. The Šreĭder representation of linear transformations. Suppose that M is an L-space on a measurable space X and that B is a Banach space with a separable predual, B_*. Let $\mathscr{B}(X, B)$ denote the set of maps $f: X \to B$ which are bounded in B-norm and measurable when B is given the weak* topology from B_*. If $f \in \mathscr{B}(X, B)$ and $\mu \in M$, there is a unique element $\int f\, d\mu \in B$ defined by the relation

$$\forall b_* \in B_* \ \left\langle b_*, \int f\, d\mu \right\rangle = \int_X \langle b_*, f(x)\rangle\, d\mu(x).$$

If D is a countable dense set in the unit ball of B_*, then the equation

$$\|f(x)\|_B = \sup_{b_* \in D} |\langle b_*, f(x)\rangle|$$

shows that $\|f(\cdot)\|_B$ is measurable. It is clear that

$$\left\| \int f\, d\mu \right\|_B \le \int \|f\|_B\, d|\mu|.$$

The set of equivalence classes of $\mathscr{B}(X, B)$ under equality μ-a.e. will be denoted $\mathscr{B}(X, B)_\mu$, although this distinction will often be ignored.

The following theorem, which we shall term the *Šreĭder representation*, associates to each element of $L(M, B)$ a certain family of maps in $\mathscr{B}(X, B)$. We denote the image of $\mu \in M$ under $\Sigma \in L(M, B)$ by Σ_μ.

THEOREM 2.1. *Let M be an L-space and B a Banach space with a separable predual. There is a bijection between $L(M, B)$ and the set of elements $\{b_{\cdot,\mu}\}_{\mu\in M} \in \prod_{\mu\in M} \mathscr{B}(X, B)_\mu$ which satisfy*

(i) $$\sup_{\mu\in M} \| \|b_{x,\mu}\|_B \|_{L^\infty(\mu)} < \infty$$

and

(ii) $$\forall \nu \ll \mu \in M \ b_{x,\nu} = b_{x,\mu} \quad \nu\text{-a.e. } [x]$$

such that if Σ *corresponds to* $\{b._{,\mu}\}_{\mu\in M}$ *(written* $\Sigma \sim b._{,}.$*), then*

$$\text{(iii)} \qquad \forall\mu \in M \quad \Sigma_\mu = \int b_{x,\mu}\, d\mu(x)$$

and

$$\text{(iv)} \qquad \|\Sigma\|_{L(M,B)} = \sup_{\mu\in M} \|\, \|b_{x,\mu}\|_B\|_{L^\infty(\mu)}\,.$$

Proof. Given $\{b._{,\mu}\}$ satisfying (i) and (ii), define Σ by (iii). If $\mu, \nu \in M$, then by (ii), we have $b_{x,\mu} = b_{x,|\mu|+|\nu|}$ μ-a.e., whence $\Sigma_\mu = \int b_{x,|\mu|+|\nu|}\, d\mu(x)$. In conjunction with similar equations for Σ_ν and $\Sigma_{\mu+\nu}$, this equation shows that $\Sigma_\mu + \Sigma_\nu = \Sigma_{\mu+\nu}$. Similarly, for $\alpha \in \mathbf{C}$, $\Sigma_{\alpha\mu} = \alpha\Sigma_\mu$, whence Σ is linear. Let K denote the quantity in (i). Then

$$\begin{aligned}\|\Sigma\| = \sup_{\|\mu\|\le 1} \|\Sigma_\mu\| &= \sup_{\|\mu\|\le 1} \left\|\int b_{x,\mu}\, d\mu(x)\right\| \\ &\le \sup_{\|\mu\|\le 1} \int \|b_{x,\mu}\|\, d|\mu|(x) \le K.\end{aligned}$$

To show that $\|\Sigma\| = K$, choose any nonzero $\mu \in M$ and $\varepsilon > 0$. Let $0 \neq \nu \in L(\mu)$ be such that $\left\|\, \|b._{,\mu}\|_B - \|\, \|b._{,\mu}\|_B\|_{L^\infty(\mu)} \right\|_{L^\infty(\nu)} < \varepsilon$. Let S be the unit sphere of B. Since the unit ball of B is weak* compact, there exists a finite number of elements, $b_*^1, \ldots, b_*^n$, of the unit ball of B_* such that

$$S = \bigcup_{i=1}^{n} \{b \in S \colon |\langle b_*^i, b\rangle - 1| < \varepsilon\}.$$

Therefore $\exists 0 < \omega \in L(\nu)$ $\exists i$ $\|\langle b_*^i, b_{x,\mu}/\|b_{x,\mu}\|_B\rangle - 1\|_{L^\infty(\omega)} < \varepsilon$. We have

$$\begin{aligned}\|\Sigma\| \ge \frac{\|\Sigma_\omega\|}{\|\omega\|} \ge \frac{1}{\|\omega\|}|\langle b_*^i, \Sigma_\omega\rangle| &= \frac{1}{\|\omega\|}\left|\int \langle b_*^i, b_{x,\mu}\rangle\, d\omega(x)\right| \\ &\ge \frac{1}{\|\omega\|}\int \|b_{x,\mu}\|_B\, d\omega(x) - \varepsilon K \ge \left\|\, \|b._{,\mu}\|_B\right\|_{L^\infty(\mu)} - \varepsilon(K+1).\end{aligned}$$

Thus $\|\Sigma\| = K$.

Conversely, let $\Sigma \in L(M, B)$. Fix $\mu \in M$. For $b_* \in B_*$, we denote by $b_* \circ \Sigma$ the map $\nu \mapsto \langle b_*, \Sigma_\nu\rangle$. Restricted to $L(\mu)$, this map is a bounded linear functional and hence can be represented by a function $g_{b_*} \in L^\infty(\mu)$. Choose a countable linearly independent set D whose linear span over $\mathbf{Q}$, D', is dense in B_*. If $b_* = \sum_{i=1}^n \alpha_i d_*^i$, $d_*^i \in D$, $\alpha_i \in \mathbf{Q}$, define

$$h_{b_*} = \sum_{i=1}^{n} \alpha_i g_{d_*^i}.$$

Then $b_* \mapsto h_{b_*}(x)$ is rational-linear on D' for every $x \in X$. Furthermore, $h_{b_*} = g_{b_*}$ μ-a.e., whence by countability of D',

$$\forall b_* \in D' \ |h_{b_*}(x)| \leq \|b_* \circ \Sigma\| \leq \|b_*\| \cdot \|\Sigma\| \tag{2.1}$$

for μ-a.e. x. Now for every x such that (2.1) holds, $b_* \mapsto h_{b_*}(x)$ extends from D' to all of B_* as a bounded linear functional, hence element of B, call it $f(x)$. This defines $f(x)$ μ-a.e. and shows that, given any $b_* \in B_*$, if $b_* = \lim_{n\to\infty} b_*^n$ $(b_*^n \in D')$, then

