

# Perfect Matchings as IID Factors on Non-Amenable Groups

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**Abstract.** We prove that in every bipartite Cayley graph of every non-amenable group, there is a perfect matching that is obtained as a factor of independent uniform random variables. We also discuss expansion properties of factors and improve the Hoffman spectral bound on independence number of finite graphs.

## §1. Introduction.

A *perfect matching* in a graph is a set of its edges that includes each vertex exactly once. An early result guaranteeing the existence of a perfect matching is due to König (1916, 1926), who showed the sufficiency that the graph be bipartite and regular of finite degree. On the other hand, infinite graphs may come with a measurable structure and one may wish for a measurable perfect matching. That is, suppose that the bipartite graph has its two parts equal to  $[0, 1]$  and  $[2, 3]$ , with edge set a Borel subset of  $[0, 1] \times [2, 3]$ . If the graph is regular, must it have a Borel perfect matching? Laczkovich (1988) showed that the answer is no for 2-regular graphs. Kłopotowski, Nadkarni, Sarbadhikari, and Srivastava (2002) built on his example to show the same for any even degree\*. However, it is still open whether there is a measurable version of König's theorem for 3-regular graphs.

A somewhat related notion is the following. Suppose we are given a finitely generated group  $\Gamma$  and a Cayley graph  $G$  of  $\Gamma$ . In addition, we have independent uniform  $[0, 1]$  random variables assigned to each edge (or vertex). We call an instance of such an assignment a *configuration*. Note that  $\Gamma$  acts on  $G$  by automorphisms, whence it also acts on the set of configurations, as well as on perfect matchings of  $G$ . A random perfect matching of  $G$  that is obtained as some measurable function of the configuration and that commutes with the

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\* One of us (R.L.) noted that their example was incorrect for some degrees, but their example was corrected by Conley and Kechris (2009).

action of  $\Gamma$  is called a  $\Gamma$ -**factor** perfect matching of the random variables. Does one exist? In the case of the usual Cayley graph of  $\mathbb{Z}$ , the answer is no since the only invariant measure on perfect matchings is not mixing, yet every factor of independent random variables is mixing. However, this may be the only exception. Timár (2009) shows a positive answer for the usual Cayley graphs of  $\mathbb{Z}^d$  ( $d > 1$ ). Our main contribution is to prove that the answer is yes when  $\Gamma$  is non-amenable and  $G$  is bipartite:

**THEOREM 1.1.** *Let  $G$  be a bipartite non-amenable simple Cayley graph of a group  $\Gamma$ . Then there is an  $\Gamma$ -factor of independent uniform  $[0, 1]$  random variables on  $\Gamma$  that is a perfect matching of  $G$  a.s.*

In fact, we prove a slight strengthening of this in Theorem 2.4.

One connection of the two above notions is due to Kechris (personal communication, 2001; see Conley and Kechris (2009)). He attempted to show that there is no measurable version of König's theorem for 3-regular graphs by an approach that would succeed if the 3-regular tree had no perfect matching as a factor of IID. More precisely, note that the line graph  $G'$  of the 3-regular tree is the usual Cayley graph of  $\mathbb{Z}_3 * \mathbb{Z}_3$  and that a perfect matching of the tree corresponds to a set of vertices in  $G'$  that has exactly one in each triangle. His approach would succeed when there is no such set of vertices as a  $\mathbb{Z}_3 * \mathbb{Z}_3$ -factor of independent random variables with values in  $\{0, 1\}$ . This is equivalent to existence of a  $\mathbb{Z}_3 * \mathbb{Z}_3$ -factor from  $[0, 1]^{\mathbb{Z}_3 * \mathbb{Z}_3}$  by a result of Ball (2005). Our result is a factor that not only commutes with the action of the group, but with all automorphisms of the Cayley graph, whence it shows that Kechris's approach will not work, at least when sets of measure 0 are ignored.

This is somewhat surprising, actually. Consider again the case where  $G$  is the 3-regular tree. To obtain a perfect matching as a factor in  $G$ , one must have a rule for each edge  $e$  that decides whether  $e$  belongs to the matching, depending on the configuration. This rule must be the same for each edge (after action by an automorphism); being measurable, it depends only on the configuration within some distance  $R$  of the edge, up to a small error. The balls of radius  $R$  about two neighboring edges have a substantial symmetric difference, yet the rules must make consistent decisions, so this appears very hard to do. Indeed, if we wished to choose a set of vertices of  $G$  as a factor with the property that no two are adjacent, then there would be a bound to the density of such a set that is strictly less than  $1/2$ , even though  $G$  is bipartite.

As a matter of fact, there is considerable interest in finding such sets of vertices with high density on regular trees. The reason is this: First, a set of vertices no two of which are adjacent is called *independent*. The *independence ratio* of a finite graph  $G = (V, E)$

is the maximum of  $|K|/|V|$  over all independent sets  $K \subseteq V$ . An open question is to determine the limiting independence ratio for random regular graphs as the number of vertices tends to infinity. The existence of this limit, but not its value, has recently been established by Bayati, Gamarnik, and Tetali (2010). Any factor of IID random variables on a  $d$ -regular tree can be emulated on finite  $d$ -regular graphs with large girth or, more generally, with rare small cycles; this includes random regular graphs. If the factor gives an independent set on the tree, then it will give an independent set on the finite graphs of almost the same density. Furthermore, the best lower bounds on the independence ratios on all regular graphs of large girth are produced in this way by factors on regular trees (Lauer and Wormald, 2007; Hoppen, 2008). These match the best lower bounds on the independence ratios on random regular graphs (see Wormald (1999)), which were first obtained by other techniques. Furthermore, B. Szegedy (personal communication, 2009) conjectures that the possible values for the densities of independent sets in random  $d$ -regular graphs coincides with the possible densities of independent sets as IID factors in  $d$ -regular trees. (This is part of a much more general conjecture.)

