THE LOCAL STRUCTURE
OF SOME MEASURE-ALGEBRA HOMOMORPHISMS

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Extending classical theorems, we obtain representations for bounded linear transformations from $L$-spaces to Banach spaces with a separable predual. In the case of homomorphisms from a convolution measure algebra to a Banach algebra, we obtain a generalization of Šreider's representation of the Gelfand spectrum via generalized characters. The homomorphisms from the measure algebra on a LCA group, $G$, to that on the circle are analyzed in detail. If the torsion subgroup of $G$ is denumerable, one consequence is the following necessary and sufficient condition that a positive finite Borel measure on $G$ be continuous: $\exists \gamma_\alpha \to \infty$ in $\hat{G}$ such that $\forall n \neq 0 \quad \hat{\mu}(\gamma_\alpha) \to 0$.

1. Introduction. Given a measurable space $X$ and a (bounded) complex measure $\mu$ on $X$, the Banach space dual of $L^1(\mu)$ is commonly represented as $L^\infty(\mu)$. We shall call $M$ an $L$-space on $X$ if $M$ is a Banach space of complex measures on $X$ (under the measure norm) such that $\nu \ll \mu \in M \Rightarrow \nu \in M$ [Sc]. Šreider [Sr] gave a representation of the dual $M^*$ of $M$ as a space of so-called generalized functions, i.e., families of functions $f_\mu \in L^\infty(\mu)$ satisfying

(1.1) $\nu \ll \mu \Rightarrow f_\nu = f_\mu \quad \nu$-a.e.,

(1.2) $\sup_{\mu \in M} \|f_\mu\|_{L^\infty(\mu)} < \infty$.

The representation of $M^*$, like that of $L^1(\mu)^*$, is by integration:

$\mu \mapsto \int f_\mu d\mu$.

Now, given two Banach spaces, $B_1$ and $B_2$, we denote by $L(B_1, B_2)$ the Banach space of bounded linear transformations from $B_1$ to $B_2$. Since $M^* = L(M, C)$, we may ask, in generalizing the above, for a representation of $L(M, B)$, where $B$ is an arbitrary Banach space. Again, the case where $M = L^1(\mu)$ is classical [DS]; here, the hypothesis that $B$ has a separable predual is made. In §2, we extend this theorem to general $L$-spaces $M$ in a manner similar to Šreider's representation above. In essence, functions are replaced by
$B$-valued functions. Our treatment will be entirely self contained, thus giving an apparently new proof of [DS, Theorem VI.8.6]. However, another point of view could be adopted. Namely, if we use the Radon-Nikodym theorem to identify $L(\mu) = \{ \nu \ll \mu : \nu \text{ bounded} \}$ with $L^1(\mu)$, then we may regard an $L$-space $M$ as the direct limit $\lim_{\mu \in M} L^1(\mu)$, where $M$ is directed by $\ll$ and for $\nu \ll \mu$, $L^1(\nu)$ is included in $L^1(\mu)$. Now $L(\cdot, B)$ is a functor from the category of Banach spaces to its opposite category and, furthermore, is easily checked to be a left adjoint. Since left adjoints preserve direct limits and inverse limits are dual to direct limits, it follows that $L(M, B)$ is the inverse limit $\lim_{\mu \in M} L(L^1(\mu), B)$, where, for $\nu \ll \mu$, $L(L^1(\mu), B)$ is mapped by restriction to $L(L^1(\nu), B)$. Hence, given a representation of $L(L^1(\mu), B)$ (as in [DS]) and a construction of inverse limits, we may obtain a representation of $L(M, B)$. This amounts to the same as our Theorem 2.1.

Now Šreider was actually interested in representing $\Delta M$, the multiplicative linear functionals on $M$, when $M$ was a convolution measure algebra on a locally compact abelian group. He showed that in addition to (1.1) and (1.2), the following property was necessary and sufficient for $f_\mu$ to define an element of $\Delta M$:

\[(1.3) \quad \forall \mu, \nu \geq 0 f_{\mu \ast \nu}(xy) = f_\mu(x)f_\nu(y) \quad \mu \times \nu \text{-a.e. } [(x, y)].\]

We, too, are mainly interested in the subset of homomorphisms $\text{Hom}(M, B) \subseteq L(M, B)$ when $B$ is a Banach algebra. A similar condition to (1.3) is found in Theorem 3.2. In particular, when $M = M(G)$, the complex Borel measures on a locally compact abelian group, $G$, and $B = M(T)$, $T$ the circle, $\text{Hom}(M(G), M(T))$ contains in a natural way $\text{Hom}(G, T) = \hat{G}$. The closure of $\hat{G}$ in a certain weak topology is related to the behavior of Fourier transforms at infinity and contains much information about a measure $\mu$ when regarded locally, i.e., when restricted to $L(\mu)$, or, what is the same, when viewed via the Šreider representation. For example, this analysis will lead to the following surprising result: if the torsion subgroup of $G$ is denumerable, then a positive measure $\mu \in M(G)$ is continuous iff there is a net $\{\gamma_\alpha\} \subseteq \hat{G}$ tending to infinity such that for all $n \neq 0$, $\lim_\alpha \hat{\mu}(\gamma_\alpha^n) = 0$. Characterizations of certain other classes of measures are found in §4; these have proved useful in [KL] and [L4]. Other analyses of the local structure of the closure of $\hat{G}$ for certain $\mu$ can be found in [L3], [L4], and [L5]. The local structure of $\hat{G}$ is also related to asymptotic distribution; this relationship, described here, has been used in [KL] and [L4].
The Šreider representation, Theorem 3.2, has been given before in [IgK] for the case $\text{Hom}(M, M(T))$, $M$ being an $L$-subalgebra of $M(T)$, though in slightly different notation. An alternative representation for $\text{Hom}(M, M(G))$, where $M$ is a semisimple commutative convolution measure algebra in the sense of Taylor and $G$ is a compact abelian group, analogous to Taylor's representation of $\Delta M$ via a structure semigroup, has been given in [InK].

2. The Šreider representation of linear transformations. Suppose that $M$ is an $L$-space on a measurable space $X$ and that $B$ is a Banach space with a separable predual, $B_*$. Let $\mathcal{B}(X, B)$ denote the set of maps $f: X \to B$ which are bounded in $B$-norm and measurable when $B$ is given the weak* topology from $B_*$. If $f \in \mathcal{B}(X, B)$ and $\mu \in M$, there is a unique element $\int f \, d\mu \in B$ defined by the relation

$$\forall b_* \in B_* \quad \left\langle b_*, \int f \, d\mu \right\rangle = \int_X \langle b_*, f(x) \rangle \, d\mu(x).$$

If $D$ is a countable dense set in the unit ball of $B_*$, then the equation

$$\|f(x)\|_B = \sup_{b_* \in D} |\langle b_*, f(x) \rangle|$$

shows that $\|f(\cdot)\|_B$ is measurable. It is clear that

$$\left\| \int f \, d\mu \right\|_B \leq \int \|f\|_B \, d|\mu|.$$

The set of equivalence classes of $\mathcal{B}(X, B)$ under equality $\mu$-a.e. will be denoted $\mathcal{B}(X, B)_\mu$, although this distinction will often be ignored.