$$\langle b_*, f(x)\rangle = \lim_{n\to\infty} \langle b_*^n, f(x)\rangle = \lim_{n\to\infty} h_{b_*^n}(x) \tag{2.2}$$

for every x where f is defined. Write $b_{\cdot,\mu}$ for the equivalence class of f. From Equation (2.1), we see that $\|f(x)\| \leq \|\Sigma\|$ for every x where f is defined. Together with (2.2), this shows that $b_{\cdot,\mu} \in \mathscr{B}(X, B)_\mu$ and gives (i). Now for $\nu \in L(\mu)$ and $b_* \in D'$, we have

$$\left\langle b_*, \int f\, d\nu \right\rangle = \int \langle b_*, f(x)\rangle\, d\nu(x) = \int h_{b_*}(x)\, d\nu(x)$$
$$= \int g_{b_*}(x)\, d\nu(x) = \langle b_*, \Sigma_\nu\rangle .$$

Since D' is dense, (iii) follows. We claim that $b_{\cdot,\mu}$ is uniquely determined by the property just established:

$$\forall \nu \in L(\mu)\ \Sigma_\nu = \int b_{x,\mu}\, d\nu(x).$$

Indeed, if we also have that $\forall \nu \in L(\mu)\ \Sigma_\nu = \int b'_{x,\mu}\, d\nu(x)$ for some $b'_{\cdot,\mu} \in \mathscr{B}(X, B)_\mu$, then

$$\forall b_* \in D'\ \forall \nu \in L(\mu)\ \int \langle b_*, b_{x,\mu}\rangle\, d\nu(x) = \int \langle b_*, b'_{x,\mu}\rangle\, d\nu(x),$$

whence for μ-a.e. x $\forall b_* \in D'$ $\langle b_*, b_{x,\mu}\rangle = \langle b_*, b'_{x,\mu}\rangle$, i.e., $b_{x,\mu} = b'_{x,\mu}$ μ-a.e. Thus (ii) follows. The same argument shows that if $\Sigma \sim b_{\cdot,\cdot}$ and $\Sigma \sim b'_{\cdot,\cdot}$, then $b_{\cdot,\cdot} = b'_{\cdot,\cdot}$. □

We define the *weak* operator topology* (W* OT) on $L(M, B)$ as the weakest topology such that $\forall \mu \in M$ $\forall b_* \in B_*$ $\Sigma \mapsto \langle b_*, \Sigma_\mu\rangle$ is continuous. It is an elementary exercise to show that the unit ball of $L(M, B)$ is W* OT compact.

For $\mu \in M$, let $L(M, B)_\mu$ denote the set of Šreĭder representations $b._{,\mu}$ of elements of $L(M, B)$. We give $L(M, B)_\mu$ the weak topology generated by the maps $b._{,\mu} \mapsto \int \langle b_*, b_{x,\nu}\rangle \, d\nu(x)$ $(b_* \in B_*, \nu \in L(\mu))$. Thus, the W* OT is the inverse limit of these topologies, i.e., it is the weak topology generated by the maps $\Sigma \mapsto b._{,\mu}$ $(\mu \in M)$ from $L(M, B) \to L(M, B)_\mu$, where $\Sigma \sim b._{,\cdot}$.

Every decomposition $M = I \oplus J$ of M as a direct sum of closed subspaces yields an addition on $L(M, B)$ as follows: if $\Pi^1, \Pi^2 \in L(M, B)$, then we may define

$$\Sigma_\mu = \Pi^1_{\mu_I} + \Pi^2_{\mu_J}, \tag{2.3}$$

where $\mu = \mu_I + \mu_J$, $\mu_I \in I$, $\mu_J \in J$. If $\Sigma \sim b._{,\cdot}$, $\Pi^i \sim b^i._{,\cdot}$, and $I \perp J$, then $b_{x,\mu} = b^1_{x,\mu_I} + b^2_{x,\mu_J}$ μ-a.e.

The case where $B = M(Y)$, the space of complex regular Borel measures on a locally compact metric space, Y, is of interest. A predual of B is the separable space $C_0(Y)$ of continuous functions vanishing at infinity. We shall denote the Šreĭder representation of Σ by $\sigma_{x,\mu}$ in this case; thus, if $f \in C_0(Y)$ and $\mu \in M$,

$$\int_Y f \, d\Sigma_\mu = \int_X \left(\int_Y f \, d\sigma_{x,\mu} \right) d\mu(x). \tag{2.4}$$

(If Y is separable and a countable union of complete subspaces, then (2.4) holds for $f \in \mathscr{B}(Y, \mathbf{C})$ since it is preserved under bounded pointwise limits. In particular, for Borel sets $E \subseteq Y$,

$$\Sigma_\mu(E) = \int_X \sigma_{x,\mu}(E) \, d\mu(x).)$$

Let M^+ denote the nonnegative elements of M and likewise for $M^+(Y)$. We say that $\Sigma \in L(M, M(Y))$ is *positive* if it carries M^+ into $M^+(Y)$. It is easy to see from (2.4) applied to $|\mu|$ that $\Sigma \geq 0$ iff $\forall \mu \in M$ $\forall^e x[\mu]$ $\sigma_{x,\mu} \geq 0$ ("$\forall^e x[\mu]$" means "for μ-a.e. x"–see **[L1]**). It is also easy to show that if $\Sigma \geq 0$, then $\nu \ll \mu \Rightarrow \Sigma_\nu \ll \Sigma_{|\mu|}$ and $|\Sigma_\mu| \leq \Sigma_{|\mu|}$.

3. The Šreĭder representation of homomorphisms. Let G be a locally compact semigroup with separately continuous multiplication. Then $M(G)$ is a Banach algebra under convolution **[W]**. Let M be an L-subalgebra of $M(G)$, i.e., a subalgebra which is also an L-subspace, and let B be a Banach algebra with a separable predual such that multiplication is separately weak* measurable and

$$(3.1) \quad \forall f \in \mathscr{B}(G, B)\ \forall b \in B\ \forall \mu \in M$$
$$\int f(x) \cdot b\, d\mu(x) = \left(\int f\, d\mu\right) \cdot b$$
$$\& \int b \cdot f(x)\, d\mu(x) = b \cdot \int f\, d\mu .$$

In order to state some sufficient conditions that (3.1) be true, we define the following multiplication on $B^* \times B$. If $b \in B$ and $b^* \in B^*$, then $b' \mapsto \langle b' \cdot b, b^* \rangle$ is a bounded linear functional on B; we denote it by $b^* \cdot b$. Let $\overline{B}_*^{sw^*}$ be the smallest subspace of B^* containing (canonically) B_* which is closed under weak* sequential limits. Let ΔB be the subset of B^* consisting of the multiplicative linear functionals.

