Monotonicity of Average Return Probabilities for Random Walks in Random Environments

by Russell Lyons

Abstract. We extend a result of Lyons (2016) from fractional tiling of finite graphs to a version for infinite random graphs. The most general result is as follows. Let $\mathbf{P}$ be a unimodular probability measure on rooted networks (G, o) with positive weights w_G on its edges and with a percolation subgraph H of G with positive weights w_H on its edges. Let $\mathbf{P}_{(G,o)}$ denote the conditional law of H given (G, o). Assume that $\alpha := \mathbf{P}_{(G,o)}\big[o \in \mathsf{V}(H)\big] > 0$ is a constant $\mathbf{P}$-a.s. We show that if $\mathbf{P}$-a.s. whenever $e \in \mathsf{E}(G)$ is adjacent to o,

$$\mathbf{E}_{(G,o)}\big[w_H(e) \bigm| e \in \mathsf{E}(H)\big]\, \mathbf{P}_{(G,o)}\big[e \in \mathsf{E}(H) \bigm| o \in \mathsf{V}(H)\big] \le w_G(e)\,,$$

then

$$\forall t > 0 \quad \mathbf{E}\big[p_t(o; G)\big] \le \mathbf{E}\big[p_t(o; H) \bigm| o \in \mathsf{V}(H)\big]\,.$$

§1. Introduction.

Associated to a graph with nonnegative numbers on its edges such that the sum of numbers of edges incident to each given vertex is finite, there is a continuous-time random walk that, when at a vertex x, crosses each edge e incident to x at rate equal to the number on e. When all rates equal 1, this is called continuous-time simple random walk. In general, the rate at which the random walk leaves x equals the sum of the numbers on the edges incident to x.

It is well known and easy to prove that every such (weighted) random walk has the property that the probability of return to the starting vertex is a decreasing function of time. Equivalently, the return probability at any fixed time decreases if all the rates are increased by the same factor. However, the return probability is *not* a decreasing function of the set of rates in general. Indeed, the behavior of the return probabilities is not intuitive; a small

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example is shown in Figure 1.1. Examples show that the return probability to a vertex x need not be monotonic even when rates are changed only on edges not incident to x. On the other hand, on a finite graph, the *average* of the return probabilities *is* decreasing in the rates, as shown by Benjamini and Schramm (see Theorem 3.1 of Heicklen and Hoffman (2005)). Recall that on a finite graph, the stationary measure for this continuous-time random walk is uniform on the vertices.

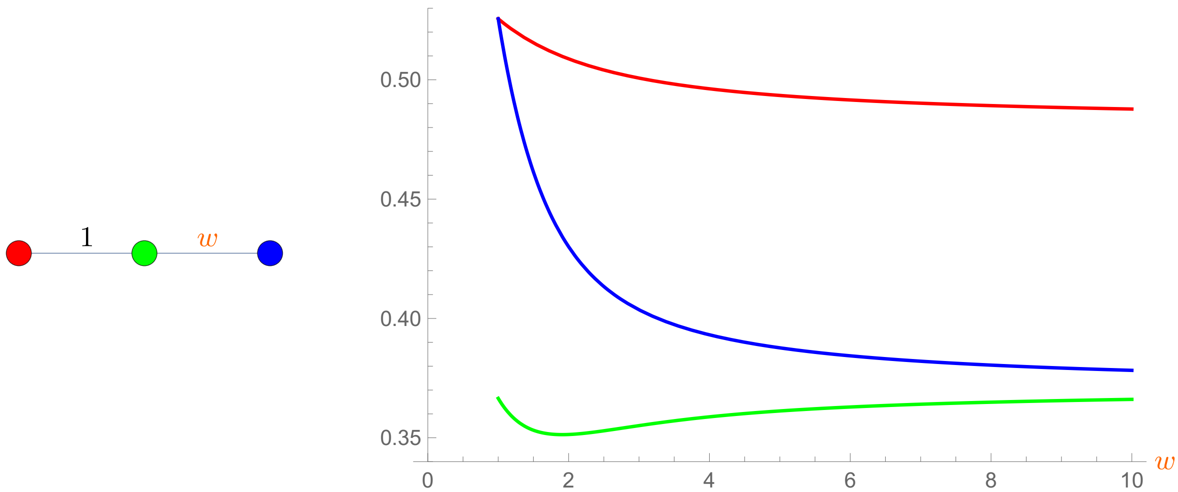

Figure 1.1. The return probabilities *at time 1* of the graph on the left as the rate w varies.

