Determinantal probability: surprising relations

Russell Lyons

Indiana University, Bloomington

ICM, Seoul, Korea, 2014

Dirichlet Sample Gates

Uniform Sample Gates

Sample Gates, Indiana University
credit: Joey Lax-Salinas


SAMPLE GATES and PLAZA
PRESENTED TO
INDIANA UNIVERSITY
IN HONOR OF
LOUISE WAITE SAMPLE
and
KIMSEY OWNBEY SAMPLE, SR.
by
THEIR SON
EDSON WAITE SAMPLE
1987

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- Clearly, $\{e_n\,;\ n \in 3\mathbb{Z}\}$ is complete in $L^2[0,1/3]$.
- Koosis (CRASP, 1960): There exists $S \subseteq \mathbb{Z}^+$ of density 0 such that $\{e_n\,;\ n \in S\}$ is complete in $L^2[0,a]$ for all $a \in (0,1)$.

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For discrete $S \subseteq \mathbb{R}$, the Beurling-Malliavin density $\mathrm{BM}(S)$ of S is the supremum of those $D \geq 0$ for which there exist disjoint nonempty intervals $I_n \subset \mathbb{R} \setminus \{0\}$ with $|S \cap I_n| \geq D|I_n|$ for all n and

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Theorem of Beurling and Malliavin (*Acta Math.*, 1967)

Let $A \subset \mathbb{R}$ be an *interval*. If $\{e_n\,;\ n \in S\}$ is complete for $L^2(A)$, then $\mathrm{BM}(S) \geq |A|$. If $\mathrm{BM}(S) > |A|$, then $\{e_n\,;\ n \in S\}$ is complete for $L^2(A)$.

(This holds for S that are not necessarily sets of integers.)

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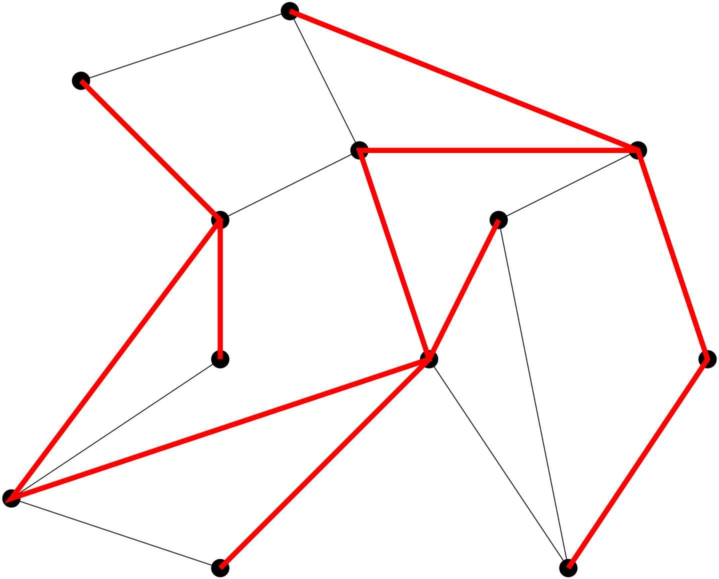
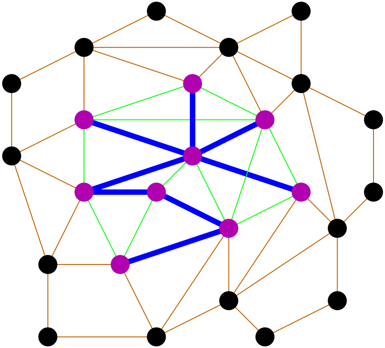
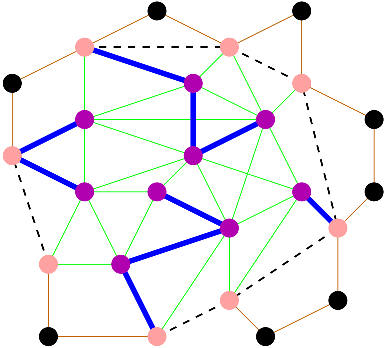
Corollary 2 (trees)





FSF



WSF

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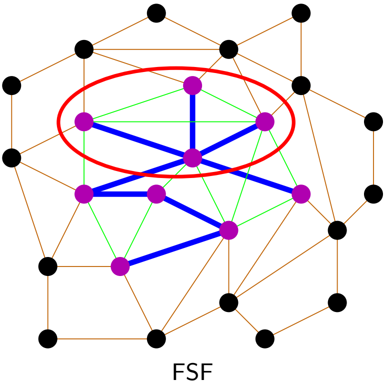

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This probability measure $\mathbf{P}^A$ is a determinantal probability measure, as is WSF_G and the law of the zero set, Z.