In a wider context, factors are a fundamental object in the ergodic theory of amenable groups. They are just beginning to be understood for non-amenable groups: see Bowen (2010) for the case of free groups.

Finally, in the continuous context, Poisson point processes are the analogue of IID random variables from the discrete setting and play a corresponding role in the ergodic theory of continuous amenable groups. There are several recent papers on factors of Poisson point processes that give graphs, including perfect matchings: see Ferrari, Landim, and Thorisson (2004), Holroyd and Peres (2003), Timár (2004), and Holroyd, Pemantle, Peres, and Schramm (2009).

In Section 2, we prove our result on perfect matchings. This depends on an expansion property of factors in the non-amenable setting. There, we also show how our theorem on perfect matchings extends to all measure-preserving equivalence relations with expansive generating graphings. Some general remarks on expansion of factors are given in Section 3. Since matchings are independent sets in line graphs, we also discuss in Section 4 some improvements in the classical Hoffman bound for independent sets. We conclude with a few open questions in Section 5.

## §2. Perfect Matchings.

There are various equivalent definitions of non-amenability. The simplest is due to Følner (1955). To state it for a graph  $G = (V, E)$ , define

$$\Phi(G) := \inf \left\{ \frac{|\partial_E K|}{|K|}; \emptyset \neq K \subset V \text{ is finite} \right\}.$$

Here,  $\partial_E K$  is the set of edges that join  $K$  to its complement. Then  $G$  is *non-amenable* if  $\Phi(G) > 0$ .

We shall give a randomized algorithm (that takes infinitely many steps) to produce (a.s.) a perfect matching in a bipartite non-amenable Cayley graph. To prove that it works, we shall need a lemma that exploits the expansion property of non-amenability in the context of factors of IID. Our proof of this expansion property depends on spectral information.

For a function  $f : \Gamma \rightarrow \mathbb{R}$  and an element  $\gamma \in \Gamma$ , write  $R_\gamma f$  for the function  $x \mapsto f(x\gamma)$ . The *right regular representation* of  $\Gamma$  is the  $\Gamma$  action on  $\ell^2(\Gamma)$  given by  $\gamma \mapsto R_\gamma \upharpoonright \ell^2(\Gamma)$ . A representation is called *subregular* if it is the restriction of the regular representation to a  $\Gamma$ -invariant subspace. The *trivial representation* is the action on  $\mathbb{R}$  that fixes all points. Let  $\mu$  be the usual product measure on  $[0, 1]^\Gamma$  with each coordinate getting Lebesgue measure. We also have the representation  $R$  on  $L^2([0, 1]^\Gamma, \mu)$  given by  $(R_\gamma F)(\omega) := F(R_{\gamma^{-1}}\omega)$ . The following theorem has been known for some time; see Proposition 3.2 and Lemma 3.3 of Kechris and Tsankov (2008) for a more general result.

**THEOREM 2.1.** *Let  $\Gamma$  be a countable group. The representation  $R$  of  $\Gamma$  on  $L^2([0, 1]^\Gamma, \mu)$  is a sum of the trivial representation, the regular representation, and subregular representations.*

(The basic idea is that if  $\{W_n\}$  is an orthonormal basis of  $L^2([0, 1])$  with  $W_0 = \mathbf{1}$  and if  $C_\gamma : [0, 1]^\Gamma \rightarrow [0, 1]$  denotes the evaluation function at the coordinate  $\gamma$ , then an orthonormal basis of  $L^2([0, 1]^\Gamma, \mu^\Gamma)$  is the set of all products  $\prod_{\gamma \in \Gamma} W_{n(\gamma)} \circ C_\gamma$  with  $n(\gamma) = 0$  for all but finitely many  $\gamma$ .)

Fix a finite set  $S \subset \Gamma$  that is closed under inverses and that generates  $\Gamma$ . We are interested in the Cayley graph  $G$  of  $\Gamma$  with respect to  $S$ . Let

$$\mathcal{P} := |S|^{-1} \sum_{s \in S} R_s$$

be the *transition operator*. Then  $\mathcal{P}$  is self-adjoint and, thus, has real spectrum, whether it acts on  $\ell^2(\Gamma)$  or on  $L^2([0, 1]^\Gamma, \mu)$ .

The following is an immediate consequence of Theorem 2.1.

COROLLARY 2.2. *The spectrum of  $\mathcal{P}$  on  $\ell^2(\Gamma)$  is the same as the spectrum of  $\mathcal{P}$  on  $\mathbf{1}^\perp$  in  $L^2([0, 1]^\Gamma, \mu)$ .*

Let  $\rho$  be the spectral radius of  $\mathcal{P}$  on  $\ell^2(\Gamma)$ . Kesten (1959) proved that  $\rho < 1$  iff  $\Gamma$  is nonamenable. Let  $X$  stand for  $[0, 1]^\Gamma$ . Write  $L_0^2(X, \mu)$  for the orthocomplement of the constants in  $L^2(X, \mu)$ .