In Theorems 4.1 and 4.2 of Lyons (2016), we extended and strengthened the theorem of Benjamini and Schramm to the case of graphs of different sizes and even to the case of one graph G that is "fractionally tiled" by a set of subgraphs H_i with a certain condition on the edge weights of G and H_i. Our purpose here is to establish a version of those results for infinite graphs.

For a very simple example of our results here, consider the square lattice $\mathbb{Z}^2$ and the subgraph H formed by deleting every vertex both of whose coordinates are odd; see Figure 1.2. There are four subgraphs of $\mathbb{Z}^2$ that are isomorphic to H. Considering those four copies of H, we find that each vertex of $\mathbb{Z}^2$ is covered three times, once by a vertex of degree 4 and twice by a vertex of degree 2. An appropriate average return probability in H is thus 1/3 that of a vertex of degree 4 plus 2/3 that of a vertex of degree 2. Consider continuous-time simple random walk on each graph, where edges are crossed at rate 1; the return probabilities are shown in Figure 1.3. As illustrated in Figure 1.4, we have for all $t \ge 0$,

$$p_t\big((0,0);\mathbb{Z}^2\big) \le \frac{1}{3}p_{3t/2}\big((0,0);H\big) + \frac{2}{3}p_{3t/2}\big((0,1);H\big)\,.$$

Effectively, we have used rates 3/2 on every edge of H. This inequality follows from Corollary 2.3. It is sharp in the following sense: if $3t/2$ is replaced by βt for some $\beta > 3/2$,

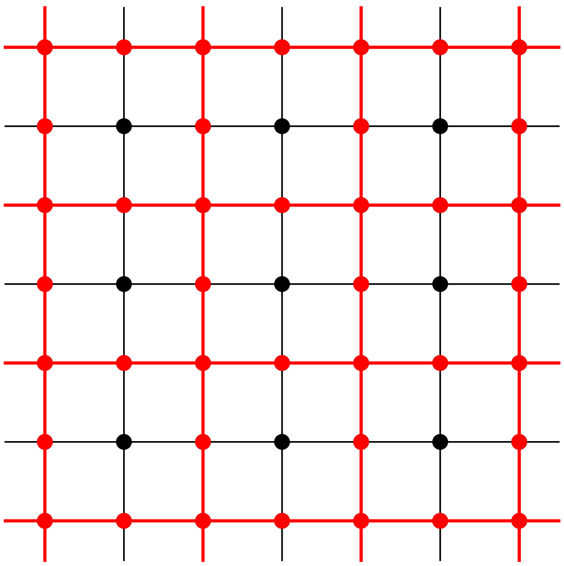

Figure 1.2. The square lattice $\mathbb{Z}^2$ and the subgraph H, thicker and in red, formed by deleting every vertex both of whose coordinates are odd.

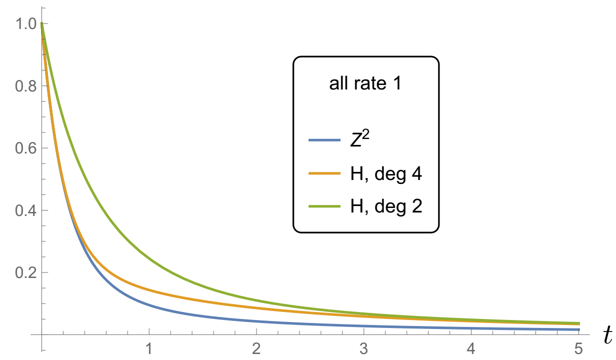

Figure 1.3. Return probabilities for continuous-time simple random walk on each graph.

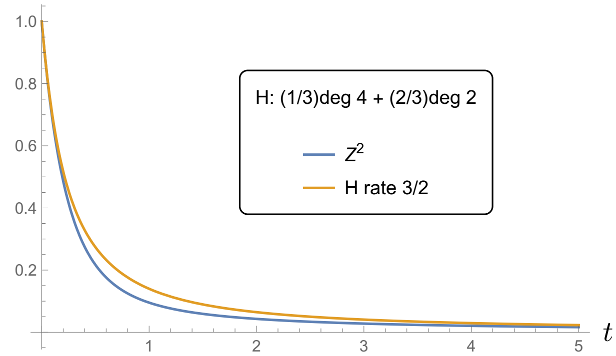

Figure 1.4. Comparison of continuous-time random walk with rates 1 on $\mathbb{Z}^2$ and rates $3/2$ on H, the latter averaged over the starting point.

then the resulting inequality fails for some $t > 0$.

