Change Intolerance in Spanning Forests

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Abstract. Call a percolation process on edges of a graph **change intolerant** if the status of each edge is almost surely determined by the status of the other edges. We give necessary and sufficient conditions for change intolerance of the wired spanning forest when the underlying graph is a spherically symmetric tree.

§1. Introduction.

An important pair of probability measures on graphs that has been studied for the last 10 years is that of free and wired uniform spanning forests. These measures have intimate connections to a number of other probabilistic models, such as random walks, domino tilings, random-cluster measures, and Brownian motion, as well as to some topics outside of or only tangentially related to probability theory, such as harmonic Dirichlet functions and ℓ^2-Betti numbers. Completely separately, notions of insertion and deletion tolerance, also known as finite energy, have been important for decades in the study of percolation and other models in statistical physics. Normally, one would expect spanning forest measures to be neither insertion nor deletion tolerant, but we shall see that this is not always the case.

We now recall briefly the definitions of these spanning forest measures. A comprehensive study of uniform spanning forests appears in Benjamini, Lyons, Peres, and Schramm (2001), hereinafter referred to as [BLPS01]; a survey appears in Lyons (1998). The origin of these measures was the proof by Pemantle (1991) of the following conjecture of Lyons: If an infinite graph G is exhausted by finite subgraphs G_n, then the uniform distributions on the spanning trees of G_n converge weakly to a measure supported on spanning forests of G; by "spanning forest", we mean a subgraph without cycles that contains every vertex. We call this limit measure the **free uniform spanning forest** measure (FSF), since there

is another natural construction where the exterior of G_n is identified to a single vertex ("wired") before passing to the limit. This second construction, which we call the **wired uniform spanning forest** measure (WSF), was implicit in Pemantle's paper and was made explicit by Häggström (1995). We shall be concerned primarily with WSF in this paper.

We next define the tolerance notions that we shall investigate. Let $G = (\mathsf{V}, \mathsf{E})$ be a connected graph, where V is the vertex set and E is the edge set. Let μ be a probability measure on $2^{\mathsf{E}} := \{0,1\}^{\mathsf{E}}$. We think of an element $\omega \in \{0,1\}^{\mathsf{E}}$ as the subset of E where $\omega(e) = 1$, and refer to ω as a **configuration**. The **status** of an edge e in a configuration ω means simply $\omega(e)$. For a subset K of E, let $\mathcal{F}(K)$ denote the μ-completion of the σ-field on 2^{E} determined by the coordinates $e \in K$. We call μ **change intolerant** if the status of each edge is determined by the rest of the configuration, i.e., if for all $e \in \mathsf{E}$, we have $\mathcal{F}(\{e\}) \subset \mathcal{F}(\mathsf{E} \setminus \{e\})$. A useful equivalent definition is that for all $e \in \mathsf{E}$, we have $\mu_e \perp \mu_{\neg e}$, where μ_e denotes μ conditioned on $\omega(e) = 1$ and restricted to $\mathcal{F}(\mathsf{E} \setminus \{e\})$, and $\mu_{\neg e}$ denotes μ conditioned on $\omega(e) = 0$ and restricted to $\mathcal{F}(\mathsf{E} \setminus \{e\})$. One way that a measure μ may be far from change intolerant is to be **insertion tolerant**, meaning that for each e, we have $\mu[\omega(e) = 1 \mid \mathcal{F}(\mathsf{E} \setminus \{e\})] > 0$ a.s., or, equivalently, $\mu_{\neg e} \ll \mu_e$. Another way that a measure μ may be far from change intolerant is to be **deletion tolerant**, meaning that for each e, we have $\mu[\omega(e) = 0 \mid \mathcal{F}(\mathsf{E} \setminus \{e\})] > 0$ a.s., or, equivalently, $\mu_e \ll \mu_{\neg e}$. Since all trees are infinite WSF-a.s., it is never the case that WSF is deletion tolerant. However, it is reasonable to ask whether WSF is **essentially deletion tolerant**, meaning that for each e, we have $\mu[\omega(e) = 0 \mid \mathcal{F}(\mathsf{E} \setminus \{e\})] > 0$ a.s. on the event that both endpoints of e belong to infinite components of $\omega \setminus \{e\}$.

