EXTRA CREDIT HOMEWORK
M564 Prof. Lyons Spring 2024

This gives a completely "soft" alternative development of the uniqueness and continuity theorems, that is, it avoids all calculations. It is motivated by functional analysis. Write λ for Lebesgue measure on $\mathbb{R}$, $L^1(\mathbb{R}) := L^1(\mathbb{R}, \lambda)$, and $L^\infty(\mathbb{R}) := L^\infty(\mathbb{R}, \lambda)$. These functions are complex valued. Denote by $M(\mathbb{R})$ the set of (finite) complex Borel measures on $\mathbb{R}$, the dual of the complex-valued $C_0(\mathbb{R})$. We define their Fourier transforms just as for probability measures: $\widehat{\mu}(t) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\mu(x)$. Again, for $f \in L^1(\mathbb{R})$, we write $\widehat{f} := \widehat{f\lambda}$.

Prove each of the following statements in turn.

1. The set $\mathcal{A} := \{\widehat{f};\ f \in L^1(\mathbb{R})\}$ is dense in $C_0(\mathbb{R})$. *Hint: Use the Stone–Weierstrass theorem. You may use without proof the facts that $L^1(\mathbb{R})$ is closed under convolution and $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.*

2. For all $\mu, \nu \in M(\mathbb{R})$, we have $\int \widehat{\nu}\,\mathrm{d}\mu = \int \widehat{\mu}\,\mathrm{d}\nu$.

3. Apply the preceding result to ν of the form $f\lambda$ for $f \in L^1(\mathbb{R})$ to deduce uniqueness, i.e., that the map $\mu \mapsto \widehat{\mu}$ is injective.

4. Let μ_n and μ be in the unit ball of $M(\mathbb{R})$. Then $\mu_n \stackrel{w^*}{\to} \mu$ iff $\widehat{\mu}_n \stackrel{w^*}{\to} \widehat{\mu}$ in $L^\infty(\mathbb{R})$ (regarded as the dual of $L^1(\mathbb{R})$).

5. Let μ_n be in the unit ball of $M(\mathbb{R})$. If $\widehat{\mu}_n \to g$ λ-a.e., then $\exists \mu \in M(\mathbb{R})$ such that $g = \widehat{\mu}$ λ-a.e. and $\mu_n \stackrel{w^*}{\to} \mu$.

6. If $g \in L^\infty(\mathbb{R})$, $h \in C(\mathbb{R})$, $g = h$ λ-a.e., and g is continuous at 0, then $g(0) = h(0)$.

7. Let μ_n be probability measures on $\mathbb{R}$ with $\widehat{\mu}_n(t) \to g(t)$ for every t. If g is continuous at 0, then there is a probability measure μ such that $\mu_n \Rightarrow \mu$. (You may use the fact from homework that for a probability measure, μ, we have $\mu_n \Rightarrow \mu$ iff $\mu_n \stackrel{w^*}{\to} \mu$.)

This completes the derivations. In case you wonder what the pointwise limit of characteristic functions can be when it exists but is not a characteristic function itself, you can also do the following.

8. Give an example of probability measures μ_n on $\mathbb{R}$ such that $\lim_{n\to\infty} \widehat{\mu}_n$ exists everywhere yet is nowhere continuous. *Hint: Start with a discrete version of Example 26.2 of the book.*