This particular example can be easily derived from Theorem 4.2 of Lyons (2016). With some more work, so can all the results here when the unimodular probability measures involved are sofic. However, the general case (which is not known to be sofic) does not follow from earlier work. Nevertheless, our results and proofs are modeled on Theorem 4.2 of Lyons (2016). The challenge here was to formulate the proper statements for infinite graphs and to make the appropriate adjustments to the proofs required for using direct integrals instead of direct sums.

For a more complicated example of our results, suppose that G is the usual nearest-neighbor graph on $\mathbb{Z}^d$ ($d \geq 2$) and H is the infinite cluster of supercritical Bernoulli (site or bond) percolation on G. Let $\delta := \mathbf{E}\big[\deg_H(o) \bigm| o \in H\big]/(2d) \in (0,1)$. Then

$$\forall t \geq 0 \qquad p_t(o; \mathbb{Z}^d) \leq \mathbf{E}\big[p_{t/\delta}(o; H) \bigm| o \in \mathsf{V}(H)\big] .$$

This is obtained by using $w_H \equiv 1/\delta$ in Corollary 2.3. The preceding inequality is false for any larger value of δ.

§2. Statements of Results and Background.

Let G be a simple, locally finite graph with weights $w_G(e) \geq 0$ on the edges e. Consider the continuous-time random walk on G where edge e is crossed at rate $w_G(e)$ when the walk is incident to e. Let $p_t(x; G)$ denote the probability that a random walk started at x is found at x at time t. If Δ_G is the corresponding Laplacian, i.e., $\Delta_G(x, y) := -w(e)$ when e is an edge joining x and y with weight $w(e)$, all other off-diagonal elements of Δ_G are 0, and the row sums are 0, then $p_t(x; G)$ is the (x, x)-entry of $e^{-t\Delta_G}$. If the entries of Δ_G are unbounded, then we take the minimal Markov process, which dies after an explosion. The infinitesimal generator is then the self-adjoint extension of $-\Delta_G$ (for uniqueness of the extension, see Huang, Keller, Masamune, and Wojciechowski (2013)). For the definition of unimodular in our context, see Aldous and Lyons (2007).

Theorem 2.1. *Let* $\mathbf{P}$ *be a unimodular probability measure on rooted networks* (G, o) *with positive weights* w_G *on its edges and with a percolation subgraph* H *of* G *with positive weights* w_H *on its edges. Let* $\mathbf{P}_{(G,o)}$ *denote the conditional law of* H *given* (G, o)*. Assume that* $\alpha := \mathbf{P}_{(G,o)}\big[o \in \mathsf{V}(H)\big] > 0$ *is a constant* $\mathbf{P}$*-a.s. If* $\mathbf{P}$*-a.s. whenever* $e \in \mathsf{E}(G)$ *is adjacent to* o*,*

$$\mathbf{E}_{(G,o)}\big[w_H(e) \bigm| e \in \mathsf{E}(H)\big]\, \mathbf{P}_{(G,o)}\big[e \in \mathsf{E}(H) \bigm| o \in \mathsf{V}(H)\big] \leq w_G(e)\,, \tag{2.1}$$

then

$$\forall t > 0 \quad \mathbf{E}\big[p_t(o; G)\big] \leq \mathbf{E}\big[p_t(o; H) \bigm| o \in \mathsf{V}(H)\big]\,.$$

The case where G is finite is Theorem 4.2 of Lyons (2016), although it is disguised. The case where $G = H$ and $w_G \geq w_H$ is Theorem 5.1 of Aldous and Lyons (2007).