Let E be a denumerable set and $H \leq \ell^2(E)$ be a closed subspace. Write $P_H \colon \ell^2(E) \to \ell^2(E)$ for the orthogonal projection onto H. Write $[P_H]$ for its matrix in the standard orthonormal basis $\{\mathbf{1}_e\,;\ e \in E\}$.

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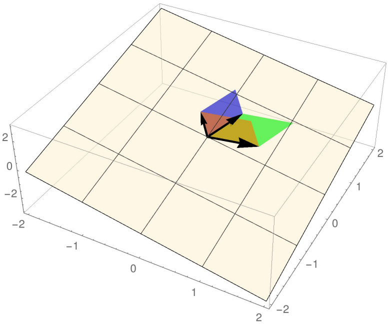
Clearly

$$H = \ell^2(E) \Longrightarrow \mathbf{P}^H[\mathfrak{B} = E] = 1\,.$$



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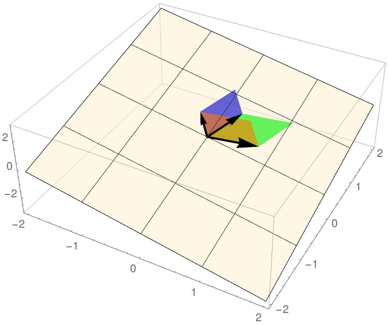

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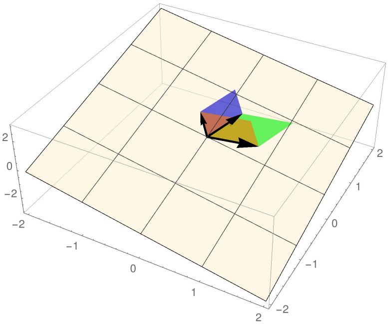

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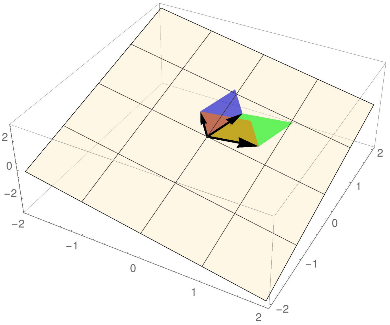

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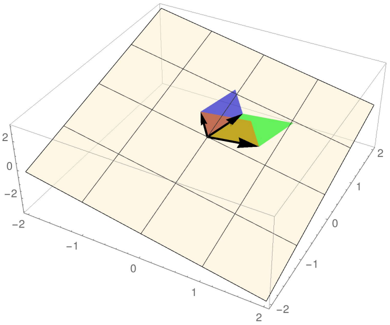

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We use $E := \mathbb{Z}$ and $H := \widehat{L^2(A)}$. Here, we think of $L^2(A)$ as the functions in $L^2[0,1]$ that vanish outside A. Let $\mathfrak{B} \sim \mathbf{P}^H$. We will show that $S := \mathfrak{B}$ works a.s. Let $f \in L^2(A)$ with $f \perp e_n$ for $n \in S$. Then $H \ni \widehat{f} \perp \mathbf{1}_n$ for $n \in S$, so by the Theorem, $\widehat{f} = \mathbf{0}$, i.e., $f = \mathbf{0}$.

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The subspace $\{P_{\ell^2(B)} h\,;\ h \in H\}$ is dense in $\ell^2(B)$ for $\mathbf{P}^H$-a.e. B, i.e., if $v \in \ell^2(B)$ satisfies $v \perp h$ for all $h \in H$, then $v = \mathbf{0}$.

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Recall: $H = \ell^2(E) \Longrightarrow \mathbf{P}^H[\mathfrak{B} = E] = 1$.

