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A LOWER BOUND ON THE CESÀRO OPERATOR

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If the sequence $a = \langle a_n \rangle_0^\infty \in l^2$, i.e., $\|a\|^2 = \sum_{n=0}^\infty |a_n|^2 < \infty$, define Sa as the sequence of averages

$$\left\langle \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\rangle_{n=0}^{\infty}.$$

It follows easily from the Marcinkiewicz Interpolation Theorem that S is a bounded operator from l^2 to l^2; this can also be proved directly using the Cauchy-Buniakowski-Schwarz inequality [**1**]. S is known as the Cesàro operator. S is, of course, not bounded below, but the following property does hold, confirming a conjecture of Allen Shields and Sheldon Axler.

THEOREM. *If* $a_0 \geqslant a_1 \geqslant \cdots \geqslant 0$, $a = \langle a_n \rangle_0^\infty$, *then* $\|Sa\|^2 \geqslant \pi^2 \|a\|^2/6$, *with equality if and only if* $a_1 = a_2 = \cdots = 0$.

PROOF. If we expand the squares and group terms, we find that

$$\|Sa\|^2 = \frac{\pi^2}{6}\|a\|^2 - \sum_{n=1}^\infty \left(\sum_{k=1}^n \frac{1}{k^2} \right) a_n^2 + \sum_{n=1}^\infty \left(2 \sum_{k=n+1}^\infty \frac{1}{k^2} \right) \sum_{j=0}^{n-1} a_j a_n$$

$$\geqslant \frac{\pi^2}{6}\|a\|^2 + \sum_{n=1}^\infty \left(2n \sum_{k=n+1}^\infty \frac{1}{k^2} - \sum_{k=1}^n \frac{1}{k^2} \right) a_n^2.$$

Since

$$2n \sum_{k=n+1}^\infty \frac{1}{k^2} > 2n \int_{n+1}^\infty \frac{dx}{x^2} = \frac{2n}{n+1} \geqslant \sum_{k=1}^n \frac{1}{k^2},$$

the result follows.

REFERENCES

1. Arlen Brown, P. R. Halmos and A. L. Shields, *Cesàro operators*, Acta Sci. Math. (Szeged) **26** (1965), 125–137.

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