Remark 2.2. Theorem 2.1 is sharp in a sense: if

$$\mathbf{E}\Big[\sum_{e \sim o} \mathbf{E}_{(G,o)}\big[w_H(e) \bigm| e \in \mathsf{E}(H)\big]\, \mathbf{P}_{(G,o)}\big[e \in \mathsf{E}(H) \bigm| o \in \mathsf{V}(H)\big]\Big] > \mathbf{E}\Big[\sum_{e \sim o} w_G(e)\Big]\,, \tag{2.2}$$

then for all sufficiently small, positive t,

$$\mathbf{E}\big[p_t(o; G)\big] > \mathbf{E}\big[p_t(o; H) \bigm| o \in \mathsf{V}(H)\big]\,. \tag{2.3}$$

For example, let (G, o) be any unimodular random rooted graph and consider Bernoulli(α) site percolation on G. Let H be the induced subgraph. Then

$$\forall t \geq 0 \qquad \mathbf{E}\big[p_t(o; G)\big] \leq \mathbf{E}\big[p_{t/\alpha}(o; H) \bigm| o \in \mathsf{V}(H)\big]\,.$$

This is obtained by using $w_H \equiv 1/\alpha$. This is sharp: for all $\beta < \alpha$, there is some t such that $\mathbf{E}\big[p_t(o; G)\big] > \mathbf{E}\big[p_{t/\beta}(o; H)\big]$.

The following corollary is immediate from Theorem 2.1.

Corollary 2.3. *Let G be a unimodular transitive graph and H be a random subgraph of G with edge weights w_H such that the law of (H, w_H) is* $\mathrm{Aut}(G)$*-invariant. If*

$$\forall e \sim o \qquad \mathbf{E}\big[w_H(e) \bigm| e \in H\big]\, \mathbf{P}[e \in H \mid o \in H] \le 1\,,$$

then continuous-time simple random walk on G and the continuous-time network random walk on (H, w_H) satisfy

$$\forall t > 0 \quad p_t(o; G) \le \mathbf{E}\big[p_t(o; H, w_H) \bigm| o \in \mathsf{V}(H)\big]\,.$$

One might expect also the following as a corollary: Suppose that G is a fixed Cayley graph and w_1, w_2 are two random fields of positive weights on its edges with the properties that each field w_i has an invariant law and a.s. $w_1(e) \ge w_2(e)$ for each edge e. Then $\mathbf{E}\big[p_{1,t}(o; G)\big] \le \mathbf{E}\big[p_{2,t}(o, G)\big]$ for all $t > 0$, where $p_{i,t}$ denotes the return probability to a fixed vertex o at time t with the weights w_i. This is indeed known to be true for amenable G (Fontes and Mathieu, 2006) and also when the pair (w_1, w_2) is invariant (Aldous and Lyons, 2007). However, it is open in general and was asked by Fontes and Mathieu (personal communication). Even more generally, the following question is open, even for finite graphs where it was raised by Lyons (2016):

Question 2.4. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are two unimodular probability measures on rooted graphs with positive edge weights for which there is a coupling that is carried by the set of pairs $\big((G, o, w_G), (H, o, w_H)\big)$ with H a subgraph of G and $w_H(e) \le w_G(e)$ for all $e \in \mathsf{E}(H)$. Is $\mathbf{E}_1\big[p_t(o; G)\big] \le \mathbf{E}_2\big[p_t(o; H)\big]$ for all $t > 0$?

We prove Theorem 2.1 and Remark 2.2 in the following section. Here we present the background required, especially regarding von Neumann algebras.

We will use the notation $A \le B$ for self-adjoint operators A and B to mean that $B - A$ is positive semidefinite. Sometimes we regard the edges of a graph as oriented, where we choose one orientation (arbitrarily) for each edge. In particular, we do this whenever we consider the ℓ^2-space of the edge set of a graph. In this case, we denote the tail and the head of e by e^- and e^+. Define $d_G \colon \ell^2\big(\mathsf{V}(G)\big) \to \ell^2\big(\mathsf{E}(G)\big)$ by

$$d_G(a)(e) := \sqrt{w_G(e)}\,\big[a(e^-) - a(e^+)\big]\,.$$

Then $\Delta_G = d_G^* d_G$.