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Proof.

Let $G = (\mathsf{V}, \mathsf{E})$. Use $E := \mathsf{E}$, regarding a spanning forest as a set of edges.

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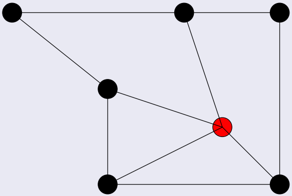

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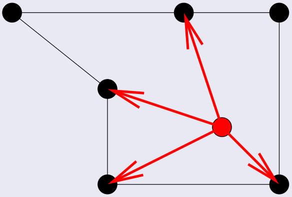

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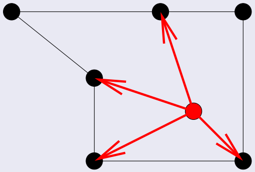

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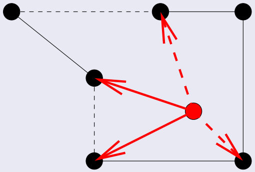

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Discrete vs. continuous. Duality.

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The point process is determinantal if $\exists K\colon E^2 \to \mathbb{C}$ such that

$$\forall k \geq 1 \quad \rho_k(x_1, \ldots, x_k) = \det \left[K(x_i, x_j)\right]_{1 \leq i,j \leq k}$$

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We call ρ_k the k-point intensity function if

$$\text{for all Borel } A \subseteq E^k \setminus \Delta_k(E) \qquad \mathbf{E}\left[\mathfrak{X}^k(A)\right] = \int_A \rho_k \, d\mu^k \,,$$

where $\Delta_k(E) := \left\{(x_1, \ldots, x_k) \in E^k \,;\; i \neq j \implies x_i \neq x_j\right\}$.

For us, $K(\cdot,\cdot)$ will be locally square integrable, Hermitian, and define an orthogonal projection onto some closed subspace $H \leq L^2(E,\mu)$ via

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$$K(x,z) = \int_E K(x,y)K(y,z)\,d\mu(y) \quad \mu^2\text{-a.e.},$$

so we can redefine K so that it holds everywhere. Writing

$$K_z(x) := K(x,z),$$

we obtain $K_z \in H$. When $h \in H$, we have

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For us, $K(\cdot,\cdot)$ will be locally square integrable, Hermitian, and define an orthogonal projection onto some closed subspace $H \le L^2(E,\mu)$ via

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i.e., K is a reproducing kernel for H.

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Recall the Bergman space $A^2(\mathbb{D})$ = the analytic functions in $L^2(\mathbb{D})$.

Conjectured Corollary 3

Let Z be the set of zeroes of a random complex-Gaussian power series in the unit disk. Then a.s., the only function in the Bergman space $A^2(\mathbb{D})$ that vanishes on Z is the zero function.

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Let μ be Lebesgue measure on $\mathbb{R}$ and $K(x,y) := \sin \pi(x-y)/\big(\pi(x-y)\big)$, which projects onto a Paley-Wiener space and defines the sine-kernel process. Then a.s., for $\mathbf{P}^K$-a.e. $\mathfrak{X}$, the only $h \in L^2[0,1]$ such that $\widehat{h}{\restriction}\mathfrak{X} = 0$ is $h = \mathbf{0}$.

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Let μ be standard Gaussian measure on $\mathbb{C}$ and $K(z, w) := e^{z\bar{w}}$, which projects onto the Bargmann-Fock space $B^2(\mathbb{C})$ consisting of the entire functions that lie in $L^2(\mathbb{C}, \mu)$, and defines the Ginibre process. Then a.s., for $\mathbf{P}^K$-a.e. $\mathfrak{X}$, the only $h \in B^2(\mathbb{C})$ such that $h{\restriction}\mathfrak{X} = 0$ is $h = \mathbf{0}$.

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The two instances above that have been proved by Ghosh follow from his more general result that our conjecture holds whenever $\mathbf{P}^K$ is rigid, which means that $\mathfrak{X}(B)$ is measurable with respect to the $\mathbf{P}^K$-completion of $\mathscr{F}(E \setminus B)$ for every ball $B \subset E$.

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