PROPOSITION 3.1. *Let B be a Banach algebra with a separable predual. Right multiplication on B is weak* measurable and the first equation of* (3.1) *holds if any of the following conditions is satisfies*:

(i) $B_* \cdot B \subseteq \overline{B}_*^{sw^*}$.
(ii) *Right multiplication is weak* continuous.*
(iii) *Right multiplication is weak* measurable and $\overline{B}_*^{sw^*} \cap \Delta B$ separates points in B.*

Proof. The class of $b^* \in B^*$ such that $b \mapsto \langle b, b^* \rangle$ is weak* measurable contains B_* and is closed under weak* sequential limits. Thus, all elements of $\overline{B}_*^{sw^*}$ are weak* measurable. Now right multiplication is weak* measurable iff $\forall b \in B\ \ \forall b_* \in B_*\ \ b' \mapsto \langle b_*, b' \cdot b \rangle$ is weak* measurable. But $\langle b_*, b' \cdot b \rangle = \langle b', b_* \cdot b \rangle$, whence this condition is equivalent to all elements of $B_* \cdot B$ being weak* measurable. The sufficiency of (i) for measurability is now obvious. Also, the class of weak* measurable $b^* \in B^*$ such that

$$\left\langle \int f\, d\mu, b^* \right\rangle = \int \langle f, b^* \rangle\, d\mu$$

is closed under weak* sequential limits by the bounded convergence theorem, hence contains $\overline{B}_*^{sw^*}$. Thus, if (i) holds, then $\forall b_* \in B_*$ $\forall b \in B$

$$\left\langle b_*, \int f \cdot b\, d\mu \right\rangle = \int \langle b_*, f \cdot b \rangle\, d\mu = \int \langle f, b_* \cdot b \rangle\, d\mu$$
$$= \left\langle \int f\, d\mu, b_* \cdot b \right\rangle = \left\langle b_*, \left(\int f\, d\mu\right) \cdot b \right\rangle ,$$

whence the first equation of (3.1).

Now (ii) is equivalent to $B_* \cdot B \subseteq B_*$ since B_* is the set of weak* continuous linear functionals on B. Thus, sufficiency follows from that of (i). Finally, if (iii) holds, then for $f \in \mathscr{B}(G, B)$, $b \in B$, $\mu \in M$, and $b^* \in \overline{B}_*^{sw^*} \cap \Delta B$, we have

$$\begin{aligned}\left\langle \int f \cdot b \, d\mu , b^* \right\rangle &= \int \langle f \cdot b , b^* \rangle \, d\mu = \int \langle f , b^* \rangle \langle b , b^* \rangle \, d\mu \\ &= \int \langle f , b^* \rangle \, d\mu \cdot \langle b , b^* \rangle = \left\langle \int f \, d\mu , b^* \right\rangle \cdot \langle b , b^* \rangle \\ &= \left\langle \left(\int f \, d\mu \right) \cdot b , b^* \right\rangle ,\end{aligned}$$

from which the first equation of (3.1) follows. □

Let $\mathscr{B}_0(G, B)$ denote the Baire-measurable functions from G to B, where B is given the weak* topology. For $\mu, \nu \in M(G)$, let $\mu \times \nu$ denote, besides the usual product measure, also its unique extension to a regular Borel measure in $M(G \times G)$. If $f \in \mathscr{B}_0(G, B)$ and $\mu, \nu \in M(G)$, then

$$\begin{aligned}\int f \, d\mu * \nu &= \int f(xy) \, d\mu \times \nu(x, y) \\ &= \iint f(xy) \, d\mu(x) \, d\nu(y),\end{aligned}$$

as can be seen by applying any $b_* \in B_*$ **[W]**.

The Šreĭder representation of $\mathrm{Hom}(M, B)$, the continuous homomorphisms from M to B, satisfies one property additional to those in Theorem 2.1.

THEOREM 3.2. *Let G be a locally compact semigroup with separately continuous multiplication and M an L-subalgebra of $M(G)$. Let B be a Banach algebra with a separable predual and separately weak* measurable multiplication satisfying* (3.1). *Let $\Sigma \in L(M, B)$ and choose $b_{\cdot, \mu} \in \mathscr{B}_0(G, B)$ $(\mu \in M)$ so that $\Sigma \sim b_{\cdot,\cdot}$. Then $\Sigma \in \mathrm{Hom}(M, B)$ iff*

(3.2) $\quad \forall \mu, \nu \in M^+ \; b_{xy, \mu * \nu} = b_{x, \mu} \cdot b_{y, \nu}$ *for $\mu \times \nu$-a.e. (x, y).*

Proof. Suppose first that (3.2) is satisfied. Then for $\mu, \nu \in M$,

$$\begin{aligned}\Sigma_{\mu*\nu} &= \int b_{t,|\mu|*|\nu|}\, d\mu*\nu(t) = \iint b_{xy,|\mu|*|\nu|}\, d\mu(x)\, d\nu(y)\\ &= \iint b_{x,|\mu|}\cdot b_{y,|\nu|}\, d\mu(x)\, d\nu(y)\\ &= \int \left(\int b_{x,|\mu|}\, d\mu(x)\right)\cdot b_{y,|\nu|}\, d\nu(y)\\ &= \int b_{x,|\mu|}\, d\mu(x)\cdot \int b_{y,|\nu|}\, d\nu(y) = \Sigma_\mu\cdot\Sigma_\nu .\end{aligned}$$

Conversely, if $\Sigma \in \mathrm{Hom}(M, B)$, then given $\mu, \nu \in M^+$, we have for all $\mu' \in L(\mu)$ and $\nu' \in L(\nu)$,

$$\begin{aligned}\int b_{xy,\mu*\nu}\, d\mu'\times\nu'(x, y) &= \int b_{t,\mu*\nu}\, d\mu'*\nu'(t) = \Sigma_{\mu'*\nu'}\\ &= \Sigma_{\mu'}\cdot\Sigma_{\nu'} = \int b_{x,\mu}\, d\mu'(x)\cdot\int b_{y,\nu}\, d\nu'(y)\\ &= \iint b_{x,\mu}\cdot b_{y,\nu}\, d\mu'(x)\, d\nu'(y)\\ &= \int b_{x,\mu}\cdot b_{y,\nu}\, d\mu'\times\nu'(x, y).\end{aligned}$$

Since the span of $L(\mu)\times L(\nu)$ is dense in $L(\mu\times\nu)$, (3.2) follows. □