Consider the Hilbert space $\mathscr{G} := \int^{\oplus} \ell^2\big(\mathsf{V}(G)\big)\, d\mathbf{P}(G, o)$; see Section 5 of Aldous and Lyons (2007) for details of this direct integral. Let Tr denote the normalized trace corresponding to $\mathbf{P}$, as in Section 5 of Aldous and Lyons (2007). That is, given an equivarant operator $T = \int^{\oplus} T_G \, d\mathbf{P}(G,o)$ on $\mathscr{G}$ in the von Neumann algebra Alg associated by those authors to $\mathbf{P}$, we define

$$\mathrm{Tr}(T) := \int (T_G \mathbf{1}_{\{o\}}, \, \mathbf{1}_{\{o\}}) \, d\mathbf{P}(G,o) \, .$$

This trace on Alg is obviously finite. A closed densely defined operator is ***affiliated*** with Alg if it commutes with all unitary operators that commute with Alg. Write AffAlg for the set of all such operators. An operator $T \in \mathsf{AffAlg}$ is called Tr-***measurable*** if for all $\epsilon > 0$, there is an orthogonal projection $E \in \mathsf{Alg}$ whose image lies in the domain of T and $\mathrm{Tr}(E^{\perp}) < \epsilon$. For example, Δ_G is Tr-measurable because if E_n denotes the orthogonal projection to the space of functions that are nonzero only on those (G,o) where the sum of the edge weights at o is at most n, then $\lim_{n \to \infty} \mathrm{Tr}(E_n^{\perp}) = 0$ and $\|\Delta_G E_n\| \le 2n$.

We will need another representation of the trace. For $s \in [0,1]$ and a Tr-measurable operator $T \ge 0$ with spectral resolution E_T, define

$$m_s(T) := \inf\bigl\{\lambda \ge 0 \, ; \; \mathrm{Tr}\bigl(E_T(\lambda, \infty)\bigr) \le 1 - s\bigr\} \, ;$$

see Remark 2.3.1 of Fack and Kosaki (1986). By Lemma 2.5(iii) of Fack and Kosaki (1986), if $0 \le S \le T$ are Tr-measurable, then

$$\forall s \in [0,1] \qquad m_s(S) \le m_s(T) \, . \tag{2.4}$$

A proof similar to that of Corollary 2.8 of Fack and Kosaki (1986) shows that for bounded monotone $f \colon \mathbb{R} \to \mathbb{R}$ and $T \in \mathsf{AffAlg}$, we have

$$\mathrm{Tr}\bigl(f(T)\bigr) = \int_0^1 f\bigl(m_s(T)\bigr) \, ds \, . \tag{2.5}$$

From (2.5) and (2.4), we obtain

$$\mathrm{Tr}\, f(S) \le \mathrm{Tr}\, f(T) \tag{2.6}$$

for bounded increasing $f \colon \mathbb{R} \to \mathbb{R}$ and $0 \le S \le T$ that are Tr-measurable operators in AffAlg. Furthermore, if f is strictly increasing, then equality holds in (2.6) iff $S = T$: if equality holds, then $f(S) = f(T)$ (because Tr is faithful by Lemma 2.3 of Aldous and Lyons (2007)), whence $f^{-1}\bigl(f(S)\bigr) = f^{-1}\bigl(f(T)\bigr)$.

Let $w_{G,n}$ denote the weights on G when for every $x \sim y$, the edge weight $w_G(x,y)$ is replaced by 0 if the sum of the weights incident to x and y is larger than n. We claim that

$$\lim_{n \to \infty} \int p_t(o; w_{G,n}) \, d\mathbf{P}(G,o) = \int p_t(o; w_G) \, d\mathbf{P}(G,o) \, . \tag{2.7}$$

To see this, let E_n denote, as before, the orthogonal projection to the space of functions that are nonzero only on those (G, o) where the sum of the edge weights at o is at most n. Then $\Delta_{G,w_{G,n}} E_n = \Delta_{G,w_G} E_n$ for all n. Since $\lim_{n\to\infty} \operatorname{Tr}(E_n^\perp) = 0$, it follows that $\lim_{n\to\infty} \Delta_{G,w_{G,n}} = \Delta_{G,w_G}$ in the measure topology (Definition 1.5 of Fack and Kosaki (1986)). Since $\Delta_{G,w_{G,n}} \le \Delta_{G,w_G}$, we have $m_s\big(\Delta_{G,w_{G,n}}\big) \le m_s\big(\Delta_{G,w_G}\big)$ by (2.4). Therefore, $\lim_{n\to\infty} m_s\big(\Delta_{G,w_{G,n}}\big) = m_s\big(\Delta_{G,w_G}\big)$ by Lemma 3.4(ii) of Fack and Kosaki (1986). Now use $f(\lambda) := e^{-t\lambda}$ in (2.5) to obtain $\lim_{n\to\infty} \operatorname{Tr}\big(e^{-t\Delta_{G,w_{G,n}}}\big) = \operatorname{Tr}\big(e^{-t\Delta_{G,w_G}}\big)$, which is the same as (2.7).

