# Spectral Embedding Bounds Random Walk Eigenvalues and Return Probabilities

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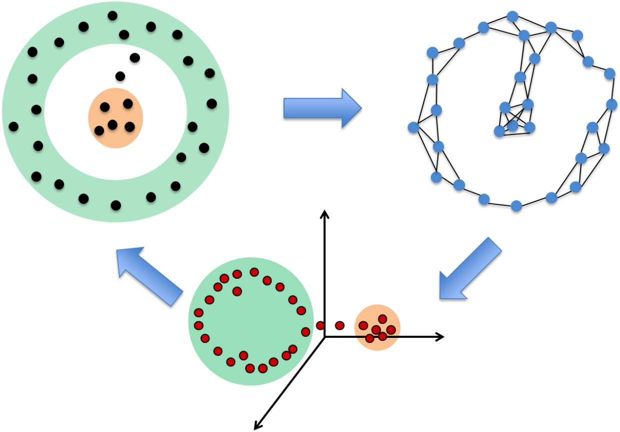
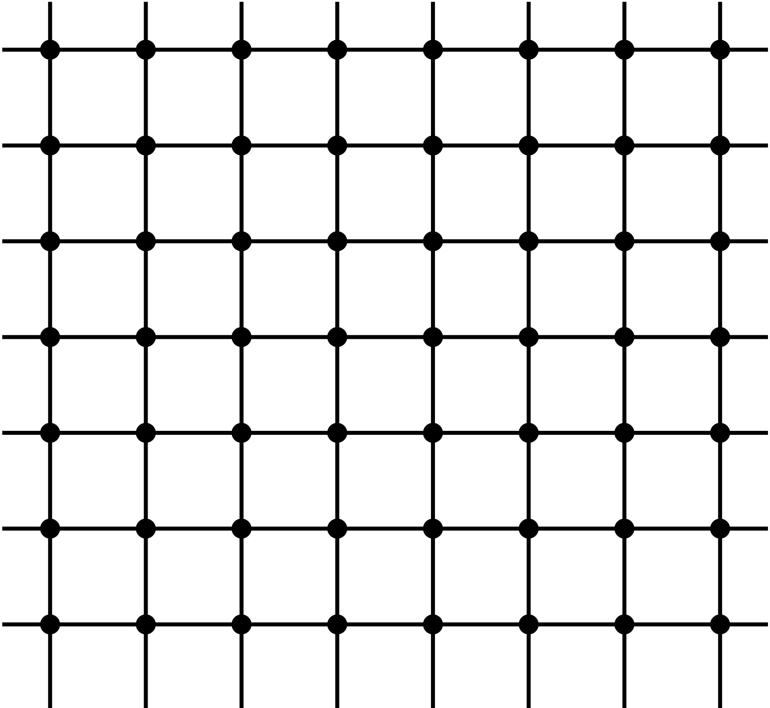
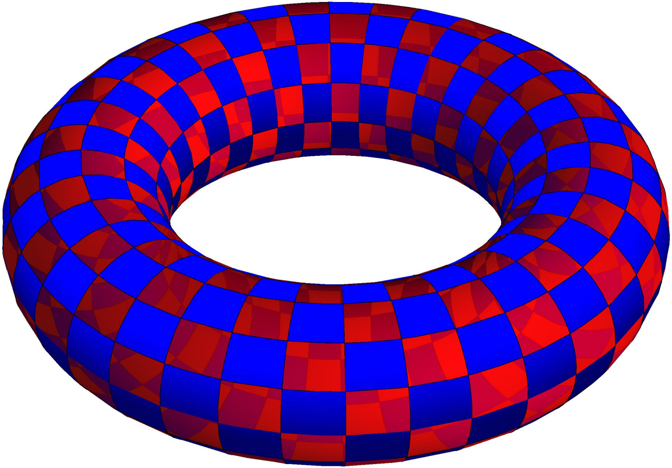
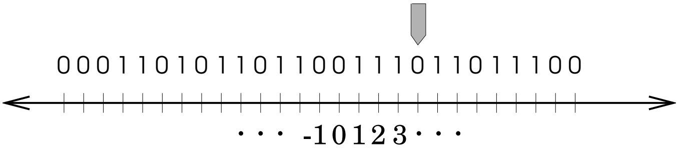


(Ng-Jordan-Weiss (2001))








The Lamplighter group over $\mathbb{Z}$.