We shall show that there are trees T for which WSF on T is change intolerant, as well as T for which WSF is insertion tolerant and essentially deletion tolerant. In fact, we shall give a simple necessary and sufficient condition for a spherically symmetric tree T to have a change-intolerant WSF. Although it cannot happen for a spherically symmetric tree, there are trees on which WSF is change intolerant at certain edges and insertion tolerant at others: simply take two trees, one of which is change intolerant, the other of which is insertion tolerant, and join them by identifying a vertex of one with a vertex of the other.

There are many examples of graphs for which WSF is change intolerant. Indeed, it is at first surprising that there should be any graphs for which WSF is not change intolerant. It would be interesting to know the situation on $T \times \mathbb{Z}$ when T is a tree on which WSF is not change intolerant.

EXAMPLE 1.1. Let G be a finite graph. Then the uniform spanning tree is change intoler-

ant, since the number of edges in a spanning tree is constant and equal to $|\mathsf{V}| - 1$.

For the next example we need some definitions. An infinite path in a tree that starts at any vertex and does not backtrack is called a **ray**. Two rays are **equivalent** if they have infinitely many vertices in common. An equivalence class of rays is called an **end**.

EXAMPLE 1.2. Let G be the Cayley graph of a group which is not a finite extension of $\mathbb{Z}$. Then by Theorem 10.1 of [BLPS01], every component tree in the WSF has exactly one end. Since adding an edge e to a configuration will either form a cycle or connect two components, the resulting configuration would not be a forest with trees having only one end each. Likewise, deleting an edge would necessarily leave a finite component. Thus, WSF is change intolerant.

EXAMPLE 1.3. Let G be two copies of $\mathbb{Z}^3$ attached by an additional edge at their respective origins. By Theorem 9.4 of [BLPS01], the WSF has two components a.s., each with one end. Hence adding an edge to a configuration would either form a cycle or yield only one component, which cannot happen. Similarly, deleting an edge would leave a finite component. Thus, WSF is change intolerant.

More generally, WSF is change intolerant on any graph for which the WSF has a finite number of components (since the number of components is an a.s. constant by Theorem 9.4 of [BLPS01]) or for which each component has the same a.s. constant finite number of ends. Also, recall that the WSF is a single tree a.s. on any graph for which simple random walk is recurrent ([BLPS01], Proposition 5.6).

In order to state our principal theorem, we need some notation. Given a tree T with root o and given $k \in \mathbb{N}$, the **level** T_k is the set of vertices of T at distance k from o. The tree is called **spherically symmetric** if for each k, all vertices in T_k have the same number of children (the same degree). It is well known that simple random walk on T is transient iff $\sum_m 1/|T_m| < \infty$.

THEOREM 1.4. *Let T be a spherically symmetric transient tree and $L_n := \sum_{m>n} 1/|T_m|$. Consider the series*

$$\sum_{n \geq 1} \frac{1}{|T_n|^2 L_n L_{n-1}} \,. \tag{1.1}$$

(i) *If this series diverges, then the WSF on T is change intolerant.*

(ii) *If this series converges, then the WSF on T is insertion tolerant.*

(iii) *If this series converges and T has bounded degree, then the WSF on T is essentially deletion tolerant.*

Note that the series (1.1) converges if $|T_n|/n^\gamma$ is bounded above and below by positive constants for some $\gamma > 1$, while the series diverges if $\liminf_{n\to\infty} |T_{n+1}|/|T_n| \in (1,\infty)$. Statement (iii) is not vacuous because when the series converges, a.s. all components have more than one end by Corollary 11.4 of [BLPS01], which is reproduced as Corollary 2.3 below.