Suppose that Φ is a positive, unital, linear map from a unital C^*-algebra A to a von Neumann algebra with finite trace, Tr. The proof of Theorem 3.9 of Antezana, Massey, and Stojanoff (2007) shows that

$$\operatorname{Tr} j\big(\Phi(T)\big) \le \operatorname{Tr} \Phi\big(j(T)\big) \tag{2.8}$$

for self-adjoint operators $T \in \mathsf{A}$ and functions $j \colon \mathbb{R} \to \mathbb{R}$ that are convex on the convex hull of the spectrum of T. (In fact, those authors show the more general inequality $\operatorname{Tr} k\Big(j\big(\Phi(T)\big)\Big) \le \operatorname{Tr} k\Big(\Phi\big(j(T)\big)\Big)$ for every increasing convex k.)

§3. Proofs.

Proof of Theorem 2.1. Suppose first that the entries of Δ_G and Δ_H are uniformly bounded, so that $\Delta_{\mathscr{G}}$ and $\Delta_{\mathscr{H}}$ are bounded oeprators in Alg.

In addition to the Hilbert space $\mathscr{G} := \int^\oplus \ell^2\big(\mathsf{V}(G)\big)\, d\mathbf{P}(G, o)$ we considered in the preceding section, also let

$$\mathscr{H} := \int^\oplus \int^\oplus \ell^2\big(\mathsf{V}(H)\big)\, d\mathbf{P}_{(G,o)}(H)\, d\mathbf{P}(G, o)\,.$$

By Lemma 2.3 of Aldous and Lyons (2007), we have that

$$\mathbf{P}\Big[\forall x \in \mathsf{V}(G) \;\; \mathbf{P}_{(G,o)}\big[x \in \mathsf{V}(H)\big] = \alpha\Big] = 1\,. \tag{3.1}$$

Similarly, (2.1) implies that a.s.

$$\forall e \in \mathsf{E}(G) \qquad \alpha^{-1} \int_{e \in \mathsf{E}(H)} w_H(e)\, d\mathbf{P}_{(G,o)} \le w_G(e)\,. \tag{3.2}$$

By (3.1), for every $f = \int^\oplus f(G, o)\, d\mathbf{P}(G, o) \in \mathscr{G}$, we have that

$$\phi(f) := \alpha^{-1/2} \int^\oplus \int^\oplus \sum_{x \in \mathsf{V}(H)} f(G, o)(x) \mathbf{1}_{\{x\}}\, d\mathbf{P}_{(G,o)}(H)\, d\mathbf{P}(G, o) \in \mathscr{H}$$

has the same norm as f. Moreover, $\phi\colon \mathscr{G} \to \mathscr{H}$ defines an isometry, i.e., $\phi^*\phi$ is the identity map. Define $\Phi\colon \mathcal{L}(\mathscr{H}) \to \mathcal{L}(\mathscr{G})$ by $\Phi T := \phi^* T \phi$. Then Φ is a positive unital map.

Consider quadruples (G, H, w_G, w_H) of graphs G and H and weight functions w_G and w_H with H a subgraph of G. An isomorphism of a pair of such quadruples is defined in the obvious way. As before, however, we will generally omit including the weight functions in the notations for networks. Similarly to how Alg is defined, let A be the von Neumann algebra of (equivalence classes of) bounded linear maps $T = \int^{\oplus}\int^{\oplus} T_{(G,o,H)}\, d\mathbf{P}_{(G,o)}(H)\, d\mathbf{P}(G,o) \in \mathcal{L}(\mathscr{H})$ that are equivariant in the sense that for all isomorphisms $\psi\colon (G_1, H_1) \to (G_2, H_2)$, all $o_1 \in \mathsf{V}(G_1)$, $o_2 \in \mathsf{V}(G_2)$, and all $x, y \in \mathsf{V}(H_1)$, we have $(T_{(G_1,o_1,H_1)}x, y) = (T_{(G_2,o_2,H_2)}\psi x, \psi y)$; in particular, $T_{(G,o,H)}$ does not depend on o. Then Φ maps A into Alg.

