Comparing Return Probabilities

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$$\Delta(x, y) := \begin{cases} -w(x, y) & \text{if } x \neq y \text{ and } x \sim y, \\ 0 & \text{if } x \neq y \text{ and } x \not\sim y, \\ w(x) & \text{if } x = y. \end{cases}$$

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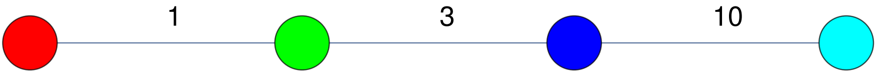
- Proof 2:

$$\frac{d}{dt}\langle e^{-t\Delta}\mathbf{1}_x, \mathbf{1}_x \rangle = \langle -\Delta e^{-t\Delta}\mathbf{1}_x, \mathbf{1}_x \rangle = -\langle \Delta e^{-t\Delta/2}\mathbf{1}_x, e^{-t\Delta/2}\mathbf{1}_x \rangle \leq 0 \,.$$

Individual Return Probabilities



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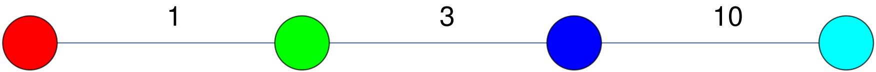

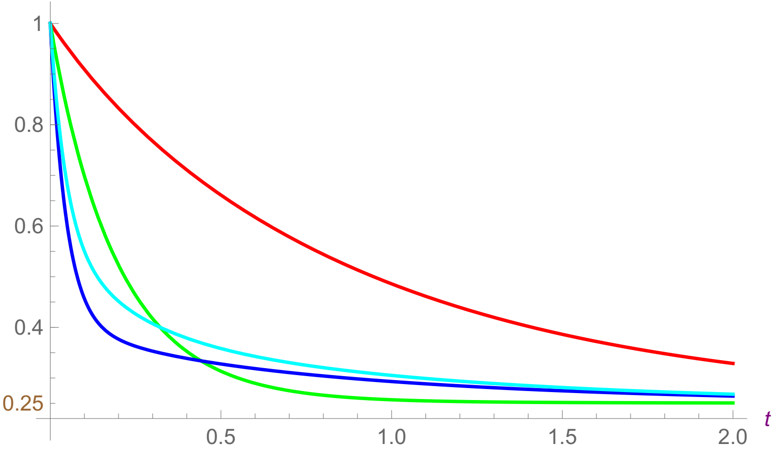

Individual Return Probabilities: Time 1

Are they monotonic in w?

Note that multiplying all weights by a constant is equivalent to multiplying time by that same constant.