In the next section, we shall use the analysis of the WSF on trees given by [BLPS01] to reduce Theorem 1.4 to questions involving random walk and percolation (Lemma 2.1). In Section 3, motivated by Lyons, Pemantle, and Peres (1995), we introduce the martingales needed to analyze these new questions and prove Theorem 1.4. (Actually, we extend Theorem 1.4 to networks where the conductance of an edge can depend on its distance to the root, as explained in Section 2.) In the last section we give an application to determinantal probability measures and answer a question of Lyons (2002).

§2. Preliminary Reduction.

If x and y are endpoints of an edge $[x,y]$, we write $x \sim y$ and call x, y **neighbors**. A **network** is a pair (G, C), where G is a connected graph with at least two vertices and C is a function from the unoriented edges of G to the positive reals. The quantity $C(e)$ is called the **conductance** of e. The network is finite if G is finite. We assume that $\sum_{y\sim x} C([x,y])$ is finite for all $x \in \mathsf{V}$. The **network random walk** on (G, C) is the nearest-neighbor random walk on G with transition probabilities proportional to the conductances. The most natural network on G is the default network $(G, \mathbf{1})$, for which the network random walk is simple random walk. For each edge $e \in \mathsf{E}$, the quantity $R(e) := 1/C(e)$ is the **resistance** of e. For general networks, we use a measure on spanning trees adapted to the conductances. That is, for a finite network, choose a spanning tree proportional to its weight, where the **weight** of a spanning tree T is $\prod_{e\in T} C(e)$. The proof in Pemantle (1991) of the existence of a limit of such measures when an infinite graph is approximated by finite subgraphs, either wired or not, extends to general networks. Explicit details are given in [BLPS01].

Given a tree T, choose arbitrarily a vertex o of T, which we call the **root**. If x is a vertex of T other than the root, write $\hat{x}$ for the **parent** of x, i.e., the next vertex after x on the shortest path from x to the root. We also call x a **child** of $\hat{x}$. If x and y are two vertices, $x \wedge y$ denotes the most recent common ancestor to x and y.

Let (T, C) be a transient network whose underlying graph is a tree. For each vertex x, consider an independent network random walk Z_x on T starting at x and stopped when it reaches $\hat{x}$, if ever. (Note that Z_o is never stopped.) Write ζ_x for the set of edges crossed

an odd number of times by Z_x and Ξ for the set of vertices x such that for all y, if x is an endpoint of an edge in ζ_y and ζ_y is infinite, then $x = y$. Then Section 11 of [BLPS01] shows that the law of $\bigcup_{x \in \Xi} \zeta_x$ is WSF. In particular, if the deletion of an edge e from T leaves a recurrent component, then e belongs to the wired spanning forest a.s. Therefore, we shall henceforth restrict consideration to networks that are **fully transient**, meaning that there are no edges whose deletion leaves a recurrent component.

Note that ζ_o is a ray starting at o. The law of ζ_o is denoted Ray_o. The law of the connected component of o in $\big\{[x, \hat{x}]; \; x \neq o, \zeta_x = \{[x, \hat{x}]\}\big\}$ is denoted Perc_o (since the latter is an independent percolation on T). Write $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$ for the law of $\xi \cup \omega$ when (ξ, ω) has the law $\mathsf{Ray}_o \otimes \mathsf{Perc}_o$. By the description above, the component $\tau(o)$ of o in the WSF has the same law as $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$.