The following theorem, which we shall term the Šreider representation, associates to each element of $L(M, B)$ a certain family of maps in $\mathcal{B}(X, B)$. We denote the image of $\mu \in M$ under $\Sigma \in L(M, B)$ by $\Sigma\mu$.

**Theorem 2.1.** Let $M$ be an $L$-space and $B$ a Banach space with a separable predual. There is a bijection between $L(M, B)$ and the set of elements $\{b_*, \mu\}_{\mu \in M} \in \prod_{\mu \in M} \mathcal{B}(X, B)_\mu$ which satisfy

(i) $\sup_{\mu \in M} \|b_{x, \mu}\|_{L^\infty(\mu)} < \infty$

and

(ii) $\forall \nu \ll \mu \in M \quad b_{x, \nu} = b_{x, \mu} \quad \nu$-a.e. [x]
such that if $\Sigma$ corresponds to $\{b., \mu\}_{\mu \in M}$ (written $\Sigma \sim b.$), then

$$(iii) \quad \forall \mu \in M \quad \Sigma_{\mu} = \int b_{x, \mu} \, d\mu(x)$$

and

$$(iv) \quad \|\Sigma\|_{L(M, B)} = \sup_{\mu \in M} \|b_{x, \mu}\|_{L^\infty(\mu)}.$$

**Proof.** Given $\{b., \mu\}$ satisfying (i) and (ii), define $\Sigma$ by (iii). If $\mu, \nu \in M$, then by (ii), we have $b_{x, \mu} = b_{x, |\mu|+|\nu|}$ $\mu$-a.e., whence $\Sigma_{\mu} = \int b_{x, |\mu|+|\nu|} \, d\mu(x)$. In conjunction with similar equations for $\Sigma_{\nu}$ and $\Sigma_{\mu+\nu}$, this equation shows that $\Sigma_{\mu} + \Sigma_{\nu} = \Sigma_{\mu+\nu}$. Similarly, for $\alpha \in \mathbb{C}$, $\Sigma_{\alpha \mu} = \alpha \Sigma_{\mu}$, whence $\Sigma$ is linear. Let $K$ denote the quantity in (i). Then

$$\|\Sigma\| = \sup_{\|\mu\| \leq 1} \|\Sigma_{\mu}\| = \sup_{\|\mu\| \leq 1} \left\|\int b_{x, \mu} \, d\mu(x)\right\| \leq \sup_{\|\mu\| \leq 1} \int \|b_{x, \mu}\| \, |d\mu(x)| \leq K.$$

To show that $\|\Sigma\| = K$, choose any nonzero $\mu \in M$ and $\varepsilon > 0$. Let $0 \neq \nu \in L(\mu)$ be such that $\|b., \mu\|_{B} - \|b., \mu\|_{L^\infty(\mu)} \|L^\infty(\nu) < \varepsilon$. Let $S$ be the unit sphere of $B$. Since the unit ball of $B$ is weak* compact, there exists a finite number of elements, $b^1_*, \ldots, b^n_*$, of the unit ball of $B_*$ such that

$$S = \bigcup_{i=1}^{n} \{b \in S : |\langle b^i_*, b \rangle - 1| < \varepsilon\}.$$

Therefore $\exists \omega \in L(\nu)$ $\forall i \quad |\langle b^i_*, b, \mu \rangle | b_{x, \mu} \|B\rangle - 1\|L^\infty(\omega) < \varepsilon$. We have

$$\|\Sigma\| \geq \|\Sigma_{\omega}\| \geq \frac{1}{\|\omega\|} |\langle b^i_*, \Sigma_{\omega} \rangle | = \frac{1}{\|\omega\|} \left\|\int b^i_*, b_{x, \mu} \, d\omega(x)\right\| \geq \frac{1}{\|\omega\|} \int \|b_{x, \mu}\|_{B} \, d\omega(x) - \varepsilon K \geq \|b., \mu\|_{L^\infty(\mu)} - \varepsilon (K + 1).$$

Thus $\|\Sigma\| = K$.

Conversely, let $\Sigma \in L(M, B)$. Fix $\mu \in M$. For $b_* \in B_*$, we denote by $b_* \circ \Sigma$ the map $\nu \mapsto \langle b_*, \Sigma_{\nu} \rangle$. Restricted to $L(\mu)$, this map is a bounded linear functional and hence can be represented by a function $g_{b_*} \in L^\infty(\mu)$. Choose a countable linearly independent set $D$ whose
linear span over $\mathbb{Q}$, $D'$, is dense in $B_*$. If $b_* = \sum_{i=1}^{n} \alpha_i d_i^*$, $d_i^* \in D$, $\alpha_i \in \mathbb{Q}$, define

$$h_{b_*} = \sum_{i=1}^{n} \alpha_i g_{d_i^*}.$$ 

Then $b_* \mapsto h_{b_*}(x)$ is rational-linear on $D'$ for every $x \in X$. Furthermore, $h_{b_*} = g_{b_*} \mu$-a.e., whence by countability of $D'$,

$$\forall b_* \in D' \mid h_{b_*}(x) \leq \|b_* \cdot \Sigma\| \leq \|b_*\| \cdot \|\Sigma\|$$

for $\mu$-a.e. $x$. Now for every $x$ such that (2.1) holds, $b_* \mapsto h_{b_*}(x)$ extends from $D'$ to all of $B_*$ as a bounded linear functional, hence element of $B$, call it $f(x)$. This defines $f(x)$ $\mu$-a.e. and shows that, given any $b_* \in B_*$, if $b_* = \lim_{n \to \infty} b_n^*$ ($b_n^* \in D'$), then

$$\langle b_*, f(x) \rangle = \lim_{n \to \infty} \langle b_n^*, f(x) \rangle = \lim_{n \to \infty} h_{b_n^*}(x)$$

for every $x$ where $f$ is defined. Write $b_*, \mu$ for the equivalence class of $f$. From Equation (2.1), we see that $\|f(x)\| \leq \|\Sigma\|$ for every $x$ where $f$ is defined. Together with (2.2), this shows that $b_*, \mu \in \mathcal{B}(X, B)_\mu$ and gives (i). Now for $\nu \in L(\mu)$ and $b_* \in D'$, we have

$$\left\langle b_*, \int f \, d\nu \right\rangle = \left\langle b_*, f(x) \right\rangle \, d\nu(x) = \int h_{b_*}(x) \, d\nu(x) = \int g_{b_*}(x) \, d\nu(x) = \langle b_*, \Sigma \nu \rangle.$$ 

Since $D'$ is dense, (iii) follows. We claim that $b_*, \mu$ is uniquely determined by the property just established:

$$\forall \nu \in L(\mu) \Sigma \nu = \int b_{x, \mu} \, d\nu(x).$$ 

Indeed, if we also have that $\forall \nu \in L(\mu) \Sigma \nu = \int b'_{x, \mu} \, d\nu(x)$ for some $b'_{x, \mu} \in \mathcal{B}(X, B)_\mu$, then

$$\forall b_* \in D' \forall \nu \in L(\mu) \int \langle b_*, b_{x, \mu} \rangle \, d\nu(x) = \int \langle b_*, b'_{x, \mu} \rangle \, d\nu(x),$$

whence for $\mu$-a.e. $x$ $\forall b_* \in D'$ $\langle b_*, b_{x, \mu} \rangle = \langle b_*, b'_{x, \mu} \rangle$, i.e., $b_{x, \mu} = b'_{x, \mu}$ $\mu$-a.e. Thus (ii) follows. The same argument shows that if $\Sigma \sim b_* \cdot \mu$ and $\Sigma \sim b'_{x, \mu}$, then $b_* = b'_{x, \mu}$.