A measurable function  $\phi : X \rightarrow \{0, 1\}^\Gamma$  or  $\phi : X \rightarrow \{0, 1\}^{\mathbf{E}(G)}$  is called a  $\Gamma$ -**factor** if  $R_\gamma(\phi(\omega)) = \phi(R_\gamma\omega)$  for all  $\gamma \in \Gamma$  and  $\omega \in X$ . More generally, if  $\Gamma'$  is a group of automorphisms of  $G$  that commutes with  $\phi$ , then  $\phi$  is called a  $\Gamma'$ -**factor**. The full group of automorphisms is denoted  $\text{Aut}(G)$ . To any factor with range  $\{0, 1\}^\Gamma$ , we associate the set

$$B := \{\omega; (\phi(\omega))(\text{id}) = 1\},$$

where  $\text{id}$  denotes the identity element of  $\Gamma$ . Conversely, given any measurable  $B \subseteq X$ , there is an associated  $\Gamma$ -factor defined by  $(\phi(\omega))(\gamma) := \mathbf{1}_{\{R_\gamma\omega \in B\}}$ . We think of the image  $\phi(\omega)$  of a factor  $\phi$  as subset of  $\Gamma$ , namely, those  $\gamma \in \Gamma$  where  $(\phi(\omega))(\gamma) = 1$  and also write  $\gamma \in \phi(\omega)$  when  $(\phi(\omega))(\gamma) = 1$ . We sometimes omit parentheses and write  $\phi\omega$  for  $\phi(\omega)$ . We also think of

$$b := \mu(B)$$

as the density of the factor. Write

$$f_B := \mathbf{1}_B - \mu(B)\mathbf{1} \in L_0^2(X, \mu).$$

We have

$$\|f_B\|_2^2 = b(1 - b).$$

LEMMA 2.3. *Let  $G = (\Gamma, S)$  be a Cayley graph. Let  $\phi : (X, \mu) \rightarrow \{0, 1\}^\Gamma$  be a  $\Gamma$ -factor. Define  $\phi'\omega$  to consist of all the vertices that are adjacent to some vertex in  $\phi\omega$ . Then*

$$b' := \mathbf{P}[\text{id} \in \phi'\omega] \geq \frac{1}{\rho^2(1 - b) + b} \cdot b. \quad (2.1)$$

*Proof.* Let  $A := \{\omega; \phi'\omega(\text{id}) = 1\}$ . Since  $\mathbf{1}_{A^c} \cdot \mathcal{P}\mathbf{1}_B = \mathbf{0}$ , we have

$$b = (\mathbf{1}_B, \mathcal{P}\mathbf{1}) = (\mathcal{P}\mathbf{1}_B, \mathbf{1}) = (\mathcal{P}\mathbf{1}_B, \mathbf{1}_A).$$

Therefore,

$$b^2 \leq \|\mathcal{P}\mathbf{1}_B\|^2 \|\mathbf{1}_A\|^2.$$

Now  $\|\mathbf{1}_A\|^2 = b'$  and

$$\|\mathcal{P}\mathbf{1}_B\|^2 = \|\mathcal{P}(f_B + b\mathbf{1})\|^2 = \|\mathcal{P}f_B\|^2 + \|b\mathbf{1}\|^2 \leq \rho^2\|f_B\|^2 + b^2 = \rho^2b(1 - b) + b^2$$

since  $f_B \perp \mathbf{1}$  and  $\mathcal{P}$  preserves  $L_0^2(X, \mu)$ . Therefore,

$$\frac{b'}{b} \geq \frac{1}{\rho^2(1-b) + b}. \quad \blacksquare$$

We also need the following general tool (see, e.g., Benjamini, Lyons, Peres, and Schramm (1999)), whose proof we include for the convenience of the reader:

THE MASS-TRANSPORT PRINCIPLE FOR COUNTABLE GROUPS. *Let  $\Gamma$  be a countable group. If  $f : \Gamma \times \Gamma \rightarrow [0, \infty]$  is diagonally invariant, then*

$$\sum_{x \in \Gamma} f(\text{id}, x) = \sum_{x \in \Gamma} f(x, \text{id}).$$

*Proof.* Just note that  $f(\text{id}, x) = f(x^{-1}\text{id}, x^{-1}x) = f(x^{-1}, \text{id})$  and that summation of  $f(x^{-1}, \text{id})$  over all  $x^{-1}$  is the same as  $\sum_{x \in \Gamma} f(x, \text{id})$  since inversion is a bijection of  $\Gamma$ .  $\blacksquare$

In this context, we often use  $f(x, y) = \mathbf{E}F(x, y, \omega)$ , where  $F$  is defined on a probability space whose measure is  $\Gamma$ -invariant. If  $F$  is diagonally invariant, then so is  $f$ . We then call  $F(x, y, \omega)$  the **mass transported from  $x$  to  $y$** .

We are now ready to prove our main theorem.