If multiplication in B is jointly weak* continuous (for example, if $B_*\cap\Delta B$ separates points in B), then the unit ball in $\mathrm{Hom}(M, B)$ is easily shown to be $\mathrm{W^*OT}$ compact. An example where compactness fails is $\mathrm{Hom}(M(\mathbf{R}), M(\mathbf{R}))$: define T_n $(n \geq 1)$ in the unit ball by

$$\int_{\mathbf{R}} f(x)\, d(T_n)_\mu(x) = \int_{\mathbf{R}} f(nx)\, d\mu(x) \qquad (f \in C_0(\mathbf{R}))$$

and let $\Sigma\colon \mu \mapsto \mu(\{0\})\delta(0)$, where $\delta(0)$ is the Dirac measure at 0. Then $T_n \to \Sigma$ in $\mathrm{W^*OT}$, but

$$\Sigma \in L(M(\mathbf{R}), M(\mathbf{R}))\setminus \mathrm{Hom}(M(\mathbf{R}), M(\mathbf{R})).$$

We define the following multiplication on $L(M, B)$: if $\Sigma \sim b_{\cdot,\cdot}$ and $\Pi \sim b'_{\cdot,\cdot}$, then $\Sigma\cdot\Pi$ is defined by its Šreĭder representation $b_{x,\mu}\cdot b'_{x,\mu}$. When B is commutative, $\mathrm{Hom}(M, B)$ is closed under multiplication. It is easily verified that if multiplication in B is separately weak* continuous, then multiplication in $L(M, B)$ is separately $\mathrm{W^*OT}$ continuous.

Suppose that $M = I \oplus J$, where I is a closed ideal and J is a closed subalgebra. If $\Pi^1, \Pi^2 \in \mathrm{Hom}(M, B)$ satisfy

$$\forall \mu \in I\ \forall \nu \in J\ \Pi^1_{\mu * \nu} = \Pi^1_\mu \cdot \Pi^2_\nu \quad \& \quad \Pi^1_{\nu * \mu} = \Pi^2_\mu \cdot \Pi^1_\mu, \tag{3.3}$$

then the "sum" Σ of Π^1 and Π^2 defined in (2.3) is a homomorphism.

4. Limit points of group homomorphisms. If H is a locally compact group, then convolution is separately weak* continuous in $M(H)$. Indeed, if $\mu_\alpha, \mu, \nu \in M(H)$ with $\mu_\alpha \xrightarrow{w^*} \mu$, then for $f \in C_0(H)$, the map $x \mapsto \int f(xy)\, d\nu(y)$ lies in $C_0(H)$, whence

$$\begin{aligned}\int f\, d\mu_\alpha * \nu &= \iint f(xy)\, d\nu(y)\, d\mu_\alpha(x) \\ &\to \iint f(xy)\, d\nu(y)\, d\mu(x) = \int f\, d\mu * \nu,\end{aligned}$$

which is to say that $\mu_\alpha * \nu \xrightarrow{w^*} \mu * \nu$. A similar argument applies to $\nu * \mu_\alpha$. Thus, if G is a locally compact semigroup with separately continuous multiplication and H is a locally compact metrizable group, then the preceding section applied to $\mathrm{Hom}(M, M(H))$ for any L-subalgebra M of $M(G)$. Every continuous homomorphism $\varphi\colon G \to H$ yields an element of $\mathrm{Hom}(M, M(H))$, which we also denote by φ, defined by $\langle f, \varphi_\mu \rangle = \langle f \circ \varphi, \mu \rangle$ for $f \in C_0(H)$. The Šreĭder representation of such a φ is particularly simple: $\varphi \sim \delta(\varphi(x))$ (independent of μ), where $\delta(t)$ denotes the Dirac measure at t.

We identify $\mathrm{Hom}(G, H)$ with a subset of $\mathrm{Hom}(M(G), M(H))$ in the above manner. Our aim is to study the set

$$\Lambda = \overline{\mathrm{Hom}(G, H)} \setminus \mathrm{Hom}(G, H)$$

and its local structure

$$\Lambda(\mu) = \{\Sigma_\mu : \Sigma \in \Lambda\}, \qquad \breve{\Lambda}(\mu) = \{\sigma_{\cdot, \mu} : \sigma_{\cdot, \cdot} \in \breve{\Lambda}\},$$

where $\breve{\Lambda}$ consists of the Šreĭder representations of elements of Λ. Since all elements of $\mathrm{Hom}(G, H)$ are positive and lie in the unit ball, the same holds for Λ. (In fact, every positive homomorphism lies in the unit ball: if $0 \le \Sigma \in \mathrm{Hom}(M(G), M(H))$, then for $\mu \in M(G)$ and $n \ge 1$, we have

$$\|\Sigma_\mu\|^n \le \|\Sigma_{|\mu|}\|^n = \|\Sigma^n_{|\mu|}\| = \|\Sigma_{|\mu|^n}\| \le \|\Sigma\| \cdot \|\,\mu|^n\| = \|\Sigma\| \cdot \|\mu\|^n,$$

whence $\|\Sigma\| \le 1$.)

We are particularly interested in the case where G is a locally compact abelian group and H is a circle group, $\mathbf{T}$. In this case,

$\mathrm{Hom}(G, \mathbf{T}) = \widehat{G}$, the dual of G, and the identification of $\widehat{G}$ as a subset of $\mathrm{Hom}(M(G), M(\mathbf{T}))$ preserves the usual topology of $\widehat{G}$ (of uniform convergence on compact subsets). Furthermore, as $\widehat{G}$ lies in the unit ball of $\mathrm{Hom}(M(G), M(\mathbf{T}))$, it follows that $\overline{\widehat{G}} = \widehat{G} \cup \Lambda$ is a compactification of $\widehat{G}$.

Recall that a sequence $\{x_k\}_{k=1}^{\infty} \subseteq G$ is said to have an *asymptotic distribution* σ, written $\{x_k\} \sim \sigma$, if

$$\frac{1}{K}\sum_{k=1}^{K} \delta(x_k) \xrightarrow{w^*} \sigma \quad \text{as} \quad K \to \infty.$$

For $n \in \mathbf{Z}$ and $\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T}))$, define $\widehat{\Sigma}(n) \in \Delta M(G)$ by $\langle \mu, \widehat{\Sigma}(n)\rangle = \widehat{\Sigma}_\mu(n)$. We write the Šreĭder representation of $\chi \in \Delta M(G)$ as $\chi_\mu(x)$. Thus, if $\Sigma \sim \sigma_{\cdot,\cdot}$ and $\chi = \widehat{\Sigma}(n)$, then

$$\chi_\mu(x) = \hat{\sigma}_{x,\mu}(n).$$

Note that for all n, the map $\Sigma \mapsto \widehat{\Sigma}(n)$ from $(\mathrm{Hom}(M(G), M(\mathbf{T})),$ W* OT) to $\Delta M(G)$ (with its usual Gelfand topology) is continuous. We regard the Fourier transform as a restriction of the Gelfand transform; thus, in accordance with the Šreĭder representation, we have $\hat{\mu}(\gamma) = \int \gamma \, d\mu$ for $\gamma \in \widehat{G}$.