We define the weak* operator topology (W* OT) on $L(M, B)$ as the weakest topology such that $\forall \mu \in M \forall b_* \in B_* \Sigma \mapsto \langle b_*, \Sigma \mu \rangle$ is continuous. It is an elementary exercise to show that the unit ball of $L(M, B)$ is W* OT compact.
For $\mu \in M$, let $L(M, B)_\mu$ denote the set of Šreider representations $b.,_\mu$ of elements of $L(M, B)$. We give $L(M, B)_\mu$ the weak topology generated by the maps $b.,_\mu \mapsto \int \langle b.*, b_x, \nu \rangle \, d\nu(x)$ ($b.* \in B.*, \nu \in L(\mu)$). Thus, the $W^*$ OT is the inverse limit of these topologies, i.e., it is the weak topology generated by the maps $\Sigma \mapsto b.,_\mu$ ($\mu \in M$) from $L(M, B) \to L(M, B)_\mu$, where $\Sigma \sim b.,$.

Every decomposition $M = I \oplus J$ of $M$ as a direct sum of closed subspaces yields an addition on $L(M, B)$ as follows: if $\Pi^1, \Pi^2 \in L(M, B)$, then we may define

$$\Sigma_\mu = \Pi^1_{\mu^1} + \Pi^2_{\mu^2},$$

where $\mu = \mu^1 + \mu^2$, $\mu^1 \in I$, $\mu^2 \in J$. If $\Sigma \sim b.,$, $\Pi^i \sim b.,_i$, and $I \perp J$, then $b_x,\mu = b^1_{x,\mu^1} + b^2_{x,\mu^2}$ $\mu$-a.e.

The case where $B = M(Y)$, the space of complex regular Borel measures on a locally compact metric space, $Y$, is of interest. A predual of $B$ is the separable space $C_0(Y)$ of continuous functions vanishing at infinity. We shall denote the Šreider representation of $\Sigma$ by $\sigma_{x,}\mu$ in this case; thus, if $f \in C_0(Y)$ and $\mu \in M$,

$$\int_Y f \, d\Sigma_\mu = \int_X \left( \int_Y f \, d\sigma_{x,}\mu \right) \, d\mu(x).$$

(If $Y$ is separable and a countable union of complete subspaces, then (2.4) holds for $f \in \mathcal{B}(Y, \mathbb{C})$ since it is preserved under bounded pointwise limits. In particular, for Borel sets $E \subseteq Y$,

$$\Sigma_\mu(E) = \int_X \sigma_{x,}\mu(E) \, d\mu(x).$$

Let $M^+$ denote the nonnegative elements of $M$ and likewise for $M^+(Y)$. We say that $\Sigma \in L(M, M(Y))$ is positive if it carries $M^+$ into $M^+(Y)$. It is easy to see from (2.4) applied to $|\mu|$ that $\Sigma \geq 0$ iff $\forall \mu \in M \forall x[\mu] \sigma_{x,}\mu \geq 0$ ("$\forall x[\mu]$" means "for $\mu$-a.e. $x$"—see [L1]). It is also easy to show that if $\Sigma \geq 0$, then $\nu \ll \mu \Rightarrow \Sigma_\nu \ll \Sigma_{|\mu|}$ and $|\Sigma_\mu| \leq \Sigma_{|\mu|}$. 

3. The Šreider representation of homomorphisms. Let $G$ be a locally compact semigroup with separately continuous multiplication. Then $M(G)$ is a Banach algebra under convolution [W]. Let $M$ be an $L$-subalgebra of $M(G)$, i.e., a subalgebra which is also an $L$-subspace, and let $B$ be a Banach algebra with a separable predual such that
multiplication is separately weak* measurable and

\[(3.1) \quad \forall f \in \mathcal{B}(G, B) \forall b \in B \forall \mu \in M \quad \int f(x) \cdot b \, d\mu(x) = \left( \int f \, d\mu \right) \cdot b \]

\& \quad \int b \cdot f(x) \, d\mu(x) = b \cdot \int f \, d\mu.

In order to state some sufficient conditions that (3.1) be true, we define the following multiplication on $B^* \times B$. If $b \in B$ and $b^* \in B^*$, then $b^* \cdot (b^* \cdot b, b^*)$ is a bounded linear functional on $B$; we denote it by $b^* \cdot b$. Let $\overline{B}_{sw}^*$ be the smallest subspace of $B^*$ containing (canonically) $B$, which is closed under weak* sequential limits. Let $\Delta B$ be the subset of $B^*$ consisting of the multiplicative linear functionals.

**Proposition 3.1.** Let $B$ be a Banach algebra with a separable pre-dual. Right multiplication on $B$ is weak* measurable and the first equation of (3.1) holds if any of the following conditions is satisfies:

(i) $B_+ \cdot B \subseteq \overline{B}_{sw}^*$.

(ii) Right multiplication is weak* continuous.

(iii) Right multiplication is weak* measurable and $\overline{B}_{sw}^* \cap \Delta B$ separates points in $B$.

**Proof.** The class of $b^* \in B^*$ such that $b \mapsto \langle b, b^* \rangle$ is weak* measurable contains $B_+$ and is closed under weak* sequential limits. Thus, all elements of $\overline{B}_{sw}^*$ are weak* measurable. Now right multiplication is weak* measurable iff $\forall b \in B \forall b^* \in B_+ b^* \cdot (b^*, b^* \cdot b)$ is weak* measurable. But $\langle b^*, b^* \cdot b \rangle = \langle b', b^* \cdot b \rangle$, whence this condition is equivalent to all elements of $B_+ \cdot B$ being weak* measurable. The sufficiency of (i) for measurability is now obvious. Also, the class of weak* measurable $b^* \in B^*$ such that

$$\langle \int f \, d\mu, b^* \rangle = \int \langle f, b^* \rangle \, d\mu$$

is closed under weak* sequential limits by the bounded convergence theorem, hence contains $\overline{B}_{sw}^*$. Thus, if (i) holds, then $\forall b^* \in B_+ \forall b \in B$

$$\langle b^*, \int f \cdot b \, d\mu \rangle = \int \langle b^*, f \cdot b \rangle \, d\mu = \int \langle f, b^* \cdot b \rangle \, d\mu$$

$$= \langle \int f \, d\mu, b^* \cdot b \rangle = \langle b^*, (\int f \, d\mu) \cdot b \rangle,$$

whence the first equation of (3.1).
Now (ii) is equivalent to $B_\ast \cdot B \subseteq B_\ast$ since $B_\ast$ is the set of weak* continuous linear functionals on $B$. Thus, sufficiency follows from that of (i). Finally, if (iii) holds, then for $f \in \mathcal{B}(G, B)$, $b \in B$, $\mu \in M$, and $b^* \in \overline{B_\ast} \cap \Delta B$, we have

$$\langle \int f \cdot b \, d\mu, b^* \rangle = \int \langle f \cdot b, b^* \rangle \, d\mu = \int \langle f, b^* \rangle \langle b, b^* \rangle \, d\mu$$

$$= \int \langle f, b^* \rangle \, d\mu \cdot \langle b, b^* \rangle = \left\langle \int f \, d\mu, b^* \right\rangle \cdot \langle b, b^* \rangle$$

$$= \left\langle \left( \int f \, d\mu \right) \cdot b, b^* \right\rangle,$$

from which the first equation of (3.1) follows. \qed

Let $\mathcal{B}_0(G, B)$ denote the Baire-measurable functions from $G$ to $B$, where $B$ is given the weak* topology. For $\mu, \nu \in M(G)$, let $\mu \times \nu$ denote, besides the usual product measure, also its unique extension to a regular Borel measure in $M(G \times G)$. If $f \in \mathcal{B}_0(G, B)$ and $\mu, \nu \in M(G)$, then

$$\int f \, d\mu \ast \nu = \int f(x) \, d\mu \times \nu(x, y)$$

$$= \iint f(x) \, d\mu(x) \, d\nu(y),$$

as can be seen by applying any $b_\ast \in B_\ast$. [W].