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## Modern Example (Revelle 2003)

On the Lamplighter group over $\mathbb{Z}$, $\qquad p_t(x,x) \sim Ct^{1/6} e^{-C' t^{1/3}}$.

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**Theorem. (Diaconis and Saloff-Coste, 1994)**

Similar bounds hold for $p_t(o,o) - 1/n$ on finite groups of size $n$.

## Example

For finite transitive graphs with $n$ vertices, the inequality

$$p_t(o,o) - 1/n \le C' t^{-D/2}$$

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To see the connection and to state the corresponding inequality for infinite graphs, we use spectral measure.

First, finite graphs. For any function $f$, expand it in eigenfunctions as

$$f = \sum_{k=1}^{n} \langle f, g_k \rangle g_k \,.$$

Example:

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Thus,

$$P^t \mathbf{1}_x \qquad = \ (I - \mathcal{L}/2)^t \mathbf{1}_x \qquad = \sum_{k=1}^{n} (1 - \lambda_k/2)^t \, g_k(x) \quad g_k$$

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where the measure/c.d.f.

$$\mu^*(\lambda) := |\{k \,;\; 0 < \lambda_k \le \lambda\}|/n$$

puts mass $1/n$ at each $\lambda_k$ for $k \ge 2$ (with multiplicity).

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For infinite $G$ ($n = \infty$), we have the same, but where the measure/c.d.f.

$$\mu^*(\lambda) := \langle I^*_{\mathcal{L}}(\lambda)\mathbf{1}_o, \mathbf{1}_o\rangle$$

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For finite $G$, if we diagonalize $\mathcal{L} = \sum \lambda P_\lambda$, then $I_{\mathcal{L}}^*(\delta) = \sum_{0<\lambda\le\delta} P_\lambda$.

Recall $\mathrm{vol}(r) := |B(o, r)|$.

**Theorem. (L.-Oveis Gharan, 2012)**

If $G$ is transitive of degree $d$, then for all $\delta \in (0, 2)$,

$$\mu^*(\delta) \le \frac{4}{\mathrm{vol}\left(\frac{1}{\sqrt{2d\delta}}\right)} .$$

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### Sample Corollary.

If $G$ is transitive with $\mathrm{vol}(r) \ge Cr^D$ $(0 \le r \le \mathrm{diam}\, G \le \infty)$, then for all $\delta \in (0, 2)$,

$$\mu^*(\delta) \le C'\delta^{D/2}$$

and lazy simple random walk satisfies, for every $t \ge 1$,

$$p_t(o, o) - \frac{1}{n} \le C'' t^{-D/2}$$

$(n \le \infty)$.

## To show the corollary:

$$\mu^*(\delta) \leq \frac{4}{\mathrm{vol}\left(\frac{1}{\sqrt{2d\delta}}\right)} \text{ and } \mathrm{vol}(r) = \Omega(r^{D}) \text{ implies } \quad \mu^*(\delta) = O\big(\delta^{D/2}\big)$$

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## Proof.

$\mu^*(\delta) = O\big(\delta^{D/2}\big)$ is immediate. This gives

$$\begin{aligned} p_t(o,o) - \frac{1}{n} &= O\Big(\int_0^2 (1-\lambda/2)^t \, d(\lambda^{D/2})\Big) \\ &= O\Big(\int_0^2 e^{-\lambda t/2} \, d(\lambda^{D/2})\Big) \\ &= O\Big(t^{-D/2} \int_0^\infty e^{-s/2} \, d(s^{D/2})\Big) \\ &= O\big(t^{-D/2}\big)\,. \end{aligned}$$

$\square$

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For $\mathbf{0} \neq g \in \ell^2(V)$, define the Rayleigh quotient of $g$