PROPOSITION 4.1. *Let G be a locally compact abelian group and $\Lambda = \overline{\widehat{G}}\backslash\widehat{G}$ in* $\mathrm{Hom}(M(G), M(\mathbf{T}))$. *Then*

(i) Λ *is closed topologically and under multiplication by elements of* $\overline{\widehat{G}}$;

(ii) *if* $\sigma_x, \tau_x \in \breve{\Lambda}(\mu)$, *then* $\sigma_x * \tau_x \in \breve{\Lambda}(\mu)$;

(iii) $\Lambda(\mu) = \{\nu \in M(\mathbf{T})\colon \exists$ *net* $\{\gamma_\alpha\} \subseteq \widehat{G}$ $(\gamma_\alpha \to \infty \,\&\, \forall n \in \mathbf{Z}$ $\hat{\mu}(\gamma_\alpha^n) \to \hat{\nu}(n))\}$;

(iv) $\breve{\Lambda}(\mu) = \{\sigma_{\cdot} \in \mathscr{B}(G, M(\mathbf{T}))_\mu\colon \exists$ *net* $\{\gamma_\alpha\} \subseteq \widehat{G}$ $(\gamma_\alpha \to \infty \,\&\, \forall n \in \mathbf{Z}$ $\gamma_\alpha^n \to \hat{\sigma}_{\cdot}(n)$ *weak* in* $L^\infty(\mu))\}$;

(v) *if G is metrizable, then the nets in* (iii) *and* (iv) *can be replaced by sequences and* $\breve{\Lambda}(\mu) = \{\sigma_{\cdot} \in \mathscr{B}(G, M(\mathbf{T}))_\mu\colon \exists \gamma_j \in \widehat{G}$ $(\gamma_j \to \infty \,\&$ *for every subsequence* γ_{j_k}, $\forall^e x[\mu]$ $\{\gamma_{j_k}(x)\}_{k=1}^{\infty} \sim \sigma_x)\}$.

Proof. Suppose that $\Sigma \in \Lambda$ is the limit of a net $\{\gamma_\alpha\} \subseteq \widehat{G}$. Then $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n$ in $\Delta M(G)$ for all $n \in \mathbf{Z}$. Now if $\gamma_\alpha \to \gamma \in \widehat{G}$, then $\gamma_\alpha^n \to \gamma^n$, whence $\Sigma = \gamma$. But since $\Lambda \cap \widehat{G} = \varnothing$, this is impossible, and so $\gamma_\alpha \to \infty$ in $\widehat{G}$. In particular, $\widehat{\Sigma}(1)$ is 0 on $L^1(G)$ **[HMP,**

p. 136, Proposition 4] and consequently Λ is closed. It is clear that $\Lambda \cdot \widehat{G} \subseteq \Lambda$, from which (i) now follows. Statement (ii) ensues as well. Now if $\nu \in \Lambda(\mu)$, then let $\widehat{G} \ni \gamma_\alpha \to \Sigma \in \Lambda$ be such that $\nu = \Sigma_\mu$. Then $\gamma_\alpha \to \infty$ and $(\gamma_\alpha)_\mu \xrightarrow{w^*} \Sigma_\mu = \nu$, which gives the inclusion $\subseteq$ of (iii). On the other hand, if $\gamma_\alpha \to \infty$ and $\forall n \;\; \hat{\mu}(\gamma_\alpha^n) \to \hat{\nu}(n)$, then by compactness of $\overline{\widehat{G}}$, we can choose a subnet $\{\gamma'_\beta\}$ of $\{\gamma_\alpha\}$ converging to some Σ. Since $\gamma'_\beta \to \infty$, it follows that $\Sigma \in \Lambda$ and $\nu = \Sigma_\mu \in \Lambda(\mu)$. This completes the proof of (iii). The proof of (iv) is analogous. Finally, if G is metrizable, then $L^1(\mu)$ is separable for $\mu \in M(G)$ and so $L(M(G), M(\mathbf{T}))_\mu$ is metrizable. Thus, if $\mu \in M(G)$ and $\gamma_\alpha \to \Sigma \sim \sigma_{\cdot,\cdot}$, pick any non-zero $\rho \in L^1(G)$ and a subsequence $\{\delta(\gamma_{\alpha_j}(\cdot))\}$ converging to $\sigma_{\cdot,|\mu|+|\rho|}$ in $L(M(G), M(\mathbf{T}))_{|\mu|+|\rho|}$. Then $\gamma_{\alpha_j} = \delta(\gamma_{\alpha_j}(\cdot))^\frown(1) \xrightarrow{w^*} (\widehat{\Sigma}(1))_\rho = 0$ in $L^\infty(\rho)$, whence $\gamma_{\alpha_j} \to \infty$ in $\widehat{G}$, and $\gamma_{\alpha_j}^n \xrightarrow{w^*} (\widehat{\Sigma}(n))_\mu = \hat{\sigma}_{\cdot,\mu}(n)$ in $L^\infty(\mu)$. This shows the sufficiency of sequences for (iii) and (iv). Furthermore, if $\forall n \;\; \gamma_j^n \to \hat{\sigma}_\cdot(n)$ weak* in $L^\infty(\mu)$, then by [**L2**, Lemma 5], there is a subsequence $\{\gamma'_j\}$ of $\{\gamma_j\}$ such that every further subsequence $\{\gamma'_{j_k}\}$ satisfies

$$\forall^e x[\mu] \; \{\gamma'_{j_k}(x)\}_{k=1}^\infty \sim \sigma_x . \tag{4.1}$$

Conversely, if $\{\gamma_j\}$ is a sequence, every subsequence of which satisfies (4.1), then we claim $\gamma_j^n \to \hat{\sigma}_\cdot(n)$ weak* for every n. If not, then for some n there would be a subsequence $\{\gamma_{j_k}^n\}$ converging to a different limit χ. Then also

$$\frac{1}{K}\sum_{k=1}^K \gamma_{j_k}^n \xrightarrow{w^*} \chi$$

and by (4.1),

$$\frac{1}{K}\sum_{k=1}^K \gamma_{j_k}^n \xrightarrow{w^*} \hat{\sigma}_\cdot(n).$$

Therefore $\chi = \hat{\sigma}_\cdot(n)$, a contradiction. Thus (v) follows from (iv). □

When $\widehat{G}$ is regarded as a subset of $\Delta M(G)$, we shall use the notation Γ rather than $\widehat{G}$ to avoid confusion. Let $T_n \in \mathrm{Hom}(G, G)$ denote the map $x \mapsto x^n$ $(n \in \mathbf{Z})$, as well as the corresponding map induced in $\mathrm{Hom}(M(G), M(G))$. Thus, for $\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T}))$, we obtain $\Sigma \circ T_n \in \mathrm{Hom}(M(G), M(\mathbf{T}))$; note that if $\Sigma = \gamma \in \widehat{G}$, then $\gamma \circ T_n = \gamma^n$.

