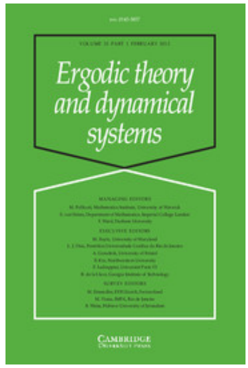

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Fixed price of groups and percolation

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Abstract. We prove that for every finitely generated group Γ, at least one of the following holds: (1) Γ has fixed price; (2) each of its Cayley graphs G has infinitely many infinite clusters for some Bernoulli percolation on G.

1. *Introduction*

Let Γ be an infinite finitely generated group. Consider an essentially free measure-preserving action of Γ on a standard probability space (X, μ). A *graphing* of this action is a graph (X, E), where $E \subseteq X \times X$ is measurable and symmetric and such that for every $x \in X$, the vertices of the connected component of x are the same as the orbit Γx. The *cost* of such a graphing equals $(1/2) \int_X |\{\gamma \in \Gamma;\ (x, \gamma x) \in E\}|\, d\mu(x)$. The *cost* of the action is the infimum of the costs of its graphings. See Gaboriau [**7**] for background information on cost. Following Gaboriau [**6**], we say that Γ has *fixed price* if all its (essentially free measure-preserving) actions have the same cost. One of the major open questions of the theory of cost is whether every group has fixed price. It is known that all amenable groups have fixed price with cost 1. Some other restricted classes and examples of groups of fixed price are also known; see Gaboriau [**6**, §VII].

Now let G be a Cayley graph of Γ. Bernoulli(p) site *percolation* on G is the subgraph of G induced by a random subset of its vertices, where each vertex is in the subset with probability p independently of other vertices. The connected components of this graph are called *clusters*. If Γ is amenable, then Burton and Keane [**5**] and Gandolfi *et al* [**8**] proved that there is at most one infinite cluster almost surely. A major open conjecture of Benjamini and Schramm [**3**] is the converse, that if Γ is non-amenable, then, for some interval of p, there are infinitely many infinite clusters almost surely. By work of Benjamini *et al* [**2**], it is known that this is equivalent to the existence of a single value of p at which there are infinitely many infinite clusters almost surely. Lyons [**9**] observed that *every* Cayley graph has this property when Γ has cost > 1; no other groups are known such that all their Cayley graphs have this property. Pak and Smirnova-Nagnibeda [**11**] proved that Γ has *some* Cayley graph with this property for *bond* percolation when Γ is non-amenable, where, in bond percolation, it is the edges, not the vertices, that are kept with probability p independently.

Here we note that if Γ does not have fixed price with cost 1, then, for each of its Cayley graphs G, there exists a p where Bernoulli(p) percolation has infinitely many infinite clusters almost surely. We prove this for site percolation, but essentially the same proof applies for bond percolation. Thus, our result here includes that of Lyons [**9**], but actually the method of proof (not written anywhere) is the same. What is new is the combination of that method with a result of Abért and Weiss [**1**].

2. *Proof*

Consider a measurable equivalence relation $\mathscr{R}$ with countable equivalence classes on a standard probability space (X, μ). A *graphing* of $\mathscr{R}$ is a graph (X, E), where $E \subseteq \mathscr{R}$ is measurable and symmetric and such that for every $x \in X$, the vertices of the connected component of x are the same as the equivalence class of x. The *cost* of such a graphing equals $(1/2) \int_X |\{y;\, (x, y) \in E\}|\, d\mu(x)$. The *cost* of $\mathscr{R}$ is the infimum of the costs of its graphings.

Abért and Weiss [**1**] proved that every Bernoulli action of Γ has the maximum cost among all essentially free measure-preserving actions of Γ. Let G be a Cayley graph of a non-amenable group Γ with respect to a generating set S. Let $\theta(p)$ denote the probability that a given vertex of G belongs to an infinite cluster in Bernoulli(p) percolation; this is the same for all vertices. Define $p_c(G) := \inf\{p \in [0, 1];\, \theta(p) > 0\}$. If there is no p with infinitely many infinite clusters almost surely, then, for all $p > p_c(G)$, there is a unique infinite cluster almost surely by a theorem of Newman and Schulman [**10**]. Furthermore, Benjamini *et al* [**2**] proved that $\theta(p_c(G)) = 0$, whence, by van den Berg and Keane [**4**], $\lim_{p \downarrow p_c(G)} \theta(p) = 0$.

Consider the Bernoulli action of Γ on $(X, \mu) := ([0, 1]^\Gamma, \lambda^\Gamma)$, where λ is Lebesgue measure on $[0, 1]$. The *Cayley graphing* of X is the graph (X, E_S), where $E_S := \{(x, sx);\, x \in X,\, s \in S \cup S^{-1}\}$. For $x \in X$ and $p \in [0, 1]$, let

$$V_p(x) := \{\gamma^{-1}x;\, \gamma \in \Gamma,\, x(\gamma) \le p\}.$$

Let $Y_p \subseteq X$ denote the set of points x that belong to an infinite cluster in the graph induced by E_S on $V_p(x)$. The orbit equivalence relation induces an equivalence relation $\mathscr{R}_p$ on Y_p. Then $\mu(Y_p) = \theta(p)$ and, for $p > p_c(G)$, the cost of $(Y_p, \mathscr{R}_p, (\mu \restriction Y_p)/\mu(Y_p))$ is at most $|S|$. Hence, by Proposition II.6 of Gaboriau [**6**], the cost of the Bernoulli action of Γ on (X, μ) is at most $1 + \theta(p)(|S| - 1)$ for $p > p_c(G)$, whence the cost equals 1. Since this is the maximum cost of any action of Γ, while the minimum cost is 1 for any infinite group, it follows that all costs are 1 and thus that Γ has fixed price.

We remark that one can replace the use of $\lim_{p \downarrow p_c(G)} \theta(p) = 0$ by the fact that $\mathscr{R}_{p_c}$ is hyperfinite, as pointed out to us by Gaboriau. Indeed, the clusters induced by E_S on $V_p(x)$ for $p < p_c(G)$ are all finite by definition for μ-almost every x, whence their increasing union is hyperfinite and therefore has cost at most 1. It follows that the cost of the Bernoulli action of Γ on (X, μ) is at most $1 + (p - p_c(G))|S|$ for $p > p_c(G)$, whence the cost equals 1.

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