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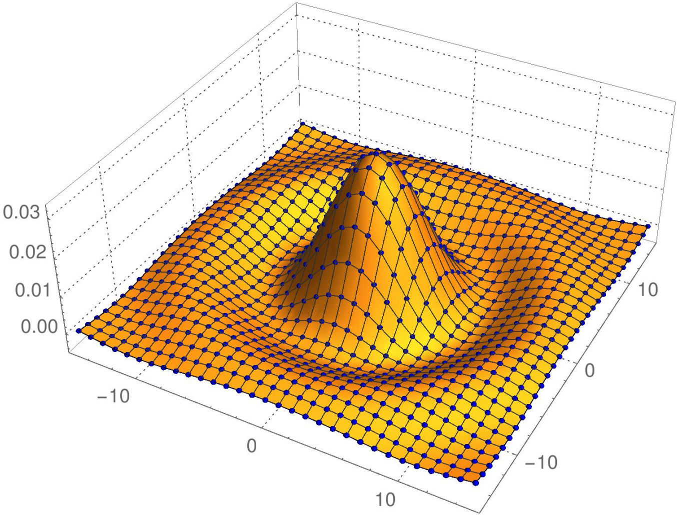
Notice $F(x) \in \operatorname{img} I^*(\delta)$.




spectral embedding $F\big((0,0)\big) \in \ell^2(\mathbb{Z}^2)$ of $\mathbb{Z}^2$ for $\delta = 0.1$
$F\big((j,k)\big)$ is the same but shifted to $(j,k)$

## Theorem restated. (L.-Oveis Gharan, 2012)

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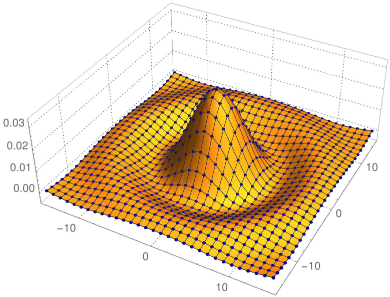


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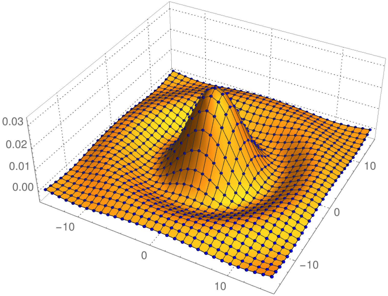


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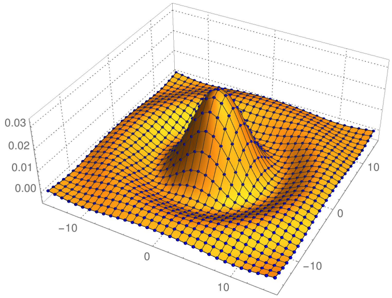


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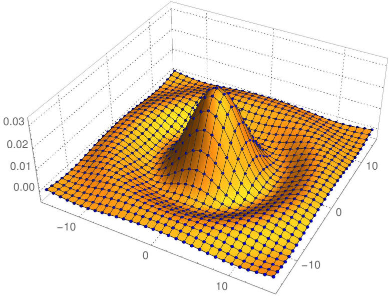


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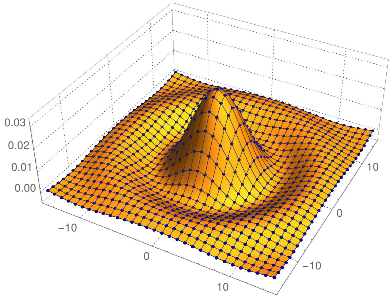


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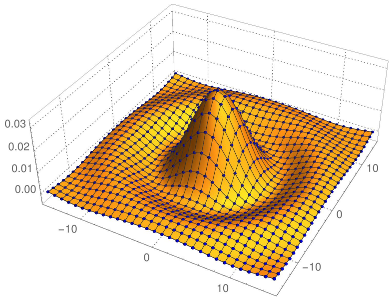


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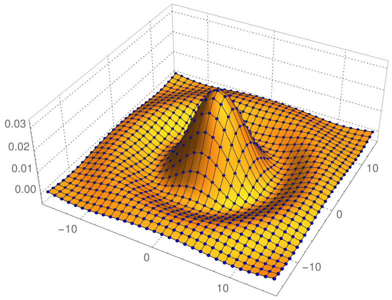


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&= \langle I^*(\delta)\mathbf{1}_o, \mathbf{1}_o\rangle =: \mu^*(\delta) \\
&= F(o)(o)\,.
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Second item: Transitivity means the graph looks the same from every vertex.

## Lemma

For all $x$, we have $\|F(o)\| = \|F(x)\|$ and for every automorphism $\phi$,

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