For neighbors x, y in T, write $T_{x,y}$ for the component of x in $T \setminus [x, y]$. We take the root of $T_{x,y}$ to be x. Let $\mathsf{Ray}_{x,y}$ and $\mathsf{Perc}_{x,y}$ denote Ray_x and Perc_x, respectively, on $T_{x,y}$. Let $\tau(x, y) := \tau(x) \cap T_{x,y}$, where $\tau(x)$ is the component of x in the WSF on all of T. Let $\alpha_{x,y}$ be the probability that a network random walk on T started at x is in $T_{x,y}$ from some time onwards. The above alternative description of the WSF shows that the $\mathsf{WSF}_{\neg[x,y]}$-law of $\big(\tau(x, y), \tau(y, x)\big)$ is

$$\left(\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y}\right) \otimes \left(\mathsf{Ray}_{y,x} \oplus \mathsf{Perc}_{y,x}\right), \tag{2.1}$$

while the $\mathsf{WSF}_{[x,y]}$-law of $\big(\tau(x, y), \tau(y, x)\big)$ is the mixture

$$\alpha_{x,y}(\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y}) \otimes \mathsf{Perc}_{y,x} + (1 - \alpha_{x,y})\mathsf{Perc}_{x,y} \otimes (\mathsf{Ray}_{y,x} \oplus \mathsf{Perc}_{y,x}). \tag{2.2}$$

Note that $\alpha_{x,y} > 0$ since $T_{x,y}$ is transient and $\alpha_{x,y} < 1$ since $T_{y,x}$ is transient. Furthermore, the configuration of the WSF on the edges that do not touch either $\tau(x)$ or $\tau(y)$ is independent of the status of $[x, y]$ (given $\tau(x) \cup \tau(y)$). We shall term the event $\{\omega \text{ is infinite}\}$ **survival** by analogy with branching processes. This gives us the following criteria, where $A_{x,y} := \mathsf{Perc}_{x,y}(\text{survival})$ and $\mathsf{Perc}^*_{x,y}$ denotes $\mathsf{Perc}_{x,y}$ conditioned on survival when $A_{x,y} > 0$.

LEMMA 2.1. *Let (T, C) be a fully transient network on a tree.*
  (i) WSF *is change intolerant iff for all neighbors x, y,*

$$\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y} \perp \mathsf{Perc}_{x,y}.$$

  (ii) WSF *is insertion tolerant iff for all neighbors x, y,*

$$\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y} \ll \mathsf{Perc}_{x,y}.$$

(iii) WSF *is essentially deletion tolerant iff for all neighbors* x, y,

$$A_{x,y} > 0 \quad and \quad \mathsf{Perc}^*_{x,y} \ll \mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y} \, .$$

Proof. This is a straightforward comparison of (2.1) and (2.2), and so we make only a few remarks on the proof. For example, change intolerance is equivalent to the mutual singularity of (2.1) and (2.2). Each term of the mixture (2.2) must be singular to (2.1). Since x and y may be switched, we may consider only the second term. Since $\alpha_{x,y} < 1$, this singularity is

$$(\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y}) \otimes (\mathsf{Ray}_{y,x} \oplus \mathsf{Perc}_{y,x}) \perp \mathsf{Perc}_{x,y} \otimes (\mathsf{Ray}_{y,x} \oplus \mathsf{Perc}_{y,x}) \, ,$$

which is the same as $\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y} \perp \mathsf{Perc}_{x,y}$.

In (iii), the word "essentially" implies that we are concerned only with the event that both x and y belong to infinite components when $[x, y]$ is removed from the wired spanning forest. This condition leads therefore to consideration of $\mathsf{Perc}^*_{x,y}$. ∎

Let $\mathcal{P}(T)$ denote the set of unit flows on T from o to infinity, i.e., the set of nonnegative functions θ on the vertices of T such that $\theta(o) = 1$ and for each vertex x, the sum of $\theta(y)$ over all children y of x equals $\theta(x)$. Let $h(x)$ be the probability that a network random walk starting at x ever visits o. Theorem 11.1 of [BLPS01] contains the following information:

THEOREM 2.2. *Let* (T, C) *be a transient network on a tree. If for all* $\theta \in \mathcal{P}(T)$, *the sum*

$$\sum_{x \neq o} \theta(x)^2 [h(x)^{-1} - h(\hat{x})^{-1}] \tag{2.3}$$

diverges, then all components of the WSF *on* T *have one end a.s.; if this sum converges for some* $\theta \in \mathcal{P}(T)$, *then a.s. the* WSF *on* T *has components with more than one end.*