The Šreider representation of $\text{Hom}(M, B)$, the continuous homomorphisms from $M$ to $B$, satisfies one property additional to those in Theorem 2.1.

**Theorem 3.2.** Let $G$ be a locally compact semigroup with separately continuous multiplication and $M$ an $L$-subalgebra of $M(G)$. Let $B$ be a Banach algebra with a separable predual and separately weak* measurable multiplication satisfying (3.1). Let $\Sigma \in L(M, B)$ and choose $b_\ast, \mu \in \mathcal{B}_0(G, B)$ ($\mu \in M$) so that $\Sigma \sim b_\ast, \mu$. Then $\Sigma \in \text{Hom}(M, B)$ iff

$$\forall \mu, \nu \in M^+ \quad b_{xy, \mu + \nu} = b_{x, \mu} \cdot b_{y, \nu} \quad \text{for } \mu \times \nu\text{-a.e. } (x, y).$$
Proof. Suppose first that (3.2) is satisfied. Then for $\mu, \nu \in M$,
\[
\Sigma_{\mu * \nu} = \int b_{t, |\mu|*|\nu|} d\mu * \nu(t) = \int \int b_{xy, |\mu|*|\nu|} d\mu(x) d\nu(y)
\]
\[
= \int \int b_{x, |\mu|} \cdot b_{y, |\nu|} d\mu(x) d\nu(y)
\]
\[
= \int (\int b_{x, |\mu|} d\mu(x)) \cdot b_{y, |\nu|} d\nu(y)
\]
\[
= \int b_{x, |\mu|} d\mu(x) \cdot \int b_{y, |\nu|} d\nu(y) = \Sigma_{\mu} \cdot \Sigma_{\nu}.
\]
Conversely, if $\Sigma \in \text{Hom}(M, B)$, then given $\mu, \nu \in M^+$, we have for all $\mu' \in L(\mu)$ and $\nu' \in L(\nu)$,
\[
\int b_{xy, \mu * \nu} d\mu' \times \nu'(x, y) = \int b_{t, \mu * \nu} d\mu' * \nu'(t) = \Sigma_{\mu' * \nu'}
\]
\[
= \Sigma_{\mu'} \cdot \Sigma_{\nu'} = \int b_{x, \mu} d\mu'(x) \cdot \int b_{y, \nu} d\nu'(y)
\]
\[
= \int \int b_{x, \mu} \cdot b_{y, \nu} d\mu'(x) d\nu'(y)
\]
\[
= \int b_{x, \mu} \cdot b_{y, \nu} d\mu' \times \nu'(x, y).
\]
Since the span of $L(\mu) \times L(\nu)$ is dense in $L(\mu \times \nu)$, (3.2) follows. □

If multiplication in $B$ is jointly weak* continuous (for example, if $B_* \cap \Delta B$ separates points in $B$), then the unit ball in $\text{Hom}(M, B)$ is easily shown to be W* OT compact. An example where compactness fails is $\text{Hom}(M(\mathbb{R}), M(\mathbb{R}))$: define $T_n$ ($n \geq 1$) in the unit ball by
\[
\int_{\mathbb{R}} f(x) d(T_n)_\mu(x) = \int_{\mathbb{R}} f(nx) d\mu(x) \quad (f \in C_0(\mathbb{R}))
\]
and let $\Sigma: \mu \mapsto \mu(\{0\})\delta(0)$, where $\delta(0)$ is the Dirac measure at 0. Then $T_n \to \Sigma$ in W* OT, but
\[
\Sigma \in L(M(\mathbb{R}), M(\mathbb{R}))\setminus \text{Hom}(M(\mathbb{R}), M(\mathbb{R})).
\]

We define the following multiplication on $L(M, B)$: if $\Sigma \sim b_\cdot \cdot$, and $\Pi \sim b_\cdot \cdot$, then $\Sigma \cdot \Pi$ is defined by its Šreider representation $b_{x, \mu} \cdot b_{x, \mu}'$. When $B$ is commutative, $\text{Hom}(M, B)$ is closed under multiplication. It is easily verified that if multiplication in $B$ is separately weak* continuous, then multiplication in $L(M, B)$ is separately W* OT continuous.
Suppose that \( M = I \oplus J \), where \( I \) is a closed ideal and \( J \) is a closed subalgebra. If \( \Pi^1, \Pi^2 \in \text{Hom}(M, B) \) satisfy
\[
\forall \mu \in I \forall \nu \in J \Pi^1_{\mu \ast \nu} = \Pi^1_\mu \cdot \Pi^2_\nu \quad \& \quad \Pi^1_{\nu \ast \mu} = \Pi^2_\mu \cdot \Pi^1_\nu,
\]
then the “sum” \( \Sigma \) of \( \Pi^1 \) and \( \Pi^2 \) defined in (2.3) is a homomorphism.

4. Limit points of group homomorphisms. If \( H \) is a locally compact group, then convolution is separately weak* continuous in \( M(H) \). Indeed, if \( \mu, \nu \in M(H) \) with \( \mu \stackrel{w^*}{\longrightarrow} \mu \), then for \( f \in C_0(H) \), the map \( x \mapsto \int f(xy) d\nu(y) \) lies in \( C_0(H) \), whence
\[
\int f d\mu_\alpha \ast \nu = \int \int f(xy) d\nu(y) d\mu_\alpha(x) \rightarrow \int \int f(xy) d\nu(y) d\mu_\alpha(x) = \int f d\mu_\alpha \ast \nu,
\]
which is to say that \( \mu_\alpha \ast \nu \stackrel{w^*}{\longrightarrow} \mu \ast \nu \). A similar argument applies to \( \nu \ast \mu_\alpha \). Thus, if \( G \) is a locally compact semigroup with separately continuous multiplication and \( H \) is a locally compact metrizable group, then the preceding section applied to \( \text{Hom}(M, M(H)) \) for any \( L \)-subalgebra \( M \) of \( M(G) \). Every continuous homomorphism \( \phi: G \to H \) yields an element of \( \text{Hom}(M, M(H)) \), which we also denote by \( \phi \), defined by \( \langle f, \phi_\mu \rangle = \langle f \circ \phi, \mu \rangle \) for \( f \in C_0(H) \). The Šreider representation of such a \( \phi \) is particularly simple: \( \phi \sim \delta(\phi(x)) \) (independent of \( \mu \)), where \( \delta(t) \) denotes the Dirac measure at \( t \).