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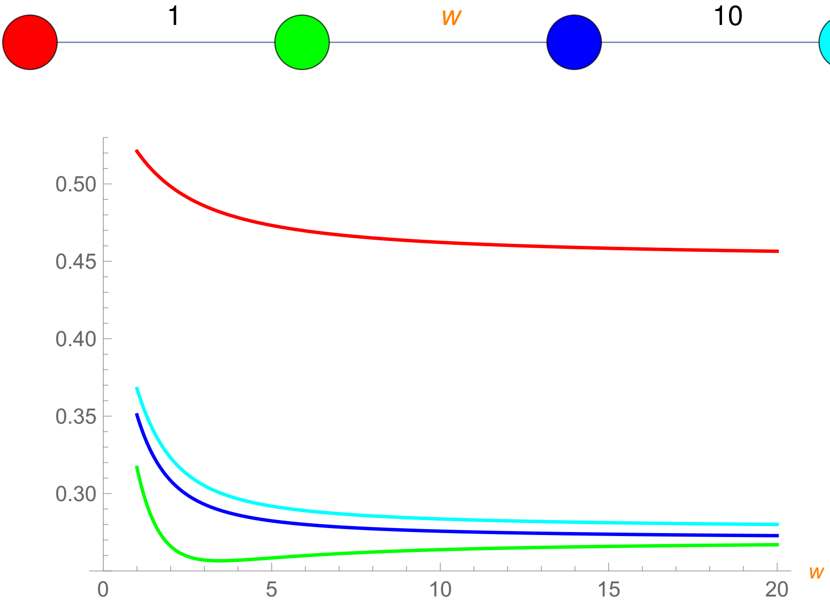
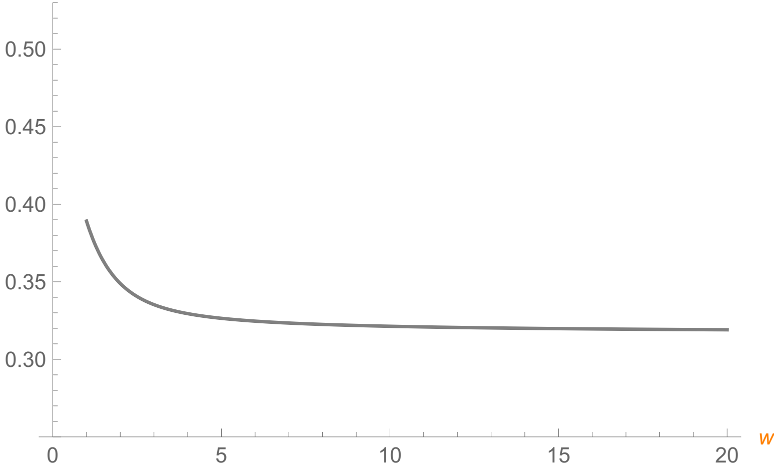


Average Return Probability: Time 1



Theorem, Benjamini and Schramm (pub. by Heicklen and Hoffman (2005))

On every finite graph, G, the average return probability at each time is monotonic decreasing in the weights, w_G.

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The average return probability equals $|\mathsf{V}(G)|^{-1} \operatorname{tr} e^{-t\Delta_G} =: \operatorname{Tr} e^{-t\Delta_G}$. If (G, w_G) is (vertex-)transitive, then this equals $p_t(o, o)$.

Open Question, L.

If G is a transitive infinite graph, is $p_t(o, o)$ monotonic decreasing in the weights, w_G, among transitive weight functions?

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This is because for equivariant operators A, $\operatorname{Tr} A := \langle A\mathbf{1}_o, \mathbf{1}_o \rangle$ defines a normalized trace, meaning that $A \mapsto \operatorname{Tr} A$ is linear, $\operatorname{Tr} A \geq 0$ for $A \geq 0$, $\operatorname{Tr} I = 1$, and $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.

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Open Question, Fontes and Mathieu

If G is an infinite Cayley graph, is $\mathbf{E}\big[p_t(o, o)\big]$ monotonic decreasing in the weights, w_G, among random weight functions with invariant law?

Theorem, Fontes and Mathieu (2006)

If G is an amenable Cayley graph, then $\mathbf{E}\big[p_t(o,o)\big]$ monotonic decreasing in the weights, w_G, among random invariant weight functions.

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This depends on the fact that for equivariant (random) operators A, we have a trace $A \mapsto \mathbf{E}\big[\langle A\mathbf{1}_o, \mathbf{1}_o\rangle\big]$.

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This depends on the fact that for equivariant (random) operators A, we have a trace $A \mapsto \mathbf{E}\big[\langle A\mathbf{1}_o, \mathbf{1}_o\rangle\big]$. More generally, if μ is a unimodular probability measure on rooted networks, then the class of equivariant operators $(G,o) \mapsto A_{(G,o)} = A_G$ has a trace

$$\mathrm{Tr}_\mu : A \mapsto \int \langle A_G \mathbf{1}_o, \mathbf{1}_o\rangle \, d\mu(G,o) = \mathbf{E}\big[\langle A\mathbf{1}_o, \mathbf{1}_o\rangle\big] .$$