A network on a spherically symmetric tree is itself called **spherically symmetric** if for all k, every edge connecting T_{k-1} with T_k has the same resistance, which we shall denote r_k. Corollary 11.4 of [BLPS01] specializes Theorem 2.2 to spherically symmetric trees:

COROLLARY 2.3. *Let* (T, C) *be a spherically symmetric network on a tree. Assume that the resulting network is transient, i.e.,* $\sum_m r_m / |T_m| < \infty$. *Denote* $L_n := \sum_{m > n} r_m / |T_m|$. *If the sum*

$$\sum_{n \geq 1} \frac{r_n}{|T_n|^2 L_n L_{n-1}} \tag{2.4}$$

diverges, then all components of the WSF on T have one end a.s.; if this series converges, then a.s. all components of the WSF on T have uncountably many ends.

More specifically, the sum (2.4) is the minimum of the sum (2.3) over $\mathcal{P}(T)$; the minimum is achieved for the equally splitting flow $\theta(x) := |T_{|x|}|^{-1}$, where $|x|$ denotes the distance of x to the root o.

§3. Martingales.

Write $I(x) := \mathsf{Ray}_o(\{\xi;\ x \in \xi\})$ for the harmonic measure of the set of rays that pass through x. For any subtree t of T containing o and any $x \in t$, write $\nu[t, x]$ for the probability that when (ξ, ω) has the law $\mathsf{Ray}_o \otimes \mathsf{Perc}_o$, we have $x \in \xi$ and the first $|x|$ levels of $\xi \cup \omega$ agree with those of t. Since

$$h(x) = \mathsf{Perc}_o\big[x \in \omega\big],$$

we have

$$\nu[t, x] = \frac{I(x)}{h(x)}\mathsf{Perc}_o([t]_{|x|}),$$

where $[t]_n$ denotes the set of subtrees of T (rooted at o) whose first n levels agree with those of t. Therefore, if T is transient,

$$(\mathsf{Ray}_o \oplus \mathsf{Perc}_o)([t]_n) = \sum_{x \in t_n} \nu[t, x] = W_n(t)\mathsf{Perc}_o([t]_n), \tag{3.1}$$

where

$$W_n(t) := \sum_{x \in t_n} \frac{I(x)}{h(x)}.$$

Let $\mathcal{F}_n$ be the σ-algebra generated by the statuses of the edges in T whose endpoints are at distance at most n from o. Equation (3.1) says that W_n is the Radon-Nikodým derivative of $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$ restricted to $\mathcal{F}_n$ with respect to Perc_o restricted to $\mathcal{F}_n$. Therefore, the sequence $\langle W_n, \mathcal{F}_n \rangle$ is a martingale with respect to Perc_o. Hence, $\langle W_n^2, \mathcal{F}_n \rangle$ is a submartingale with respect to Perc_o, which is the same thing as saying that $\langle W_n, \mathcal{F}_n \rangle$ is a submartingale with respect to $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$. Write $W(t) := \limsup_{n \to \infty} W_n(t)$. We shall use the following general lemma (see Durrett (1996), Chapter 4, Theorem 3.3, p. 242, or Lyons (2003)).

Lemma 3.1. *Let κ be a finite measure and λ a probability measure on a σ-field $\mathcal{G}$. Suppose that $\mathcal{G}_n$ are increasing sub-σ-fields whose union generates $\mathcal{G}$ and that $(\kappa{\restriction}\mathcal{G}_n)$ is absolutely continuous with respect to $(\lambda{\restriction}\mathcal{G}_n)$ with Radon-Nikodým derivative X_n. Set $X := \limsup_{n\to\infty} X_n$. Then*

$$\kappa \ll \lambda \iff X < \infty \quad \kappa\text{-a.e.}$$

and

$$\kappa \perp \lambda \iff X = \infty \quad \kappa\text{-a.e.}$$

Write $A_o := \mathsf{Perc}_o(\text{survival})$ and, when $A_o > 0$, let Perc_o^* be Perc_o conditioned on survival.