PROPOSITION 4.2. *Let G be a LCA group and*

$$\Sigma \in \mathrm{Hom}(M(G), M(\mathbf{T})).$$

Then $\Sigma \in \overline{\widehat{G}}$ iff $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n \in \mathbf{Z}$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$. The map $\Sigma \mapsto \widehat{\Sigma}(1)$ is an isomorphism from $\overline{\widehat{G}}$ onto $\overline{\Gamma}$ sending $\widehat{G}$ to Γ.

Proof. If $\Sigma \in \overline{\widehat{G}}$, let $\widehat{G} \ni \gamma_\alpha \xrightarrow{\mathrm{W^*\,OT}} \Sigma$. Since $\hat{\gamma}_\alpha(n) = \gamma_\alpha^n$, we have $\gamma_\alpha^n \to \widehat{\Sigma}(n)$ for all n. In particular, $\widehat{\Sigma}(1) \in \overline{\Gamma}$. Also, $\widehat{\Sigma}(n) = \lim \gamma_\alpha^n = \lim \gamma_\alpha \circ T_n = (\lim \gamma_\alpha) \circ T_n = \widehat{\Sigma}(1) \circ T_n$. Conversely, if $\widehat{\Sigma}(1) \in \overline{\Gamma}$ and $\forall n$ $\widehat{\Sigma}(n) = \widehat{\Sigma}(1) \circ T_n$, then let $\gamma_\alpha \to \widehat{\Sigma}(1)$. Choose a convergent subnet $\gamma'_\beta \to \Pi$ in $\mathrm{Hom}(M(G), M(\mathbf{T}))$. Then from the above, $\widehat{\Pi}(n) = \widehat{\Pi}(1) \circ T_n = \widehat{\Sigma}(1) \circ T_n = \widehat{\Sigma}(n)$ for all n, whence $\Sigma = \Pi \in \overline{\widehat{G}}$.

It follows from this that the map $\Sigma \mapsto \widehat{\Sigma}(1)$ is injective. Surjectivity onto $\overline{\Gamma}$ is proved by a compactness argument similar to the above. □

We write $M(G) = M_c(G) \oplus M_d(G)$ for the decomposition of a measure into its continuous and discrete parts. Then $h_d \colon \mu \mapsto \int_G d\mu_d = \hat{\mu}_d(0)$ is in $\overline{\Gamma} \backslash \Gamma$ [**HMP**, pp. 136–7, (4.1.4)]. We denote the element of Λ corresponding to h_d by Π^d. If G has at most countably many torsion elements, then we claim that

$$\widehat{\Pi}^d(n) = \begin{cases} 0 & \text{if } n = 0, \\ h_d & \text{if } n \neq 0, \end{cases}$$

whence

$$\Pi^d_\mu = \hat{\mu}_c(0)\lambda + \hat{\mu}_d(0)\delta(0),$$

where λ is Lebesgue measure on $\mathbf{T}$. To see this, note first that

$$\widehat{\Pi}^d(0) \colon \mu \mapsto (\mu \circ T_0^{-1})^\wedge(0) = \hat{\mu}(0).$$

Second, if $n \neq 0$, then for all $g \in G$, there are, by the supposition, denumerably many $x \in G$ such that $x^n = g$. Therefore

$$(\mu \circ T_n^{-1})(\{g\}) = \sum_{x^n = g} \mu(\{x\}),$$

whence

$$\begin{aligned} \widehat{\Pi}^d(n) \colon \mu \mapsto & \sum_{g \in G} (\mu \circ T_n^{-1})(\{g\}) \\ & = \sum_{g \in G} \sum_{x^n = g} \mu(\{x\}) = \sum_{x \in G} \mu(\{x\}) = \hat{\mu}_d(0). \end{aligned}$$

This proves the claim.

Related elements of Λ are $\Sigma \cdot \Pi^d$ for $\Sigma \in \overline{\overline{\widehat{G}}}$; if, as above, the torsion subgroup of G is denumerable, then

$$(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \Sigma_{\mu_d}.$$

Thus, if we set $\Pi: \mu \mapsto \hat{\mu}(0)\lambda$, then $\Sigma \cdot \Pi^d$ is the sum of Π and Σ defined by (2.3) and (3.3) from the decomposition $M = M_c \oplus M_d$. An interesting example is $G = \mathbf{T}$ and $\Sigma: \mu \mapsto \mu$; in this case, $(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0)\lambda + \mu_d$.

Provided still that G has a denumerable torsion subgroup, the Šreĭder representation $\pi^d_{\cdot,\cdot}$ of Π^d is given by

$$\pi^d_{x,\mu} = \begin{cases} \lambda & \text{if } \mu(\{x\}) = 0, \\ \delta(0) & \text{if } \mu(\{x\}) \neq 0. \end{cases} \tag{4.2}$$

Let $\boldsymbol{\lambda} \in \mathscr{B}(G, M(\mathbf{T}))_\mu$ be defined by $\boldsymbol{\lambda}(x) \equiv \lambda$. Then from [**HMP**, p. 70, Corollaire 2] and Proposition 4.2 (or from (4.2) and the following proposition),

$$\mu \in M_c(G) \Leftrightarrow \boldsymbol{\lambda} \in \breve{\Lambda}(\mu). \tag{4.3}$$

This yields other characterizations of $M_c(G)$ when combined with Proposition 4.1 (iv), (v). For example,

$$\begin{aligned} \mu \in M_c(G) &\Leftrightarrow \exists \gamma_\alpha \to \infty\ \forall \nu \in L(\mu)\ \forall n \neq 0\ \hat{\nu}(\gamma_\alpha^n) \to 0 \\ &\Leftrightarrow \exists \gamma_\alpha \to \infty\ \forall \gamma \in \widehat{G}\ \forall n \neq 0\ \hat{\mu}(\gamma\gamma_\alpha^n) \to 0. \end{aligned}$$

Our next proposition describes $\breve{\Lambda}(\mu)$ completely when μ is discrete (cf. [**HMP**, pp. 67–68]).