A probability measure μ on (isomorphism classes of) rooted connected networks (G, o) is called unimodular if it satisfies the following mass-transport principle: for every isomorphism-invariant nonnegative Borel function f,

$$\int \sum_{x \in \mathsf{V}(G)} f_G(o, x)\, d\mu(G, o) = \int \sum_{x \in \mathsf{V}(G)} f_G(x, o)\, d\mu(G, o)\,.$$

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Every Cayley graph (with arbitrary rooting) induces a unimodular measure: use $o :=$ the identity, $f(o, x) = f(x^{-1}o, x^{-1}x) = f(x^{-1}, o)$, and that inversion is a bijection.

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UNIMODULAR MASS TRANSPORT INC We deliver all you give us.

Company Overview

UNIMODULAR MASS TRANSPORT INC is an active carrier operating under USDOT Number 2356965.

Total Trucks	4
Tractors Owned	0
Trailer Owned	0
Total Drivers	4
USDOT	2356965
MCS-150 Mileage Year	0

Company Contact Info

UNIMODULAR MASS TRANSPORT INC
7239 Eby Dr
Merriam, KS 66204
510-688-8891

- Add Your Trucking Company
- Add Freight Broker Company
- Find Loads/Trucks
- Search Trucking Companies
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- FMCSA Cerified Medical Examiners CDL Physical Exam Locations
- Freight Factoring Companies
- Need Cash Now! Ready to grow your Business?
- Truck Driving Schools

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This leads to the following extension.

Theorem, Aldous and L. (2007)

If ν is a unimodular probability measure on rooted graphs with a pair of weight functions, w_1 and w_2, with $w_1 \leq w_2$ a.s., then $\int p_t(o, o; w_1)\, d\nu \geq \int p_t(o, o; w_2)\, d\nu$ for all $t > 0$.

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We have $\Delta_1 \leq \Delta_2$, so for all $t > 0$, we have $-t\Delta_1 \geq -t\Delta_2$. Therefore

$$\int p_t(o, o; w_1)\, d\nu = \mathrm{Tr}_\nu\, e^{-t\Delta_1} \geq \mathrm{Tr}_\nu\, e^{-t\Delta_2} = \int p_t(o, o; w_2)\, d\nu\,. \qquad \square$$

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Open Question, Aldous and L. (2007)

If μ_1 and μ_2 are unimodular probability measures on rooted networks (G, o, w_i) such that there is a coupling (G, o, w_1, w_2) that is monotone, i.e., $w_1 \le w_2$ a.s., then is $\int p_t(o,o)\,d\mu_1 \ge \int p_t(o,o)\,d\mu_2$ for all $t > 0$?

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When there is a unimodular monotone coupling ν, we have $\mathrm{Tr}_{\mu_i}\, e^{-t\Delta} = \mathrm{Tr}_\nu\, e^{-t\Delta_i}$.

What is required is to compare two different unimodular measures, each with its own trace. We attempt to attack this problem via similar questions for finite graphs.

What is required is to compare two different unimodular measures, each with its own trace. We attempt to attack this problem via similar questions for finite graphs. Since finite graphs are unimodular, this appears impossible. But the essence is to compare two different traces, so we use two different graphs.

Say that G dominates H, written $G \succcurlyeq H$, if there is a probability measure on pairs $(X, Y) \in \mathsf{V}(G) \times \mathsf{V}(H)$ such that (i) the marginal distributions of X and Y are each uniform and

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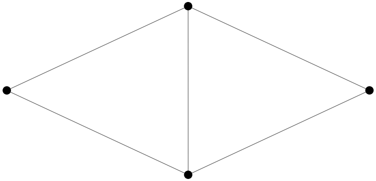
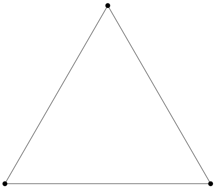
(choose X and Y independently).