We identify \( \text{Hom}(G, H) \) with a subset of \( \text{Hom}(M(G), M(H)) \) in the above manner. Our aim is to study the set
\[ \Lambda = \overline{\text{Hom}(G, H)} \setminus \text{Hom}(G, H) \]
and its local structure
\[ \Lambda(\mu) = \{ \Sigma_\mu : \Sigma \in \Lambda \}, \quad \tilde{\Lambda}(\mu) = \{ \sigma_\mu : \sigma_\nu, \nu \in \Lambda \}, \]
where \( \tilde{\Lambda} \) consists of the Šreider representations of elements of \( \Lambda \). Since all elements of \( \text{Hom}(G, H) \) are positive and lie in the unit ball, the same holds for \( \Lambda \). (In fact, every positive homomorphism lies in the unit ball: if \( 0 \leq \Sigma \in \text{Hom}(M(G), M(H)) \), then for \( \mu \in M(G) \) and \( n \geq 1 \), we have
\[ ||\Sigma_\mu||^n \leq ||\Sigma|| ||\mu||^n = ||\Sigma|| ||\mu||^n \leq ||\Sigma|| \cdot ||\mu||^n, \]
whence \( ||\Sigma|| \leq 1 \).}

We are particularly interested in the case where \( G \) is a locally compact abelian group and \( H \) is a circle group, \( \mathbb{T} \). In this case,
STRUCTURE OF SOME MEASURE-ALGEBRA HOMOMORPHISMS

Hom\((G, T) = \hat{G},\) the dual of \(G,\) and the identification of \(\hat{G}\) as a subset of Hom\((M(G), M(T))\) preserves the usual topology of \(\hat{G}\) (of uniform convergence on compact subsets). Furthermore, as \(\hat{G}\) lies in the unit ball of Hom\((M(G), M(T))\), it follows that \(\hat{G} = \hat{G} \cup \Lambda\) is a compactification of \(\hat{G}.\)

Recall that a sequence \(\{x_k\}_{k=1}^{\infty} \subseteq G\) is said to have an asymptotic distribution \(\sigma,\) written \(\{x^k\} \sim \sigma,\) if

\[
\frac{1}{K} \sum_{k=1}^{K} \delta(x_k) \stackrel{w^*}{\rightarrow} \sigma \quad \text{as} \quad K \to \infty.
\]

For \(n \in \mathbb{Z}\) and \(\Sigma \in \text{Hom}(M(G), M(T)),\) define \(\hat{\Sigma}(n) \in \Delta M(G)\) by \(\langle \mu, \hat{\Sigma}(n) \rangle = \hat{\Sigma}_\mu(n).\) We write the Šreider representation of \(\chi \in \Delta M(G)\) as \(\chi_\mu(x).\) Thus, if \(\Sigma \sim \sigma, \) and \(\chi = \hat{\Sigma}(n)\), then

\[
\chi_\mu(x) = \hat{\sigma}_{x, \mu}(n).
\]

Note that for all \(n,\) the map \(\Sigma \mapsto \hat{\Sigma}(n)\) from \((\text{Hom}(M(G), M(T)), W^* OT)\) to \(\Delta M(G)\) (with its usual Gelfand topology) is continuous. We regard the Fourier transform as a restriction of the Gelfand transform; thus, in accordance with the Šreider representation, we have

\[
\hat{\mu}(\gamma) = \int \gamma d\mu \quad \text{for} \quad \gamma \in \hat{G}.
\]

**Proposition 4.1.** Let \(G\) be a locally compact abelian group and \(\Lambda = \hat{G} \setminus \hat{G}\) in \(\text{Hom}(M(G), M(T)).\) Then

(i) \(\Lambda\) is closed topologically and under multiplication by elements of \(\hat{G};\)

(ii) if \(\sigma_x, \tau_x \in \Lambda(\mu),\) then \(\sigma_x \ast \tau_x \in \Lambda(\mu);\)

(iii) \(\Lambda(\mu) = \{\nu \in M(T): \exists \text{ net } \{\gamma_n\} \subseteq \hat{G} \quad (\gamma_n \to \infty \& \forall n \in \mathbb{Z} \quad \hat{\mu}(\gamma^n_n) \to \hat{\nu}(n)\};\)

(iv) \(\Lambda(\mu) = \{\sigma. \in \mathcal{B}(G, M(T))_\mu: \exists \text{ net } \{\gamma_n\} \subseteq \hat{G} \quad (\gamma_n \to \infty \& \forall n \in \mathbb{Z} \quad \gamma^n_n \to \hat{\sigma}(n) \text{ weak* in } L^\infty(\mu)\};\)

(v) if \(G\) is metrizable, then the nets in (iii) and (iv) can be replaced by sequences and \(\Lambda(\mu) = \{\sigma. \in \mathcal{B}(G, M(T))_\mu: \exists \gamma_j \in \hat{G} \quad (\gamma_j \to \infty \& \text{ for every subsequence } \gamma_{j_k}, \forall x[\mu] \quad \{\gamma_{j_k}(x)\}_{k=1}^{\infty} \sim \sigma_x\} \}