THEOREM 2.4. *Let  $G$  be a bipartite non-amenable simple Cayley graph. Then there is an  $\text{Aut}(G)$ -factor of  $([0, 1]^\Gamma, \mu)$  that is a perfect matching of  $G$  a.s.*

*Proof.* We shall construct the factor in infinitely many stages, each stage consisting of infinitely many steps. Since we can decompose a uniform  $[0, 1]$  random variable into an infinite sequence of independent uniform  $[0, 1]$  random variables, we shall assume that we are given such sequences at the start. We shall also make use of a reverse operation: the **composition** of a finite ordered list of numbers in  $[0, 1]$  is a number in  $[0, 1]$ . We choose this composition map to be measurable and so that given the length of the list of numbers, it is an injection except on a countable set. Each random variable will be used at most once. We shall speak of the current random variables assigned to vertices, which we throw away after use.

Suppose we have a (partial) matching. Call a path **alternating** if its edges alternate between belonging to the matching and not. Following Berge (1957), define an **(augmenting) chain** to be a simple alternating path between unmatched vertices. If we replace a chain by the same path, but with unmatched edges becoming matched and matched edges becoming unmatched, so that all the vertices of the path are now matched, we say that we **flip** the chain.

At the beginning of the first stage, we have the empty matching and all edges are chains. At the end of the  $n$ th stage, there will be no chains of length at most  $2n - 1$ , where length is measured by the number of edges. Each step in the  $n$ th stage will be a repetition of the following operation: Assign the composition of the current random variables on the vertices to each current chain of length at most  $2n - 1$ . If a chain has a larger composition than that of every other chain that it intersects, then flip that chain.

Note that once a vertex is matched in a given step, then it remains matched after all subsequent steps. Furthermore, each edge belongs to a finite number of chains of length at most  $2n - 1$ , whence it changes its status (between belonging to the matching and not) at most finitely many times during the  $n$ th stage. Finally, there is a lower bound (depending on  $n$  and  $|S|$ ) to the conditional probability that a current chain is flipped, regardless of the past, whence after infinitely many steps, there are a.s. no chains of length at most  $2n - 1$ .

In order to define the factor as a limit of the stages, we must prove that a.s. no edge changes its status infinitely many times.

Let  $\phi_n$  denote the factor matching at the end of the  $n$ th stage. Fix  $n$  and define  $\langle A_k \rangle$  recursively as follows. Let  $A_0 = A_0(\omega)$  denote the unmatched vertices in  $\phi_n \omega$ . If  $k$  is even, then let  $A_{k+1}$  be the set of vertices that have a neighbor in  $A_k$ . If  $k$  is odd, then let  $A_{k+1}$  be the set of vertices  $x$  such that for some  $y \in A_k$ , the edge  $[x, y]$  is matched in  $\phi_n \omega$ .

We claim that for every  $k \geq 1$  and every  $x \in A_k$ , there is a simple alternating path from some  $x_0 \in A_0$  to  $x$  of length at most  $k$ . Clearly, there is some alternating path  $P_x$  from some  $x_0 \in A_0$  to  $x$  of length at most  $k$ . Since  $G$  is bipartite, each edge of  $P_x$  that leads from a vertex at odd distance from  $x_0$  to a vertex at even distance from  $x_0$  is a matched edge, whence the shortest path from  $x_0$  to  $x$  contained in  $P_x$  is simple and alternating.

There are two consequences of this that we use: The first is that if  $x \in A_k$  is unmatched and  $k \geq 1$ , then there is a chain of length at most  $k$ . The second is that if for some even  $k$ , the set  $A_k$  is not independent, then there is a chain of length at most  $2k + 1$ . Indeed, suppose that  $x, y \in A_k$  are neighbors. By the above, there is some simple alternating path  $P_x$  from some  $x_0 \in A_0$  to  $x$  of even length at most  $k$  and a simple alternating path  $P_y$  from some  $y_0 \in A_0$  to  $y$  of even length at most  $k$ . Since the concatenation  $P$  of  $P_x$  followed by the edge  $(x, y)$  and then finally the reverse of  $P_y$  is a path of odd length from  $x_0$  to  $y_0$ , it follows that the distance between  $x_0$  and  $y_0$  is odd. In particular,  $x_0 \neq y_0$ . If  $P_x$  and  $P_y$  are disjoint, then since the last edge of each of these paths lies in the matching, the path  $P$  is a chain of length at most  $2k + 1$ , as desired. In case  $P_x$  and  $P_y$  are not disjoint, then their union contains a simple path  $Q$  from  $x_0$  to  $y_0$ . Since the length of  $Q$  is odd, it is easy to see that  $Q$  is alternating as well.

By the first consequence, when  $k < n$  is odd, there is a unique edge in the matching from each  $x \in A_k$  to some  $y \in A_{k+1}$ . Let  $x$  send mass 1 to  $y$  in this situation. Then by the Mass-Transport Principle,  $\mu[\text{id} \in A_{k+1}] = \mu[\text{id} \in A_k]$  for all odd  $k < n$ , where  $\mu[\text{id} \in A_k]$  means  $\mu(\{\omega; \text{id} \in A_k(\omega)\})$ . By the second consequence, for all even  $k < n$ , the set  $A_k$  is independent, which implies (for example, by Lemma 2.3) that  $\mu[\text{id} \in A_k] \leq 1/2$ . By Lemma 2.3,  $\mu[\text{id} \in A_{k+1}] \geq c\mu[\text{id} \in A_k]$  for all even  $k < n$ , where  $c := 2/(1+\rho^2)$ . Note that  $c > 1$  because  $G$  is non-amenable. (If  $G$  is a tree, then instead of using Lemma 2.3, one could deduce this expansion inequality for  $\mu[\text{id} \in A_k]$  by using the fact that regular trees are limits of finite bipartite expander graphs, in the sense that the proportion tends to 1 of vertices in those finite graphs with a large neighborhood the same as in the tree.) Since  $\mu[\text{id} \in A_{2k-1}] \geq c^k \mu[\text{id} \in A_0]$  for  $2k-1 \leq n$ , it follows that  $\mu[\text{id} \in A_0] \leq a_n := c^{-\lfloor (n+1)/2 \rfloor}$ .