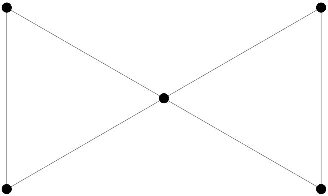
If there are weights on the edges, we require that the rooted isomorphism from (H, Y) to a subgraph of (G, X) is weight increasing.

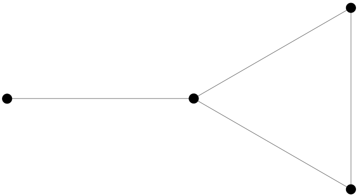




The graph on the left dominates a triangle.

The graph on the left does not dominate the graph on the right:





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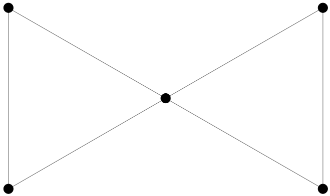
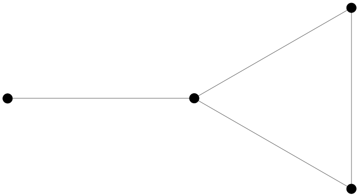


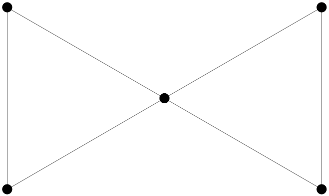


An edge fractionally tiles the graph on the left and tiles the graph on the right.

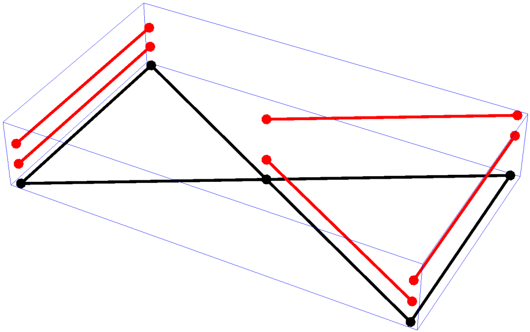
H fractionally tiles G if there is an integer number of copies of H in G such that each vertex of G is covered the same number of times by these copies of H.

If that latter number is 1, then H tiles G.

G:



If H fractionally tiles G, then $G \succcurlyeq H$:



More info

Open Question, L. (2017)

If $G \succcurlyeq H$, then does continuous-time simple random walk satisfy

$$\frac{1}{|\mathsf{V}(G)|} \sum_{x \in \mathsf{V}(G)} p_t(x, x; G) \leq \frac{1}{|\mathsf{V}(H)|} \sum_{x \in \mathsf{V}(H)} p_t(x, x; H)$$

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Theorem, L. (2017)

This inequality holds if H fractionally tiles G.

Theorem, L. (2017)

▸ More info

Let G be a finite graph with positive weights w on its edges. Suppose that H_i is a subgraph of G with positive weights w_i on its edges for $i = 1, \ldots, k$ with the following two properties:

Then for all $t > 0$, we have

$$\frac{1}{|\mathsf{V}(G)|} \sum_{x \in \mathsf{V}(G)} p_t(x; G) \leq \frac{1}{\sum_{j=1}^{k} |\mathsf{V}(H_j)|} \sum_{i=1}^{k} \sum_{x \in \mathsf{V}(H_i)} p_t(x; H_i) .$$

Theorem, L. (2017)

▸ More info

Let G be a finite graph with positive weights w on its edges. Suppose that H_i is a subgraph of G with positive weights w_i on its edges for $i = 1, \ldots, k$ with the following two properties:

- there is a constant m such that for every $x \in \mathsf{V}(G)$,

$$\left|\left\{i\,;\; x \in \mathsf{V}(H_i)\right\}\right| = m\,,$$

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$$\left|\left\{i\,;\ x \in \mathsf{V}(H_i)\right\}\right| = m\,,$$