PROPOSITION 4.3. *Let G be a LCA group. Let $\breve{\overline{\overline{\widehat{G}}}}$ denote the Šreĭder representations of $\overline{\overline{\widehat{G}}} \subseteq$ Hom$(M(G), M(\mathbf{T}))$ and, for $\mu \in M(G)$, $\breve{\overline{\overline{\widehat{G}}}}(\mu) = \{\sigma_{\cdot,\mu} : \sigma_{\cdot,\cdot} \in \breve{\overline{\overline{\widehat{G}}}}\}$. Let G_d denote G with the discrete topology and, for $\mu \in M_d(G)$, let G_d^μ denote the discrete subgroup generated by the mass-points of μ.*

(i) $\forall \Sigma \in \overline{\overline{\widehat{G}}}\ \exists \varphi \in \widehat{G}_d\ \forall \mu \in M_d(G)\ \Sigma_\mu = \sum_{x \in G} \mu(\{x\})\delta(\varphi(x))$ *and* $\sigma_{x,\mu} = \delta(\varphi(x))$, *where* $\Sigma \sim \sigma_{\cdot,\cdot}$.

(ii) $\forall \mu \in M_d(G)\ \breve{\overline{\overline{\widehat{G}}}}(\mu) \simeq \widehat{G_d^\mu}$.

(iii) $\mu \in M_d(G) \Leftrightarrow \breve{\overline{\overline{\widehat{G}}}}(\mu)$ *is a group (under the multiplication in* $L(M(G), M(\mathbf{T}))$*).*

Proof. (i) Let $\widehat{G} \ni \gamma_\alpha \xrightarrow{\text{W}^*\text{OT}} \Sigma$. Then for $x \in G$,

$$\delta(\gamma_\alpha(\cdot)) \to \sigma_{\cdot,\delta(x)} \in L(M(G), M(\mathbf{T}))_{\delta(x)},$$

i.e., $\delta(\gamma_\alpha(x)) = \sigma_{x,\delta(x)}$ eventually. Thus, $\gamma_\alpha(x)$ stabilizes at some point $\varphi(x)$ and $\sigma_{x,\delta(x)} = \delta(\varphi(x))$. The assertions now follow from linearity and properties of the Šreĭder representation.

(ii) The fact that $\breve{\overline{\widehat{G}}}(\mu)$ can be identified as a compact subgroup of $\widehat{G_d^\mu}$ follows from (i). If it were not the whole group, then there would be a nonzero $x \in G_d^\mu$ such that $\varphi(x) = 1$ for all $\varphi \in \breve{\overline{\widehat{G}}}(\mu)$. In particular, $\gamma(x) = 1$ for all $\gamma \in \widehat{G}$, whence $x = 0$, a contradiction.

(iii) This follows from [**HMP**, p. 68, Proposition 10] and (ii).□

We now arrive at the characterization of positive continuous measures mentioned in the introduction.

THEOREM 4.4. *Let G be a LCA group whose torsion subgroup is denumerable and let $\mu \in M^+(G)$ be* positive. *Then $\mu \in M_c^+(G)$ iff there is a net $\widehat{G} \ni \gamma_\alpha \to \infty$ such that for all $n \neq 0$, $\hat{\mu}(\gamma_\alpha^n) \to 0$.*

Proof. By Proposition 4.1 (iii), this is equivalent to $\mu \in M_c^+(G) \Leftrightarrow \hat{\mu}(0)\lambda \in \breve{\Lambda}(\mu)$. For $\mu \in M_c^+(G)$, this follows from $\lambda \in \breve{\Lambda}(\mu)$ (see (4.3)). If $\mu \notin M_c^+(G)$ and $\Sigma \in \Lambda$, then $\Sigma_\mu = \Sigma_{\mu_c} + \Sigma_{\mu_d} \geq \Sigma_{\mu_d}$ since $\mu_c \geq 0$ and $\Sigma \geq 0$. However, by Proposition 4.3(i), Σ_{μ_d} is nonzero and discrete; hence Σ_μ cannot equal $\hat{\mu}(0)\lambda$. □

Because of the interest this theorem may present, we provide the following "elementary" proof and strengthening for the case $G = \mathbf{T}$. If $\mu \in M_c(\mathbf{T})$, then by Wiener's theorem [**K**, p. 42], there is a sequence $\{m_k^{(1)}\}$ of density one in $\mathbf{N}$ such that $\hat{\mu}(m_k^{(1)}) \to 0$. Likewise, there is a sequence $\{m_k^{(n)}\}$ of density one such that $\hat{\mu}(nm_k^{(n)}) = \widehat{(T_n)_\mu}(m_k^{(n)}) \to 0$ since $(T_n)_\mu \in M_c$, for $n \neq 0$. By an elementary intersection argument, we obtain a sequence $\{m_k\}$, still of density one, such that for all $n \neq 0$, $\hat{\mu}(nm_k) \to 0$. (A similar argument produces a sequence $\{m_k\}$ of density one such that for $n \neq 0$ and all r, $\hat{\mu}(r + nm_k) \to 0$, i.e., $\delta(m_k x) \to \lambda$ in $L(M(\mathbf{T}), M(\mathbf{T}))_\mu$, thereby strengthening (4.3).) For the converse, we use the following proof due to Jean-François Méla. Let $K_l(x)$ be the Fejér kernel of order l. Then if $\mu \geq 0$ and if for all $n \neq 0$, $\hat{\mu}(nm_k) \to 0$, then

$$\mu(\{0\}) \le \int_{\mathbf{T}} \frac{1}{2l+1} K_l(m_k x)\, d\mu(x) \to \frac{1}{2l+1}\hat{\mu}(0) \quad \text{as } k \to \infty$$

by hypothesis. Since this is true for all l, it follows that $\mu(\{0\}) = 0$. Now apply this result to $\mu * \tilde{\mu}$, where $\tilde{\mu}(E) = \mu(-E)$.

The local structure of Λ can be used to characterize other classes of measures besides M_c and M_d. If $\mathscr{C}$ is a class of subsets of G, let

$$\mathscr{C}^{\perp} = \{\mu \in M(G) \colon \forall E \in \mathscr{C}\ |\mu|(E) = 0\}.$$

Thus, if $\mathscr{D}$ is the class of singletons, $\mathscr{D}^{\perp} = M_c(G)$.

Definition. A set $E \subseteq G$ is called an *H-set* if there is a sequence $\widehat{G} \ni \gamma_k \to \infty$ such that $\{\gamma_k(x) \colon k \ge 1, x \in E\}$ is not dense in $\mathbf{T}$. A set $E \subseteq G$ is called a *Dirichlet set* if there is a sequence $\widehat{G} \ni \gamma_k \to \infty$ such that $\lim_{k\to\infty} \sup_{x\in E} |\gamma_k(x) - 1| = 0$. A measure $\mu \in M(G)$ is called a *Dirichlet measure* if $\overline{\lim}_{\gamma\to\infty} |\widehat{|\mu|}(\gamma)| = \|\mu\|$.