Now let each endpoint of a chain that is flipped send mass 1 to each vertex in its flipped chain. Then the expected mass sent by the identity is at most  $\sum_n 2na_{n-1} < \infty$ . Each vertex receives mass equal to twice the number of times a neighboring edge changes its status. By the Mass-Transport Principle, the expected number of times an edge changes its status is finite. This proves that the limit of  $\phi_n$  exists a.s. and that a.s. all vertices are matched at the end. ■

REMARK 2.5. The same result holds for factors from  $([0, 1]^{\mathbb{E}(G)}, \nu)$ , where  $\nu$  is product measure. Indeed  $([0, 1]^\Gamma, \mu)$  is itself a factor of  $([0, 1]^{\mathbb{E}(G)}, \nu)$ . To see this, given  $\omega \in [0, 1]^{\mathbb{E}(G)}$ , define  $\xi \in [0, 1]^\Gamma$  by  $\xi(x) := \sum_{e \ni x} \omega(e) \pmod{1}$ . It is clear that each  $\xi(x)$  is uniform on  $[0, 1]$  when  $\omega$  has law  $\nu$ . To prove that  $\xi(x_1), \dots, \xi(x_n)$  are independent, we proceed by induction. Because  $G$  is infinite, we may assume that  $x_n$  belongs to an edge  $e$  whose other endpoint is not among  $x_1, \dots, x_{n-1}$ . Since  $\omega(e)$  is therefore independent of  $\xi(x_i)$  for  $i < n$ , it follows that  $\xi(x_n)$  is independent of  $\xi(x_i)$  for  $i < n$ .

REMARK 2.6. Let  $(X, \mathcal{F}, \mu)$  be a probability space and  $E \in \mathcal{F} \times \mathcal{F}$  a symmetric measurable subset of  $X \times X$ . Let  $G := (X, E)$  be the graph associated to  $E$ . Assume that all the connected components of  $G$  are bipartite and denumerable. Write  $[x] \subset X$  for the vertices in the connected component of  $x \in X$ . Suppose that  $(X, \mathcal{F}, \mu, G)$  is measure-preserving, meaning that  $\mu_L = \mu_R$ , where

$$\int_{X^2} f(x, y) d\mu_L(x, y) := \int_{x \in X} \sum_{y \in [x]} f(x, y) d\mu(x)$$

and

$$\int_{X^2} f(x, y) d\mu_R(x, y) := \int_{x \in X} \sum_{y \in [x]} f(y, x) d\mu(x)$$



for all measurable  $f : X^2 \rightarrow [0, \infty]$ . Suppose in addition that  $G$  is expansive, meaning that there exists  $c > 1$  such that for every measurable  $A \subset X$  with  $\mu(A) \leq 1/2$ , we have  $\mu(A') \geq c\mu(A)$ , where  $A'$  consists of the  $G$ -neighbors of the points in  $A$ . Then there is a  $\mu_L$ -measurable perfect matching in  $G$ . The proof is the same as that of Theorem 2.4, except that the first short part is replaced by a (similar) general argument of Elek and Lippner (2010), Proposition 1.1, which shows that there is a sequence of factors  $\phi_n$  that have the property that there is no chain of length at most  $2n - 1$  in  $\phi_n$  and such that the set of matched vertices is increasing in  $n$ .

### §3. Factor Expansion and Spectral Radius.

There is a general relationship between factors of measure-preserving actions and an associated spectral radius. It is quite analogous to expansion properties of finite graphs.

Let  $\Gamma$  be a group acting by measure-preserving transformations on a probability space  $(X, \mu)$ . We also write integration with respect to  $\mu$  as  $\mathbf{E}$ . Fix a finite  $S \subset \Gamma$ , closed under inverses and generating  $\Gamma$ . Let  $\rho$  be the spectral radius of  $\mathcal{P}$  on  $L_0^2(X, \mu)$ . In fact, for more precision, we shall use the bottom,  $-\rho_-$ , and the top,  $\rho_+$ , of the spectrum. We have  $\rho = \max(\rho_-, \rho_+)$  and

$$-\rho_- \leq (\mathcal{P}f, f) \leq \rho_+ \tag{3.1}$$

for all  $f \in L_0^2(X, \mu)$  with  $\|f\|_2 = 1$ .