**Proof.** Suppose that \(\Sigma \in \Lambda\) is the limit of a net \(\{\gamma_n\} \subseteq \hat{G}.\) Then \(\hat{\Sigma}(n) = \lim \gamma^n_n\) in \(\Delta M(G)\) for all \(n \in \mathbb{Z}.\) Now if \(\gamma_n \to \gamma \in \hat{G},\) then \(\gamma^n_n \to \gamma^n,\) whence \(\Sigma = \gamma.\) But since \(\Lambda \cap \hat{G} = \emptyset,\) this is impossible, and so \(\gamma_n \to \infty\) in \(\hat{G}.\) In particular, \(\hat{\Sigma}(1) = 0\) on \(L^1(G)\) [HMP,
p. 136, Proposition 4] and consequently \( \Lambda \) is closed. It is clear that 
\( \Lambda \cdot \hat{G} \subseteq \Lambda \), from which (i) now follows. Statement (ii) ensues as well.
Now if \( \nu \in \Lambda(\mu) \), then let \( \hat{G} \ni \gamma_\alpha \to \Sigma \in \Lambda \) be such that \( \nu = \Sigma_\mu \).
Then \( \gamma_\alpha \to \infty \) and \( (\gamma_\alpha)_\mu \xrightarrow{w^*} \Sigma_\mu = \nu \), which gives the inclusion \( \subseteq \) of (iii). On the other hand, if \( \gamma_\alpha \to \infty \) and \( \forall n \quad \hat{\nu}(\gamma_\alpha^n) \to \hat{\nu}(n) \), then by compactness of \( \hat{G} \), we can choose a subnet \( \{\gamma_\beta\} \) of \( \{\gamma_\alpha\} \) converging to some \( \Sigma \). Since \( \gamma_\beta \to \infty \), it follows that \( \Sigma \in \Lambda \) and \( \nu = \Sigma_\mu \in \Lambda(\mu) \). This completes the proof of (iii). The proof of (iv) is analogous.
Finally, if \( G \) is metrizable, then \( L^1(\mu) \) is separable for \( \mu \in \mathcal{M}(G) \) and so \( L(M(G), M(T))_\mu \) is metrizable. Thus, if \( \mu \in \mathcal{M}(G) \) and \( \gamma_\alpha \to \Sigma \sim \sigma., \) pick any non-zero \( \rho \in L^1(G) \) and a subsequence \( \{\delta(\gamma_\alpha(\cdot))\} \) converging to \( \sigma.,|\mu|+|\rho| \) in \( L(M(G), M(T))_{|\mu|+|\rho|} \). Then
\[
\gamma_\alpha = \delta(\gamma_\alpha(\cdot))(1) \xrightarrow{w^*} (\hat{\Sigma}(1))_\mu = 0 \text{ in } L^\infty(\rho), \text{ whence } \gamma_\alpha \to \infty
\]
in \( \hat{G} \), and \( \gamma_\alpha^n \xrightarrow{w^*} (\hat{\Sigma}(n))_\mu = \hat{\sigma}_\mu(n) \) in \( L^\infty(\mu) \). This shows the sufficiency of sequences for (iii) and (iv). Furthermore, if \( \forall n \quad \gamma_\alpha^n \to \hat{\sigma}_\mu(n) \) weak* in \( L^\infty(\mu) \), then by [L2, Lemma 5], there is a subsequence \( \{\gamma_j^n\} \) of \( \{\gamma_j\} \) such that every further subsequence \( \{\gamma_j^n\} \) satisfies
\[
(4.1) \quad \forall x \in [\mu] \quad \{\gamma_j^n(x)\}_k^\infty \sim \sigma_x.
\]
Conversely, if \( \{\gamma_j\} \) is a sequence, every subsequence of which satisfies (4.1), then we claim \( \gamma_j^n \to \hat{\sigma}_\mu(n) \) weak* for every \( n \). If not, then for some \( n \) there would be a subsequence \( \{\gamma_j^n\} \) converging to a different limit \( \chi \). Then also
\[
\frac{1}{K} \sum_{k=1}^K \gamma_j^n \xrightarrow{w^*} \chi
\]
and by (4.1),
\[
\frac{1}{K} \sum_{k=1}^K \gamma_j^n \xrightarrow{w^*} \hat{\sigma}_\mu(n).
\]
Therefore \( \chi = \hat{\sigma}_\mu(n) \), a contradiction. Thus (v) follows from (iv).

When \( \hat{G} \) is regarded as a subset of \( \Delta \mathcal{M}(G) \), we shall use the notation \( \Gamma \) rather than \( \hat{G} \) to avoid confusion. Let \( T_n \in \text{Hom}(G, G) \) denote the map \( x \mapsto x^n \ (n \in \mathbb{Z}) \), as well as the corresponding map induced in \( \text{Hom}(M(G), M(G)) \). Thus, for \( \Sigma \in \text{Hom}(M(G), M(T)) \), we obtain \( \Sigma \circ T_n \in \text{Hom}(M(G), M(T)) \); note that if \( \Sigma = \gamma \in \hat{G} \), then \( \gamma \circ T_n = \gamma^n \).
Proposition 4.2. Let $G$ be a LCA group and

$$\Sigma \in \text{Hom}(M(G), M(T)).$$

Then $\Sigma \in G$ iff $\hat{\Sigma}(1) \in \Gamma$ and $\forall n \in \mathbb{Z}$, $\hat{\Sigma}(n) = \hat{\Sigma}(1) \circ T_n$. The map $\Sigma \mapsto \hat{\Sigma}(1)$ is an isomorphism from $G$ onto $\Gamma$ sending $G$ to $\Gamma$.

Proof. If $\Sigma \in G$, let $\hat{G} \ni \gamma_\alpha \xrightarrow{W^* \circ T} \Sigma$. Since $\hat{\gamma}_\alpha(n) = \gamma_\alpha^n$, we have $\gamma_\alpha^n \to \hat{\Sigma}(n)$ for all $n$. In particular, $\hat{\Sigma}(1) \in \Gamma$. Also, $\hat{\Sigma}(n) = \lim_{n \to \infty} \gamma_\alpha^n = \lim_{n \to \infty} (\gamma_\alpha) \circ T_n = \hat{\Sigma}(1) \circ T_n$. Conversely, if $\hat{\Sigma}(1) \in \Gamma$ and $\forall n \in \mathbb{Z}$, $\hat{\Sigma}(n) = \hat{\Sigma}(1) \circ T_n$, then let $\gamma_\alpha \to \hat{\Sigma}(1)$. Choose a convergent subnet $\gamma'_\beta \to \Pi$ in $\text{Hom}(M(G), M(T))$. Then from the above, $\hat{\Pi}(n) = \hat{\Pi}(1) \circ T_n = \hat{\Sigma}(1) \circ T_n = \hat{\Sigma}(n)$ for all $n$, whence $\Sigma = \Pi \in G$.

It follows from this that the map $\Sigma \mapsto \hat{\Sigma}(1)$ is injective. Surjectivity onto $\Gamma$ is proved by a compactness argument similar to the above. □

We write $M(G) = M_c(G) \oplus M_d(G)$ for the decomposition of a measure into its continuous and discrete parts. Then $h_d: \mu \mapsto \int_G d\mu_d = \hat{\mu}_d(0)$ is in $\Gamma \backslash \Gamma$ [HMP, pp. 136-7, (4.1.4)]. We denote the element of $\Lambda$ corresponding to $h_d$ by $\Pi^d$. If $G$ has at most countably many torsion elements, then we claim that

$$\hat{\Pi}^d(n) = \begin{cases} 0 & \text{if } n = 0, \\ h_d & \text{if } n \neq 0, \end{cases}$$

whence

$$\Pi^d_\mu = \hat{\mu}_c(0)\lambda + \hat{\mu}_d(0)\delta(0),$$

where $\lambda$ is Lebesgue measure on $T$. To see this, note first that

$$\hat{\Pi}^d(0): \mu \mapsto (\mu \circ T_0^{-1})(0) = \hat{\mu}(0).$$

Second, if $n \neq 0$, then for all $g \in G$, there are, by the supposition, denumerably many $x \in G$ such that $x^n = g$. Therefore

$$(\mu \circ T_n^{-1})(\{g\}) = \sum_{x^n = g} \mu(\{x\}),$$

whence

$$\hat{\Pi}^d(n): \mu \mapsto \sum_{g \in G} (\mu \circ T_n^{-1})(\{g\})$$

$$= \sum_{g \in G} \sum_{x^n = g} \mu(\{x\}) = \sum_{x \in G} \mu(\{x\}) = \hat{\mu}_d(0).$$

This proves the claim.
Related elements of $\Lambda$ are $\Sigma \cdot \Pi^d$ for $\Sigma \in \overline{G}$; if, as above, the torsion subgroup of $G$ is denumerable, then

$$(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0) \lambda + \Sigma_{\mu_\lambda}.$$ 

Thus, if we set $\Pi: \mu \mapsto \hat{\mu}(0) \lambda$, then $\Sigma \cdot \Pi^d$ is the sum of $\Pi$ and $\Sigma$ defined by (2.3) and (3.3) from the decomposition $M = M_c \oplus M_d$. An interesting example is $G = \mathbb{T}$ and $\Sigma: \mu \mapsto \mu$; in this case, $(\Sigma \cdot \Pi^d)_\mu = \hat{\mu}_c(0) \lambda + \mu_d$.