Lemma 3.2. *If (T, C) is fully transient, then*

$$\mathsf{Ray}_o \oplus \mathsf{Perc}_o \perp \mathsf{Perc}_o \iff W = \infty \quad \mathsf{Ray}_o \oplus \mathsf{Perc}_o\text{-a.s.}$$

and

$$\mathsf{Ray}_o \oplus \mathsf{Perc}_o \ll \mathsf{Perc}_o \iff W < \infty \quad \mathsf{Ray}_o \oplus \mathsf{Perc}_o\text{-a.s.} \tag{3.2}$$

If also $A_o > 0$, then

$$\mathsf{Perc}_o^* \ll \mathsf{Ray}_o \oplus \mathsf{Perc}_o \iff W > 0 \quad \mathsf{Perc}_o^*\text{-a.s.} \tag{3.3}$$

Proof. The first two statements follow from Lemma 3.1 applied to $\kappa := \mathsf{Ray}_o \oplus \mathsf{Perc}_o$, $\lambda := \mathsf{Perc}_o$, and $\mathcal{G}_n := \mathcal{F}_n$, since then $X_n = W_n$. The last statement follows from Lemma 3.1 applied to $\kappa := \mathsf{Perc}_o^*$, $\lambda := \mathsf{Ray}_o \oplus \mathsf{Perc}_o$, and $\mathcal{G}_n := \mathcal{F}_n$. Indeed, by (3.1), we have

$$W_n(t)^{-1}(\mathsf{Ray}_o \oplus \mathsf{Perc}_o)([t]_n) = \mathsf{Perc}_o([t]_n)$$

when $W_n(t) \neq 0$. Our assumption of full transience implies that $W_n(t) \neq 0$ for all infinite t. Therefore,

$$\mathsf{Perc}_o^*([t]_n) = (A_o W_n(t))^{-1}(\mathsf{Ray}_o \oplus \mathsf{Perc}_o)([t]_n)\,.$$

Thus, Lemma 3.1 applies with $X_n = (A_o W_n)^{-1}$. ∎

For $x, y \in T_n$, write $[x \mid y]$ for the set of edges between $x \wedge y$ and x. For $x \in T_n$, we have

$$(\mathsf{Ray}_o \oplus \mathsf{Perc}_o)[x \in t] = \sum_{y \in T_n} \mathsf{Ray}_o[y \in \xi]\mathsf{Perc}_o\big[[x \mid y] \subseteq \omega\big]$$

$$= \sum_{y \in T_n} I(y)\frac{h(x)}{h(x \wedge y)}\,.$$

Therefore

$$
\begin{aligned}
\int W_n \, d(\mathsf{Ray}_o \oplus \mathsf{Perc}_o) &= \int \sum_{x \in T_n} \frac{I(x)}{h(x)} \mathbf{1}_{\{x \in t\}} \, d(\mathsf{Ray}_o \oplus \mathsf{Perc}_o)(t) \\
&= \sum_{x \in T_n} \frac{I(x)}{h(x)} (\mathsf{Ray}_o \oplus \mathsf{Perc}_o)[x \in t] \\
&= \sum_{x,y \in T_n} \frac{I(x)I(y)}{h(x \wedge y)} \\
&= \sum_{u \in T, \, |u| \le n} \frac{1}{h(u)} \sum_{x,y \in T_n, x \wedge y = u} I(x)I(y) \\
&= \sum_{u \in T, \, |u| \le n} \frac{1}{h(u)} \left\{ I(u)^2 - \sum_{u = \hat{v}, |v| \le n} I(v)^2 \right\} \\
&= 1 + \sum_{u \in T, \, 0 < |u| \le n} I(u)^2 \left\{ \frac{1}{h(u)} - \frac{1}{h(\hat{u})} \right\}.
\end{aligned}
\tag{3.4}
$$

Note that these are the same summands that appear in (2.3). Since $I(\cdot)$ is the equally splitting flow if T is spherically symmetric, it follows that for a spherically symmetric tree, (3.4) is bounded (in n) iff (2.4) converges.