Define the *expansion constant* of the action with respect to  $S$  by

$$\Phi(\Gamma, S, X, \mu) := \inf \left\{ \frac{1}{|S|b(1-b)} \int \sum_{s \in S} \mathbf{1}_{B \cap sB^c} d\mu; B \subset X, 0 < \mu B < 1 \right\}.$$

The following inequalities relating the expansion constant and the spectral radius are analogous to those on finite graphs, so we restrict our proofs to the essential steps. See, e.g., Levin, Peres, and Wilmer (2009), Theorem 13.14, for more details on finite graphs.

**THEOREM 3.1.** *Let  $\Gamma$  be a group acting by measure-preserving transformations on a probability space  $(X, \mu)$ . Write  $\Phi := \Phi(\Gamma, S, X, \mu)$ . Then*

$$\Phi^2/8 \leq 1 - \sqrt{1 - (\Phi/2)^2} \leq 1 - \rho \leq 1 - \rho_+ \leq \Phi.$$

There is never expansion for amenable groups, that is, for all actions of an amenable group, the spectral radius is equal to 1 by a theorem of Ornstein and Weiss (1980). Some such expansion property holds for every ergodic invariant percolation only on Kazhdan

groups. In fact, the very definition of Kazhdan's property (T) is easily seen to be equivalent to every action having spectral radius less than 1. As we saw via Corollary 2.2, the spectral radius of Bernoulli actions of non-amenable groups is strictly less than 1.

For some purposes, a notion for an action weaker than expansion is interesting, namely, the non-existence of almost invariant sets. This means that if  $\Gamma$  acts on  $(X, \mu)$  and  $B_n \subset X$  are measurable sets with  $\mu(B_n \triangle \gamma B_n) \rightarrow 0$  for all  $\gamma \in \Gamma$ , then  $\mu(B_n)(1 - \mu(B_n)) \rightarrow 0$ . See Appendix A of Hjorth and Kechris (2005) for a discussion of this and related matters.

We have

$$\Phi = \inf \frac{1}{b(1-b)} (\mathbf{1}_B, \mathcal{P}\mathbf{1}_{B^c}) = 1 - \sup \frac{(\mathcal{P}f_B, f_B)}{b(1-b)},$$

which proves that  $\Phi \geq 1 - \rho_+$ . Since  $(\mathbf{1}_B, \mathcal{P}\mathbf{1}_{B^c}) = (\mathcal{P}\mathbf{1}_B, \mathbf{1}_{B^c})$ , we also have the alternative expression

$$\Phi = \inf \left\{ \frac{1}{2|S|b(1-b)} \mathbf{E} \sum_{s \in S} (\mathbf{1}_{B \cap sB^c} + \mathbf{1}_{B^c \cap sB}); B \subset X, 0 < \mu B < 1 \right\}. \quad (3.2)$$

LEMMA 3.2. *With notation as in Theorem 3.1, if  $f \in L^2(X, \mu)$  satisfies  $f \geq 0$  a.s., then*

$$2\mu[f = 0]\Phi \int f d\mu \leq \frac{1}{|S|} \int \sum_{s \in S} |f - sf| d\mu.$$

*Proof.* For  $t > 0$ , let  $B_t := f^{-1}(t, \infty)$ . Put  $\alpha_f := \mu[f = 0]$ . Then by (3.2), we have

$$2\Phi\mu(B_t)\alpha_f|S| \leq 2\Phi\mu(B_t)\mu(B_t^c)|S| \leq \mathbf{E} \sum_s (\mathbf{1}_{\{f > t \geq sf\}} + \mathbf{1}_{\{sf > t \geq f\}}).$$

Integrating on  $t \in (0, \infty)$  with respect to Lebesgue measure gives

$$2\Phi\alpha_f|S|\mathbf{E}f \leq \mathbf{E} \sum_s (\max\{f - sf, 0\} + \max\{sf - f, 0\}) = \mathbf{E} \sum_s |f - sf|. \quad \blacksquare$$

*Proof of Theorem 3.1.* We have already proved the fourth inequality. The first inequality is elementary. To prove the second inequality, consider  $f_0 \in L_0^2$  such that  $\|f_0\| = 1$ . Define  $\lambda := (\mathcal{P}f_0, f_0)$ . Without loss of generality, we may assume that  $\mu[f_0 > 0] \leq 1/2$ . Define  $f := \max\{f_0, 0\}$ . Then checking cases shows that  $(I - \mathcal{P})f \leq (1 - \lambda)f$ , whence  $((I - \mathcal{P})f, f) \leq (1 - \lambda)\|f\|^2$ . Define

$$\beta := \mathbf{E} \sum_s |f - sf|^2 / (2|S|) = ((I - \mathcal{P})f, f).$$

Now by the lemma, since  $\alpha_f \geq 1/2$ , we have

$$\|f\|_2^4 \leq \Phi^{-2} \left( \mathbf{E} \sum_s |f^2 - sf^2| / |S| \right)^2 \leq 2\Phi^{-2}\beta \mathbf{E} \sum_s |f + sf|^2 / |S| = 2\Phi^{-2}\beta(4\|f\|_2^2 - 2\beta).$$

Therefore,

$$\lambda^2 \leq (1 - \beta/\|f\|_2^2)^2 \leq 1 - (\Phi/2)^2.$$

Now take the supremum of  $|\lambda|$  over  $f_0$ . \blacksquare

Let  $G := (\Gamma, S)$  be the right Cayley graph of  $\Gamma$  corresponding to the generating set  $S$ . When the factor is an independent set in  $G$ , we can bound its density as follows. It is analogous to the Hoffman bound (Lovász, 1979) for the independence number of a finite graph.