- for every $e \in \mathsf{E}(G)$,

$$w(e) \geq \frac{1}{m} \sum_{i\,;\ e \in \mathsf{E}(H_i)} w_i(e)\,.$$

Then for all $t > 0$, we have

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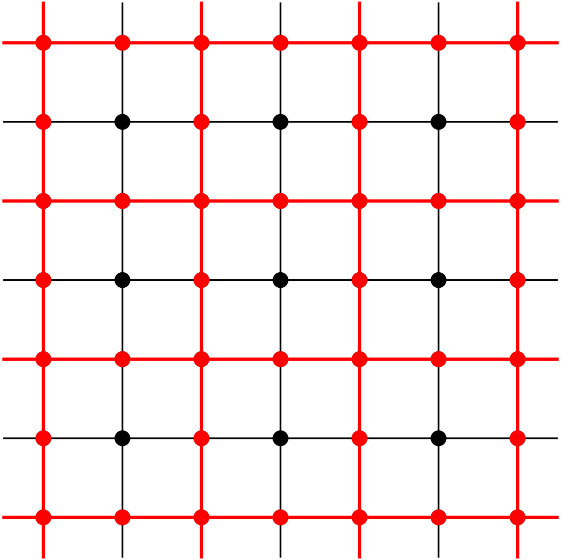
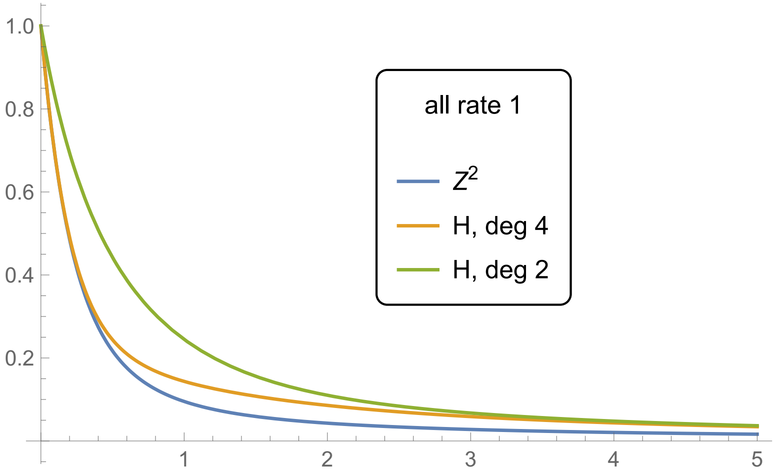
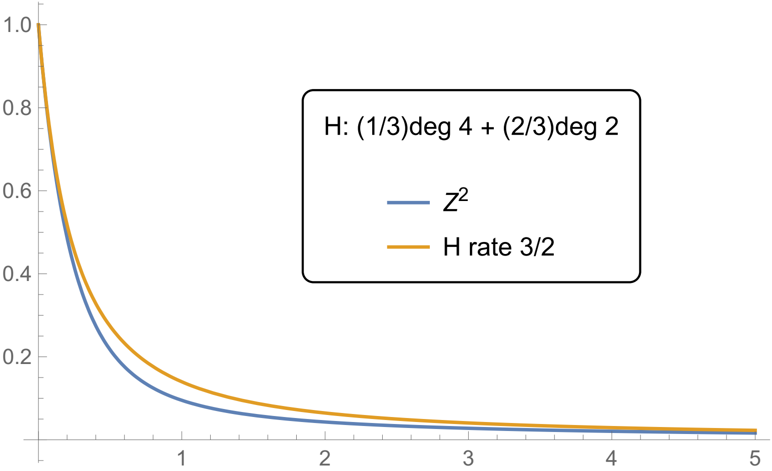


The square lattice $\mathbb{Z}^2$ and the subgraph *H* formed by deleting every vertex both of whose coordinates are odd.