THEOREM 3.3. *Let (T, C) be a spherically symmetric transient network on a tree.*
  (i) *If the series (2.4) diverges, then the* WSF *on T is change intolerant.*
  (ii) *If this series converges, then the* WSF *on T is insertion tolerant.*
  (iii) *If this series converges and T has bounded degree, then the* WSF *on T is essentially deletion tolerant.*

Proof. Note that the spherical symmetry implies that (T, C) is fully transient.

(i) In this case, Corollary 2.3 shows that each component of the WSF has only one end a.s., whence no edge can be either inserted nor deleted.

(ii) When (2.4) converges, (3.4) shows that $\langle W_n \rangle$ is bounded in expectation with respect to $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$. Since $\langle W_n \rangle$ is a submartingale, it follows that $W < \infty$ a.s. with respect to $\mathsf{Ray}_o \oplus \mathsf{Perc}_o$. In view of (3.2), we obtain $\mathsf{Ray}_o \oplus \mathsf{Perc}_o \ll \mathsf{Perc}_o$. In particular, $A_o > 0$, which we shall use in proving (iii). Now for any neighbors $x, y \in T$, the subtree $T_{x,y}$ is composed of a finite collection of spherically symmetric trees attached at the leaves of a finite tree. Thus, a similar argument shows that for all neighbors x, y, we have $\mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y} \ll \mathsf{Perc}_{x,y}$. Therefore, the WSF on T is insertion tolerant by Lemma 2.1.

(iii) Another way to regard Perc_o is as a branching process in a varying environment (BPVE). With this view, $\langle W_n, \mathcal{F}_n \rangle$ is the usual martingale in the theory of branching

processes. Theorem 4.14 of Lyons (1992) shows that when the offspring distribution of a BPVE is uniformly bounded, then $W > 0$ a.s. given the event of survival. Since $A_o > 0$ (proved for part (ii)), this proves that when T has bounded degree, $W > 0$ a.s. with respect to Perc_o^*. Because of (3.3), we obtain $\mathsf{Perc}_o^* \ll \mathsf{Ray}_o \oplus \mathsf{Perc}_o$. As for part (ii), a similar argument shows that for all neighbors x, y, we have $\mathsf{Perc}_{x,y}^* \ll \mathsf{Ray}_{x,y} \oplus \mathsf{Perc}_{x,y}$. Therefore, the WSF on T is essentially deletion tolerant by Lemma 2.1. ∎

§4. Singularity of Determinantal Probabilities.

Let (G, C) be a finite or infinite network. For this section, we choose an orientation for each edge. Identify each $e \in \mathsf{E}$ with the corresponding unit vector $\mathbf{1}_e$ in $\ell^2(\mathsf{E})$. Given two neighbors x, y, let

$$\eta^{\langle x,y \rangle} := \begin{cases} \langle x, y \rangle & \text{if } \langle x, y \rangle \in \mathsf{E} \\ -\langle x, y \rangle & \text{if } \langle y, x \rangle \in \mathsf{E}. \end{cases}$$

Let ★ denote the closure in $\ell^2(\mathsf{E})$ of the linear span of the **stars** $\sum_{y \sim x} \sqrt{C([x,y])} \eta^{\langle x,y \rangle}$ ($x \in \mathsf{V}(G)$). For a cycle of vertices $x_0, x_1, \ldots, x_n = x_0$, the function

$$\sum_{i=0}^{n-1} \eta^{\langle x_i, x_{i+1} \rangle} / \sqrt{C([x_i, x_{i+1}])}$$

is called a **cycle**. Let ◇ be the closure of the linear span of the cycles. Since each star and cycle are orthogonal to each other, we have ★ ⊥ ◇.