Provided still that $G$ has a denumerable torsion subgroup, the Šreider representation $\pi^d_\lambda$ of $\Pi^d$ is given by

$$(4.2) \quad \pi^d_\lambda, \mu = \{ \begin{array}{ll} \lambda & \text{if } \mu(\{x\}) = 0, \\ \delta(0) & \text{if } \mu(\{x\}) \neq 0. \end{array}$$

Let $\lambda \in \mathcal{B}(G, M(\mathbb{T}))_\mu$ be defined by $\lambda(x) \equiv \lambda$. Then from [HMP, p. 70, Corollaire 2] and Proposition 4.2 (or from (4.2) and the following proposition),

$$(4.3) \quad \mu \in M_c(G) \Leftrightarrow \lambda \in \overline{\Lambda}(\mu).$$

This yields other characterizations of $M_c(G)$ when combined with Proposition 4.1 (iv), (v). For example,

$$\mu \in M_c(G) \Leftrightarrow \exists \gamma_\alpha \to \infty \forall \nu \in L(\mu) \forall n \neq 0 \hat{\nu}(\gamma^n_\alpha) \to 0$$

$$\Leftrightarrow \exists \gamma_\alpha \to \infty \forall \gamma \in \hat{G} \forall n \neq 0 \hat{\mu}(\gamma \gamma^n_\alpha) \to 0.$$ 

Our next proposition describes $\overline{\Lambda}(\mu)$ completely when $\mu$ is discrete (cf. [HMP, pp. 67–68]).

**Proposition 4.3.** Let $G$ be a LCA group. Let $\overline{G}$ denote the Šreider representations of $\overline{G} \subseteq \text{Hom}(M(G), M(\mathbb{T}))$ and, for $\mu \in M(G)$, $\overline{\Gamma}(\mu) = \{ \sigma_{..\mu} : \sigma_{..} \in \overline{G} \}$. Let $G_d$ denote $G$ with the discrete topology and, for $\mu \in M_d(G)$, let $G^\mu_d$ denote the discrete subgroup generated by the mass-points of $\mu$.

(i) $\forall \Sigma \in \overline{G} \exists \varphi \in \overline{G}_d \forall \mu \in M_d(G) \Sigma_\mu = \sum_{x \in \overline{G}} \mu(\{x\}) \delta(\varphi(x))$ and $\sigma_{x, \mu} = \delta(\varphi(x))$, where $\Sigma \sim \sigma_{..}$.

(ii) $\forall \mu \in M_d(G) \overline{\Gamma}(\mu) \simeq \overline{G}^\mu_d$.

(iii) $\mu \in M_d(G) \Leftrightarrow \overline{\Gamma}(\mu)$ is a group (under the multiplication in $L(M(G), M(\mathbb{T}))$).
Proof. (i) Let \( \hat{G} \ni \gamma \overset{w^*}{\rightarrow} \Sigma \). Then for \( x \in G \),
\[
\delta(\gamma(x)) \rightarrow \sigma_x, \delta(x) \in L(M(G), M(T)) \delta(x),
\]
i.e., \( \delta(\gamma(x)) = \sigma_x, \delta(x) \) eventually. Thus, \( \gamma(x) \) stabilizes at some point \( \phi(x) \) and \( \sigma_x, \delta(x) = \delta(\phi(x)) \). The assertions now follow from linearity and properties of the Šreider representation.

(ii) The fact that \( \sim \hat{G}(\mu) \) can be identified as a compact subgroup of \( \hat{G}_d^\mu \) follows from (i). If it were not the whole group, then there would be a nonzero \( x \in G_d^\mu \) such that \( \phi(x) = 1 \) for all \( \phi \in \sim \hat{G}(\mu) \). In particular, \( \gamma(x) = 1 \) for all \( \gamma \in \hat{G} \), whence \( x = 0 \), a contradiction.

(iii) This follows from [HMP, p. 68, Proposition 10] and (ii).

We now arrive at the characterization of positive continuous measures mentioned in the introduction.

**Theorem 4.4.** Let \( G \) be a LCA group whose torsion subgroup is denumerable and let \( \mu \in M^+(G) \) be positive. Then \( \mu \in M_c^+(G) \) iff there is a net \( \hat{G} \ni \gamma \rightarrow \infty \) such that for all \( n \neq 0 \), \( \hat{\mu}(\gamma^n) \rightarrow 0 \).

Proof. By Proposition 4.1 (iii), this is equivalent to \( \mu \in M_c^+(G) \Leftrightarrow \hat{\mu}(0) \lambda \in \Lambda(\mu) \). For \( \mu \in M_c^+(G) \), this follows from \( \lambda \in \Lambda(\mu) \) (see (4.3)). If \( \mu \notin M_c^+(G) \) and \( \Sigma \in \Lambda \), then \( \Sigma = \Sigma + \Sigma_d \geq \Sigma_d \) since \( \mu_c \geq 0 \) and \( \Sigma \geq 0 \). However, by Proposition 4.3(i), \( \Sigma_d \) is nonzero and discrete; hence \( \Sigma \) cannot equal \( \hat{\mu}(0) \lambda \).

Because of the interest this theorem may present, we provide the following "elementary" proof and strengthening for the case \( G = T \). If \( \mu \in M_c(T) \), then by Wiener's theorem [K, p. 42], there is a sequence \( \{m_k^{(1)}\} \) of density one in \( N \) such that \( \hat{\mu}(m_k^{(1)}) \rightarrow 0 \). Likewise, there is a sequence \( \{m_k^{(n)}\} \) of density one such that \( \hat{\mu}(nm_k^{(n)}) = (T_n)\mu(m_k^{(n)}) \rightarrow 0 \) since \( (T_n)\mu \in M_c \), for \( n \neq 0 \). By an elementary intersection argument, we obtain a sequence \( \{m_k\} \), still of density one, such that for all \( n \neq 0 \), \( \hat{\mu}(nm_k) \rightarrow 0 \). (A similar argument produces a sequence \( \{m_k\} \) of density one such that for \( n \neq 0 \) and all \( r \), \( \hat{\mu}(r + nm_k) \rightarrow 0 \), i.e., \( \delta(m_k x) \rightarrow \lambda \) in \( L(M(T), M(T))_\mu \), thereby strengthening (4.3).) For the converse, we use the following proof due to Jean-François Méla. Let \( K_l(x) \) be the Fejér kernel of order \( l \). Then if \( \mu \geq 0 \) and if for
all \( n \neq 0 \), \( \hat{\mu}(nm_k) \to 0 \), then

\[
\mu(\{0\}) \leq \int_T \frac{1}{2l + 1} K_l(m_kx) \, d\mu(x) \to \frac{1}{2l + 1} \hat{\mu}(0) \quad \text{as} \quad k \to \infty
\]

by hypothesis. Since this is true for all \( l \), it follows that \( \mu(\{0\}) = 0 \).

Now apply this result to \( \mu \ast \hat{\mu} \), where \( \hat{\mu}(E) = \mu(-E) \).

The local structure of \( \Lambda \) can be used to characterize other classes of measures besides \( M_c \) and \( M_d \). If \( \mathcal{E} \) is a class of subsets of \( G \), let

\[
\mathcal{E}^\perp = \{ \mu \in M(G) : \forall E \in \mathcal{E} \ |\mu|(E) = 0 \}.
\]

Thus, if \( \mathcal{D} \) is the class of singletons, \( \mathcal{D}^\perp = M_c(G) \).