Continuous-time simple random walk on each graph, where edges are crossed at rate 1.



$$p_t\left((0,0);\mathbb{Z}^2\right) \leq \frac{1}{3}p_{3t/2}\left((0,0);H\right) + \frac{2}{3}p_{3t/2}\left((0,1);H\right).$$

Theorem, L. (2018)

Let G be a unimodular transitive graph and H be a random subgraph of G with edge weights w_H such that the law of (H, w_H) is $\mathrm{Aut}(G)$-invariant. If

$$\forall e \sim o \in \mathsf{V}(G) \quad \mathbf{E}\big[w_H(e) \bigm| e \in H\big]\, \mathbf{P}[e \in H \mid o \in H] \le 1\,,$$

then continuous-time simple random walk on G and the continuous-time network random walk on (H, w_H) satisfy

$$p_t(o; G) \le \mathbf{E}\big[p_t(o; H, w_H) \bigm| o \in H\big]\,.$$

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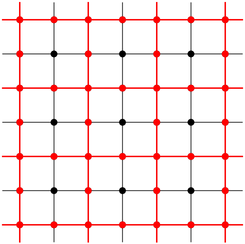

Theorem, L. (2018), restated

If

$$\forall e \sim o \in \mathsf{V}(G) \quad \mathbf{E}\big[w_H(e) \bigm| e \in H\big]\, \mathbf{P}[e \in H \mid o \in H] \le 1\,,$$

then

$$p_t(o; G) \le \mathbf{E}\big[p_t(o; H, w_H) \bigm| o \in H\big]\,.$$

For example, suppose that G is the usual nearest-neighbor graph on $\mathbb{Z}^d$ ($d \ge 2$) and H is the infinite cluster of a supercritical Bernoulli percolation on G. Let $\delta := \mathbf{E}\big[\deg_H(o) \bigm| o \in H\big]/(2d)$. Then

$$\forall t \ge 0 \quad p_t(o; \mathbb{Z}^d) \le \mathbf{E}\big[p_{t/\delta}(o; H) \bigm| o \in H\big]\,.$$

This is obtained by using $w_H \equiv 1/\delta$. The above inequality is false for any smaller value of δ.

Theorem, L. (2018)

Let $\mathbf{P}$ be a unimodular probability measure on rooted networks (G, o) with positive weights w_G on its edges and with a percolation subgraph H of G with positive weights w_H on its edges. Let $\mathbf{P}_{(G,o)}$ denote the conditional law of H given (G, o). Assume that $\alpha := \mathbf{P}_{(G,o)}\big[o \in \mathsf{V}(H)\big] > 0$ is a constant $\mathbf{P}$-a.s. If $\mathbf{P}$-a.s. whenever $e \in \mathsf{E}(G)$ is adjacent to o,

$$\mathbf{E}_{(G,o)}\big[w_H(e) \bigm| e \in \mathsf{E}(H)\big]\, \mathbf{P}_{(G,o)}\big[e \in \mathsf{E}(H) \bigm| o \in \mathsf{V}(H)\big] \le w_G(e)\,,$$

then $\mathbf{E}\big[p_t(o; G)\big] \le \mathbf{E}\big[p_t(o; H) \bigm| o \in \mathsf{V}(H)\big]$.

Example

For example, let (G, o) be any unimodular random rooted graph and consider Bernoulli(α) site percolation on G. Let H be the induced subgraph. Then

$$\forall t \geq 0 \quad \mathbf{E}\big[p_t(o; G)\big] \leq \mathbf{E}\big[p_{t/\alpha}(o; H) \bigm| o \in H\big] .$$

This is obtained by using $w_H \equiv 1/\alpha$. This is sharp: for all $\beta < \alpha$, there is some t such that $\mathbf{E}\big[p_t(o; G)\big] > \mathbf{E}\big[p_{t/\beta}(o; H) \bigm| o \in H\big]$.

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Conjecture, L. (2017)

If $G \succcurlyeq H$, then

$$\tau(G)^{1/|\mathsf{V}(G)|} \geq \tau(H)^{1/|\mathsf{V}(H)|}.$$

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This holds if either G or H is transitive, or [J. Kahn] if H fractionally tiles G.