Given any subspace $H \subseteq \ell^2(\mathsf{E})$, let P_H denote the orthogonal projection of $\ell^2(\mathsf{E})$ onto H, and let $P_H^\perp$ denote the orthogonal projection onto the orthogonal complement $H^\perp$ of H. The following result of [BLPS01] (Theorem 7.8 in an isomorphic form) extends the Transfer Current Theorem of Burton and Pemantle (1993):

THEOREM 4.1. *Given any network G and any distinct edges $e_1, \ldots, e_k \in G$, we have*

$$\mathsf{FSF}[\omega(e_1) = 1, \ldots, \omega(e_k) = 1] = \det[(P_\Diamond^\perp e_i, e_j)]_{1 \le i,j \le k}$$

and

$$\mathsf{WSF}[\omega(e_1) = 1, \ldots, \omega(e_k) = 1] = \det[(P_\bigstar e_i, e_j)]_{1 \le i,j \le k}.$$

Clearly, these formulas characterize FSF and WSF. In particular, as observed in [BLPS01], ★ ⊆ ◇$^\perp$, with equality iff $\mathsf{WSF} = \mathsf{FSF}$. Question 15.11 of [BLPS01] asks whether $\mathsf{WSF} \perp \mathsf{FSF}$ when the two measures are not equal; some cases where this is known to be true are stated there. This question remains open, but it suggested a more general possibility to Lyons (2002), which we may now show is false.

First, we give the more general context in which the question arose. Given any countable set E, identify each $e \in E$ with the corresponding unit vector $\mathbf{1}_e$ in $\ell^2(E)$. Given any closed subspace $H \subset \ell^2(E)$, there is a unique probability measure $\mathbf{P}^H$ on 2^E defined by

$$\mathbf{P}^H[\omega(e_1) = 1, \ldots, \omega(e_k) = 1] = \det[(P_H e_i, e_j)]_{1 \leq i,j \leq k}$$

for any set of distinct $e_1, \ldots, e_k \in E$; see Lyons (2002) and Daley and Vere-Jones (1988), Exercises 5.4.7–5.4.8. In case H is finite dimensional, then $\mathbf{P}^H$ is concentrated on subsets of E of cardinality equal to the dimension of H.

This suggests that in general, if $H_1 \subset H_2 \subset \ell^2(E)$ and $H_1 \neq H_2$, then $\mathbf{P}^{H_1} \perp \mathbf{P}^{H_2}$, the question asked in Lyons (2002). But this is false. To see how this follows from Theorem 1.4, we must consider the effect of conditioning on the measure WSF and its representation via determinants. This is done partly in [BLPS01] and fully in Lyons (2002). The result is that if we identify $\ell^2(\mathsf{E} \setminus \{e\})$ with $(\mathbb{R}e)^\perp \subset \ell^2(\mathsf{E})$, then $\mathsf{WSF}_e = \mathbf{P}^{H_1}$ and $\mathsf{WSF}_{\neg e} = \mathbf{P}^{H_2}$, where

$$H_1 := \bigstar \cap (\mathbb{R}e)^\perp \quad \text{and} \quad H_2 := (\bigstar + \mathbb{R}e) \cap (\mathbb{R}e)^\perp.$$

Thus, $H_1 \subseteq H_2$; furthermore, $H_1 \neq H_2$ as long as $e \notin \bigstar$, i.e., $\mathsf{WSF}[\omega(e) = 1] < 1$. This condition holds on a tree T when $e = [x, y]$ and both $T_{x,y}$ and $T_{y,x}$ are transient. Yet $\mathbf{P}^{H_1} \perp \mathbf{P}^{H_2}$ does not hold when WSF is insertion tolerant (at e), as it may be.

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