**Definition.** A set \( E \subseteq G \) is called an \( H \)-set if there is a sequence \( \hat{\gamma}_k \to \infty \) such that \( \{ \gamma_k(x) : k \geq 1, x \in E \} \) is not dense in \( T \). A set \( E \subseteq G \) is called a Dirichlet set if there is a sequence \( \hat{\gamma}_k \to \infty \) such that \( \lim_{k \to \infty} \sup_{x \in E} |\gamma_k(x) - 1| = 0 \). A measure \( \mu \in M(G) \) is called a Dirichlet measure if \( \lim_{y \to \infty} \|\mu(y)\| = \|\mu\| \).

For background on \( H \)-sets, see [Z, Chapters IX, XII]; on Dirichlet sets and measures, see [HMP, pp. 34–35, 240–247]. The following proposition is used in [KL].

**Proposition 4.5.** Let \( G \) be a LCA group.

(i) If \( G \) is metrizable, then

\[
H^\perp = \{ \mu : \forall \sigma \in \hat{\Lambda}(\mu) \ \forall x[\mu] \ \text{supp} \sigma_x = T \} = \{ \mu : \forall \Sigma \in \Lambda \ \forall \nu \in L(\mu) \ \text{supp} \Sigma_\nu = T \}.
\]

(ii) \( \mu \) is a Dirichlet measure iff the constant function \( \delta(0) \in \Lambda(\mu) \).

(iii) \( D^\perp = \{ \mu : \forall \sigma \in \hat{\Lambda}(\mu) \ \forall x[\mu] \ \sigma_x \neq \delta(0) \} \)

**Proof.** Part (i) follows from Proposition 4.1(v) and a straightforward generalization of [L4, Theorem 13]. Part (ii) follows from Proposition 4.2 and the fact that \( \mu \) is a Dirichlet measure iff the constant function \( 1 \in (\hat{T} \setminus \Gamma)(\mu) \) [HMP, p. 34, Lemma 6]. Part (iii) follows from part (ii) and the fact that \( D^\perp \) consists of the measures orthogonal to the Dirichlet measures [HMP, p. 243, Proposition 9].

Our final remarks concern the circle group.

**Definition.** A positive measure \( \mu \in M^+(T) \) is called \( C \)-quasisymmetric if for all pairs of adjacent arcs, \( I \) and \( J \), on \( T \) of equal
length, \( \mu I \leq C \cdot \mu J \). We denote the class of \( C \)-quasisymmetric measures by \( QS(C) \).

Note that quasisymmetric measures are continuous.

**Proposition 4.6.** The class \( QS(C) \) is weak* closed. If \( \mu \in QS(C) \), then \( \Lambda(\mu) \subseteq QS(C) \), \( \Lambda(\mu) \subseteq QS(C) \) in the sense that if \( \sigma \in \Lambda(\mu) \), then \( \forall x[\mu] \sigma_x \in QS(C) \), and \( \Lambda(\nu) \subseteq QS(C) \) for all \( 0 \leq \nu \in L(\mu) \).

**Proof.** Let \( QS(C) \ni \mu_\alpha \xrightarrow{w^*} \nu \). Given adjacent arcs \( I, J \) of equal length and \( \varepsilon > 0 \), pick \( f, g \in C(T) \) such that \( f \leq 1_I, 1_J \leq g, \int (1_I - f) d\nu \leq \varepsilon \), and \( \int (g - 1_J) d\nu \leq \varepsilon \). We have

\[
\nu I \leq \int f d\nu + \varepsilon = \lim \int f d\mu_\alpha + \varepsilon \leq \limsup \mu_\alpha I + \varepsilon
\]

\[
\leq C \cdot \limsup \mu_\alpha J + \varepsilon \leq C \cdot \lim \int g d\mu_\alpha + \varepsilon = C \int g d\nu + \varepsilon
\]

\[
\leq C \cdot \nu J + (C + 1)\varepsilon.
\]

Since \( \varepsilon \) was arbitrary, we see that \( \nu I \leq C \cdot \nu J \), whence \( \nu \in QS(C) \).

Choose \( \mu \in QS(C) \). Then \( \gamma_\mu \in QS(C) \) for any \( \gamma \in \hat{T} \). Since \( \Lambda(\mu) \) is contained in the weak* closure of \( \{\gamma_\mu\}_{\gamma \in \hat{T}} \), it follows that \( \Lambda(\mu) \subseteq QS(C) \). Suppose that \( E \subseteq T \) and \( \mu E > 0 \). If \( I \) and \( J \) are adjacent arcs of equal length and \( \varepsilon > 0 \), then choose \( U \), a finite union of arcs, such that \( \mu(U \Delta E) \leq \varepsilon \). By continuity of \( \mu \), we have for all large \( \gamma \),

\[
\mu(E \cap \gamma^{-1}[I]) \leq \mu(U \cap \gamma^{-1}[I]) + \varepsilon \leq C \cdot \mu(U \cap \gamma^{-1}[J]) + 2\varepsilon
\]

\[
\leq C \cdot \mu(E \cap \gamma^{-1}[J]) + (C + 2)\varepsilon.
\]

Since \( \varepsilon \) was arbitrary, it follows that \( \Lambda(\mu|_E) \subseteq QS(C) \). As \( QS(C) \) is a positive cone, we deduce that \( \Lambda(\nu) \subseteq QS(C) \) for \( 0 \leq \nu \in L(\mu) \).

Finally, let \( \sigma \in \Lambda(\mu) \). Let \( P \) be the essential range of \( \sigma \), i.e., the smallest weak* closed set \( P \) such that \( \sigma_x \in P \mu\text{-a.e.} \). Then \( P \) is contained in the weak* closure of \( \{ \int \sigma_x d\nu(x) : 0 \leq \nu \in L(\mu), \|\nu\| = 1 \} = \bigcup \{ \Lambda(\nu) : 0 \leq \nu \in L(\mu), \|\nu\| = 1 \} \), which, by the above, is contained in \( QS(C) \).

As an example of the pathology possible for \( \Lambda(\mu) \), we present the following observation.

**Proposition 4.7.** There is a measure \( \mu \in M(T) \) such that for any probability measure \( \nu \in M(T) \), there exists \( \sigma \in \Lambda(\mu) \) such that \( \sigma_x = \nu \mu\text{-a.e.} \).
Proof. Let \( \{P_k\}_{k \geq 1} \) be a set of trigonometric polynomials such that \( \{P_k \cdot \lambda\} \) is weak* dense in the set of probability measures. Let \( \{n_k\} \subseteq \mathbb{N} \) satisfy \( n_{k+1} \geq 3n_k \cdot \deg P_k \). Form the generalized Riesz product \( [\text{HMP, Chapitre 5}] \mu = \prod_{k \geq 1} P_k(n_k x) \). Then given a probability \( \nu \), let \( P_k \lambda \xrightarrow{w^*} \nu \). For any \( r, m \in \mathbb{Z} \), it is easy to see that \( \hat{\mu}(r+mn_k) \to \hat{\mu}(r)\hat{\nu}(m) \), i.e., \( \delta(n_k x) \to \nu \) in \( L(M(T), M(T))_\mu \). \( \square \)

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