PROPOSITION 3.3. *Suppose that  $\phi : (X, \mu) \rightarrow \{0, 1\}^\Gamma$  is a  $\Gamma$ -factor with the property that if  $(\phi\omega)(\text{id}) = 1$ , then  $(\phi\omega)(s) = 0$  for all  $s \in S$ . Then*

$$b \leq \rho_- / (1 + \rho_-), \quad (3.3)$$

with equality iff  $\mathcal{P}f_B = -\rho_- f_B$ .

*Proof.* We have  $(\mathcal{P}\mathbf{1}_B, \mathbf{1}_B) = 0$ , which is the same as

$$(\mathcal{P}f_B, f_B) = -b^2. \quad (3.4)$$

We deduce from this that  $b^2 \leq b(1-b)\rho_-$ , which gives the inequality. Furthermore, if equality holds, then

$$-\rho_- = \frac{(\mathcal{P}f_B, f_B)}{\|f_B\|_2^2}.$$

By (3.1), it follows that  $\mathcal{P}f_B = -\rho_- f_B$ . Conversely, if  $\mathcal{P}f_B = -\rho_- f_B$ , then it easily follows that equality holds in (3.3). ■

Other inequalities known for finite graphs can be proved as well. We illustrate with two well-known examples (see, e.g., Alon and Spencer (2008), Theorem 9.2.4 and Corollary 9.2.5).

PROPOSITION 3.4. *Let  $G = (\Gamma, S)$  be a Cayley graph. Let  $\phi : (X, \mu) \rightarrow \{0, 1\}^\Gamma$  be a  $\Gamma$ -factor. Then*

$$\mathbf{E} \left[ \left[ \frac{1}{|S|} \sum_{s \in S} \mathbf{1}_{sB} - b \right]^2 \right] \leq \rho^2 b(1-b).$$

*Proof.* This is the same as  $\|\mathcal{P}f_B\|^2 \leq \rho^2 \|f_B\|^2$ . ■

COROLLARY 3.5. *Let  $G = (\Gamma, S)$  be a Cayley graph. Let  $\phi : (X, \mu) \rightarrow (\{0, 1\}^2)^\Gamma$  be a  $\Gamma$ -factor. Define  $B_i := \{\omega; (\phi\omega)(\text{id})_i = 1\}$  for  $i = 1, 2$ . Put  $b_i := \mu(B_i)$ . Then*

$$\left| \mathbf{E} \left[ \frac{1}{|S|} \sum_{s \in S} \mathbf{1}_{B_1 \cap sB_2} \right] - b_1 b_2 \right| \leq \rho \sqrt{b_1 b_2 (1-b_1)(1-b_2)}.$$

*Proof.* The left-hand side equals  $|(f_{B_1}, \mathcal{P}f_{B_2})|$ , whence it is at most  $\|f_{B_1}\| \cdot \|\mathcal{P}f_{B_2}\|$ . Multiplying this by the same inequality with  $B_1$  and  $B_2$  reversed and using Proposition 3.4 gives the result. ■

#### §4. Improving the Hoffman Bound.

Here we discuss briefly how to improve Proposition 3.3. Our results apply to factors as well as to arbitrary regular finite graphs. One improvement holds only when  $\rho_- > 1 - 1/|S|$ ; the other holds when  $|S|\rho_- \notin \mathbb{Z}$ . In both cases, we give only sketches since we have no especially interesting applications to present. However, since the Hoffman bound has not been improved since its discovery, it seems worthwhile to explain our improvements.

There are various ways to improve the proof of Proposition 3.3. We give just one. Given the factor  $\phi$  such that  $\phi\omega$  is a.s. an independent set, define  $N(\omega) := |S \cap \phi\omega|$ . Write  $d := |S|$  and  $p := \mathbf{P}[N = d]$ . We may assume that  $\phi\omega$  is a.s. a maximal independent set, i.e., every vertex not in  $\phi\omega$  has a neighbor in  $\phi\omega$ . Consider the function

$$f(\omega) := \begin{cases} 1 & \text{if } N(\omega) = 0, \\ -a & \text{if } 1 \leq N(\omega) \leq d-1, \\ -A & \text{if } N(\omega) = d. \end{cases}$$

We choose the values of  $a$  and  $A$  so that  $f \perp \mathbf{1}$ . Then

$$d \cdot \mathcal{P}f(\omega) = \begin{cases} -ad + (a - A)|\{s \in S; N(s\omega) = d\}| & \text{if } N(\omega) = 0, \\ (1 + a)N(\omega) - ad & \text{if } N(\omega) \geq 1. \end{cases}$$