For background on H-sets, see [**Z**, Chapters IX, XII]; on Dirichlet sets and measures, see [**HMP**, pp. 34–35, 240–247]. The following proposition is used in [**KL**].

Proposition 4.5. *Let G be a LCA group.*

(i) *If G is metrizable, then*

$$\begin{aligned} H^{\perp} &= \{\mu \colon \forall \sigma.\in \breve{\Lambda}(\mu)\ \forall^e x[\mu]\ \operatorname{supp} \sigma_x = \mathbf{T}\} \\ &= \{\mu \colon \forall \Sigma \in \Lambda\ \forall \nu \in L(\mu)\ \operatorname{supp} \Sigma_\nu = \mathbf{T}\}. \end{aligned}$$

(ii) *μ is a Dirichlet measure iff the constant function $\boldsymbol{\delta}(\mathbf{0}) \in \Lambda(\mu)$.*

(iii) $D^{\perp} = \{\mu \colon \forall \sigma.\in \breve{\Lambda}(\mu)\ \forall^e x[\mu]\ \sigma_x \neq \delta(0)\}$

Proof. Part (i) follows from Proposition 4.1(v) and a straightforward generalization of [**L4**, Theorem 13]. Part (ii) follows from Proposition 4.2 and the fact that μ is a Dirichlet measure iff the constant function $\mathbf{1} \in (\overline{\Gamma}\backslash\Gamma)(\mu)$ [**HMP**, p. 34, Lemma 6]. Part (iii) follows from part (ii) and the fact that $D^{\perp}$ consists of the measures orthogonal to the Dirichlet measures [**HMP**, p. 243, Proposition 9]. □

Our final remarks concern the circle group.

Definition. A positive measure $\mu \in M^+(\mathbf{T})$ is called *C-quasi-symmetric* if for all pairs of adjacent arcs, I and J, on $\mathbf{T}$ of equal length, $\mu I \leq C \cdot \mu J$. We denote the class of C-quasisymmetric measures by $QS(C)$.

Note that quasisymmetric measures are continuous.

PROPOSITION 4.6. *The class $QS(C)$ is weak* closed. If $\mu \in QS(C)$, then $\Lambda(\mu) \subseteq QS(C)$, $\breve{\Lambda}(\mu) \subseteq QS(C)$ in the sense that if $\sigma. \in \breve{\Lambda}(\mu)$, then $\forall^e x[\mu]$ $\sigma_x \in QS(C)$, and $\Lambda(\nu) \subseteq QS(C)$ for all $0 \leq \nu \in L(\mu)$.*

Proof. Let $QS(C) \ni \mu_\alpha \xrightarrow{w^*} \nu$. Given adjacent arcs I, J of equal length and $\varepsilon > 0$, pick $f, g \in C(\mathbf{T})$ such that $f \leq \mathbf{1}_I$, $\mathbf{1}_J \leq g$, $\int(\mathbf{1}_I - f)\,d\nu \leq \varepsilon$, and $\int(g - \mathbf{1}_J)\,d\nu \leq \varepsilon$. We have

$$\begin{aligned} \nu I &\leq \int f\,d\nu + \varepsilon = \lim \int f\,d\mu_\alpha + \varepsilon \leq \overline{\lim}\,\mu_\alpha I + \varepsilon \\ &\leq C \cdot \overline{\lim}\,\mu_\alpha J + \varepsilon \leq C \cdot \lim \int g\,d\mu_\alpha + \varepsilon = C \int g\,d\nu + \varepsilon \\ &\leq C \cdot \nu J + (C+1)\varepsilon. \end{aligned}$$

Since ε was arbitrary, we see that $\nu I \leq C \cdot \nu J$, whence $\nu \in QS(C)$.

Choose $\mu \in QS(C)$. Then $\gamma_\mu \in QS(C)$ for any $\gamma \in \widehat{\mathbf{T}}$. Since $\Lambda(\mu)$ is contained in the weak* closure of $\{\gamma_\mu\}_{\gamma \in \widehat{\mathbf{T}}}$, it follows that $\Lambda(\mu) \subseteq QS(C)$. Suppose that $E \subseteq \mathbf{T}$ and $\mu E > 0$. If I and J are adjacent arcs of equal length and $\varepsilon > 0$, then choose U, a finite union of arcs, such that $\mu(U \triangle E) \leq \varepsilon$. By continuity of μ, we have for all large γ,

$$\begin{aligned} \mu(E \cap \gamma^{-1}[I]) &\leq \mu(U \cap \gamma^{-1}[I]) + \varepsilon \leq C \cdot \mu(U \cap \gamma^{-1}[J]) + 2\varepsilon \\ &\leq C \cdot \mu(E \cap \gamma^{-1}[J]) + (C+2)\varepsilon. \end{aligned}$$

Since ε was arbitrary, it follows that $\Lambda(\mu|_E) \subseteq QS(C)$. As $QS(C)$ is a positive cone, we deduce that $\Lambda(\nu) \subseteq QS(C)$ for $0 \leq \nu \in L(\mu)$. Finally, let $\sigma. \in \breve{\Lambda}(\mu)$. Let P be the essential range of $\sigma.$, i.e, the smallest weak* closed set P such that $\sigma_x \in P$ μ-a.e. Then P is contained in the weak* closure of $\{\int \sigma_x\,d\nu(x) : 0 \leq \nu \in L(\mu), \|\nu\| = 1\} = \bigcup\{\Lambda(\nu) : 0 \leq \nu \in L(\mu), \|\nu\| = 1\}$, which, by the above, is contained in $QS(C)$. □

As an example of the pathology possible for $\Lambda(\mu)$, we present the following observation.

PROPOSITION 4.7. *There is a measure $\mu \in M(\mathbf{T})$ such that for any probability measure $\nu \in M(\mathbf{T})$, there exists $\sigma. \in \breve{\Lambda}(\mu)$ such that $\sigma_x = \nu$ μ-a.e.*

Proof. Let $\{P_k\}_{k\geq 1}$ be a set of trigonometric polynomials such that $\{P_k \cdot \lambda\}$ is weak* dense in the set of probability measures. Let $\{n_k\} \subseteq \mathbf{N}$ satisfy $n_{k+1} \geq 3n_k \cdot \deg P_k$. Form the generalized Riesz product [**HMP**, Chapitre 5] $\mu = \prod_{k\geq 1} P_k(n_k x)$. Then given a probability ν, let $P_{k_l}\lambda \xrightarrow{w^*} \nu$. For any $r, m \in \mathbf{Z}$, it is easy to see that $\hat{\mu}(r+mn_{k_l}) \to \hat{\mu}(r)\hat{\nu}(m)$, i.e., $\delta(n_{k_l}x) \to \nu$ in $L(M(\mathbf{T}), M(\mathbf{T}))_\mu$. □

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