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This holds if either G or H is transitive, or [J. Kahn] if H fractionally tiles G.

Note that $\log \tau(G)^{1/\mathsf{V}(G)} = \mathsf{V}(G)^{-1} \operatorname{tr} \log \Delta_o$.

An infinitary version of the theorem holds. Define the tree entropy of μ as

$$\mathbf{h}(\mu) := \operatorname{Tr}_\mu \log \Delta .$$

Theorem, L. (2005, 2010)

If $\mu_1 \neq \mu_2$ are unimodular probability measures on rooted weighted connected infinite graphs that both satisfy

$$\int \log w_G(o)\, d\mu_i(G, o) \in [-\infty, \infty)$$

and μ_1 stochastically dominates μ_2, then $\mathbf{h}(\mu_1) > \mathbf{h}(\mu_2)$.

This depends on another representation for tree entropy:

Theorem, L. (2010)

If μ is a unimodular probability measure on rooted weighted infinite graphs that satisfies

$$\int \log w_G(o)\, d\mu(G, o) \in [-\infty, \infty)\,,$$

then

$$\mathbf{h}(\mu) = \int_0^\infty \left(\frac{s}{1+s^2} - \int R(G, o, s)\, d\mu(G, o) \right) ds\,.$$

Here, given a network G, one of its vertices x, and a positive number s, let $R(G, x, s)$ be the effective resistance between x and ∞ in the network G^s formed from G by adding an edge of conductance s between every vertex and ∞, where ∞ is also a vertex of G^s.

This allows us to use Rayleigh's monotonicity principle pointwise.

Supplementary Material

If H is transitive, then $G \succcurlyeq H$ iff every vertex of G belongs to a copy of H. If G is transitive, then $G \succcurlyeq H$ iff G contains a copy of H. In both cases, the independent coupling of roots works.

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If H fractionally tiles G, then $G \succcurlyeq H$. Conversely, if G is transitive and dominates H, then H fractionally tiles G.

Back to basics

A similar proof shows that if f is any decreasing convex function and H fractionally tiles G, then

$$\operatorname{Tr} f(\Delta_G) \leq \operatorname{Tr} f(\Delta_H)\,.$$

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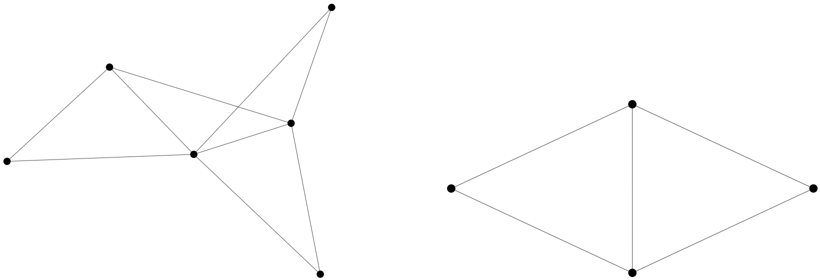
However, it is *not* true that this inequality holds whenever $G \succcurlyeq H$; a counter-example is provided by taking $f(s) := (4 - s)^+$ for these graphs:



The graph G on the left dominates the graph H on the right.

Back to theorem

Let W and V be finite sets. Suppose that $\Phi \colon \mathcal{L}\big(\ell^2(W)\big) \to \mathcal{L}\big(\ell^2(V)\big)$ is a positive unital linear map, i.e., a linear map that takes positive operators to positive operators and takes the identity map to the identity map. Antezana, Massey, and Stojanoff (2007) proved that

$$\operatorname{Tr} f\big(\Phi(A)\big) \leq \operatorname{Tr} \Phi\big(f(A)\big)$$

for self-adjoint operators $A \in \mathcal{L}\big(\ell^2(W)\big)$ and functions $f \colon \mathbb{R} \to \mathbb{R}$ that are convex on the convex hull of the spectrum of A.