Using the facts that  $\mathbf{E}[N] = db$ ,

$$\mathbf{E}[N; 1 \leq N \leq d-1] = \mathbf{E}[N] - dp = d(b-p),$$

and  $\mathbf{E}[|\{s \in S; N(\omega) = 0, N(s\omega) = d\}|] = \mathbf{E}[|\{s \in S; N(s\omega) = d\}|] = dp$ , one can calculate that

$$(\mathcal{P}f, f) = (1+a)^2(1-2b) - 1.$$

Also,

$$(f, f) = b + a^2(1-b-p) + A^2p = b + a^2(1-b-p) + [b - a(1-b-p)]^2/p.$$

Since  $(\mathcal{P}f, f) \geq -\rho_-(f, f)$  for all  $a$  and this inequality is quadratic in  $a$ , it follows that its discriminant is non-positive:

$$0 \geq b\rho_-[b(1+\rho_-) - \rho_-] + p[1 - b(1+\rho_-)(2-\rho_-)].$$

Since  $b \leq \rho_-(1+\rho_-) \leq 1/(1+\rho_-)(2-\rho_-)$ , the same inequality holds when we substitute a lower bound for  $p$ . Now  $\mathbf{E}[N; 1 \leq N \leq d-1] \leq (d-1)(1-b-p)$ , which yields  $p \geq (2d-1)d - d + 1$ , whence

$$b \leq \frac{d-1}{(1+\rho_-)[d(2-\rho_-) - 1]}. \tag{4.1}$$

As we said, (4.1) improves Proposition 3.3 only for  $\rho_- > 1 - 1/d$ . In fact, it is impossible to improve Proposition 3.3 in all cases when  $\rho_- = 1 - 1/d$  because there are cases when equality holds.

Our second improvement is as follows. We have

$$\mathcal{P}\mathbf{1}_B = \frac{q}{d}\mathbf{1}_{B^c} \quad (4.2)$$

for some integer-valued function  $q$ . Now

$$\mathbf{E}[q/|S| \mid B^c] = b/(1-b). \quad (4.3)$$

Write  $\widehat{f}_B := f_B/\|f_B\|$ . If  $\nu$  denotes the spectral measure for  $\widehat{f}_B$  with respect to  $\mathcal{P}$ , then

$$\begin{aligned} \|\mathcal{P}\widehat{f}_B + \rho_-\widehat{f}_B\|^2 &= \int_{-\rho_-}^{\rho_+} (\lambda + \rho_-)^2 d\nu(\lambda) \\ &\leq (\rho_- + \rho_+) \int_{-\rho_-}^{\rho_+} (\lambda + \rho_-) d\nu(\lambda) \\ &= (\rho_- + \rho_+) \left[-\frac{b}{1-b} + \rho_-\right]. \end{aligned}$$

On the other hand, by (4.2), one can calculate that

$$b(1-b)\|\mathcal{P}\widehat{f}_B + \rho_-\widehat{f}_B\|^2 = \text{Var}(q/d \mid B^c)(1-b) + [\rho_-(1-b) - b]^2 b/(1-b),$$

whence

$$(\rho_- + \rho_+)b[\rho_-(1-b) - b] \geq \text{Var}(q/d \mid B^c)(1-b) + [\rho_-(1-b) - b]^2 b/(1-b),$$

which simplifies to

$$\text{Var}(q/|S| \mid B^c) \leq b \left[ \rho_- - \frac{b}{(1-b)} \right] \left[ \rho_+ + \frac{b}{(1-b)} \right]. \quad (4.4)$$

Now if  $m := b/(1-b) \in (k/d, (k+1)/d)$ , then the smallest  $\text{Var}(q/d \mid B^c)$  can be is when  $q$  takes only the values  $k$  and  $k+1$  on  $B^c$ . Note that if  $\rho_- < (k+1)/d$ , then  $m < (k+1)/d$ . This gives that either  $m \leq k/d$  or

$$\text{Var}(q/d \mid B^c) \geq -\frac{k+k^2}{d^2} + \frac{1+2k}{d}m - m^2.$$

Combining this with (4.4) and, for simplicity, using  $\rho \leq 1$ , we obtain

$$m \leq \frac{k^2 + k}{d(1+2k) - d^2\rho_-},$$

which is the same as

$$b \leq \frac{k^2 + k}{k^2 + (1+2d)k + d - d^2\rho_-}.$$

REMARK 4.1. Expanding the inequality of Proposition 3.4 gives the same inequality (4.4), but with  $\rho$  in place of  $\rho_-$ , which can be significantly worse.

## §5. Open Questions.

It is interesting to consider the chromatic number with respect to invariant processes under increasing restrictions: For example, a regular tree has chromatic number 2, and there is an invariant random proper 2-coloring, which is ergodic. However, there is no such mixing 2-coloring, but there is a mixing 3-coloring. What is the minimum number of colors for a proper coloring that is an IID factor? For large degree  $d$ , it is at least  $d/(2 \log d)$  since Frieze and Łuczak (1992) proved that for large degree  $d$ , the independence ratio for large random  $d$ -regular graphs is asymptotic to  $2 \log d/d$ . The minimum number of colors as an IID factor on a Cayley graph of degree  $d$  is at most  $d + 1$ , as shown by Schramm (personal communication, 1997). This was shown more generally to hold for any factor by Kechris, Solecki, and Todorćević (1999).

Related to this is a simpler question due to Lyons and Schramm in 1997 (unpublished): If  $G$  is a Cayley graph of chromatic number  $\chi$ , then is there a random invariant  $\chi$ -coloring? It is easy to show a positive answer when  $G$  is amenable. Conley and Kechris (2009) prove some general results on invariant coloring, as well as a version of our Proposition 3.3, discovered independently. For various results on coloring Poisson-Voronoi tessellations by factors, see Angel, Benjamini, Gurel-Gurevich, Meyerovitch, and Peled (2009) and Timár (2010).

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