Let $\Delta_{\mathscr{G}} := \int^{\oplus} \Delta_G\, d\mathbf{P}(G,o) \in \mathsf{Alg}$ and $\Delta_{\mathscr{H}} := \int^{\oplus}\int^{\oplus} \Delta_H\, d\mathbf{P}_{(G,o)}(H)\, d\mathbf{P}(G,o) \in \mathsf{A}$. Then $\Phi(\Delta_{\mathscr{H}}) \in \mathsf{Alg}$ and, therefore, $j(\Delta_{\mathscr{G}}), j\big(\Phi(\Delta_{\mathscr{H}})\big) \in \mathsf{Alg}$ for all bounded Borel $j\colon \mathbb{R} \to \mathbb{R}$.

We claim that

$$\Delta_{\mathscr{G}} \geq \Phi(\Delta_{\mathscr{H}})\,. \tag{3.3}$$

To see this, let $f \in \mathscr{G}$. We have

$$\big(\Delta_{\mathscr{G}}(f),\, f\big) = \mathbf{E}\big[\|d_G f(G,o)\|^2\big] \tag{3.4}$$

and

$$\big(\Phi\Delta_{\mathscr{H}}(f),\, f\big) = \big(\phi^*\Delta_{\mathscr{H}}\phi f,\, f\big) = \big(\Delta_{\mathscr{H}}\phi f,\, \phi f\big)\,. \tag{3.5}$$

Now

$$\begin{aligned}
\big(\Delta_{\mathscr{H}}\phi f,\, \phi f\big) &= \alpha^{-1}\iint \|d_H f(G,o)\|^2\, d\mathbf{P}_{(G,o)}\, d\mathbf{P}(G,o) \\
&= \alpha^{-1}\iint \sum_{e\in\mathsf{E}(H)} w_H(e)\big(f(G,o)(e^-) - f(G,o)(e^+)\big)^2\, d\mathbf{P}_{(G,o)}\, d\mathbf{P}(G,o) \\
&= \alpha^{-1}\int \sum_{e\in\mathsf{E}(G)} \int_{\mathsf{E}(H)\ni e} w_H(e)\, d\mathbf{P}_{(G,o)} \cdot \big(f(G,o)(e^-) - f(G,o)(e^+)\big)^2\, d\mathbf{P}(G,o) \\
&\leq \int \sum_{e\in\mathsf{E}(G)} w_G(e)\cdot\big(f(G,o)(e^-) - f(G,o)(e^+)\big)^2\, d\mathbf{P}(G,o) \\
&= \mathbf{E}\big[\|d_G f(G,o)\|^2\big]
\end{aligned}$$

by (3.2). Combining this with (3.4) and (3.5), we get our claimed inequality (3.3).

By (3.3) and (2.6), we have

$$\operatorname{Tr} j(\Delta_{\mathscr{G}}) \leq \operatorname{Tr} j\big(\Phi(\Delta_{\mathscr{H}})\big)$$

for every decreasing function j. (We have strict inequality if j is strictly decreasing and we have strict inequality in (3.3).) Use $j(s) := e^{-ts}$ in this and in (2.8) to obtain

$$\operatorname{Tr} j(\Delta_{\mathscr{G}}) \le \operatorname{Tr} \Phi\big(j(\Delta_{\mathscr{H}})\big) . \tag{3.6}$$

The left-hand side equals $\mathbf{E}\big[p_t(o; G)\big]$. The right-hand side equals

$$\begin{aligned} \operatorname{Tr} \Phi\big(j(\Delta_{\mathscr{H}})\big) &= \alpha^{-1} \iint_{o \in \mathsf{V}(H)} \big(j(\Delta_H)\mathbf{1}_{\{o\}},\, \mathbf{1}_{\{o\}}\big)\, d\mathbf{P}_{(G,o)}\, d\mathbf{P}(G,o) \\ &= \mathbf{E}\big[p_t(o; H) \bigm| o \in \mathsf{V}(H)\big] , \end{aligned}$$

which completes the proof of the theorem in the case of bounded vertex weights.

We deduce the general case from this by a truncation argument. Recall (2.7) and its notation, which we use also for H. Let μ_n be the law of $(w_{G,n}, w_{H,n})$. Since the diagonal entries of $\Delta_{(G, w_{G,n})}$ and $\Delta_{(H, w_{H,n})}$ are bounded and (2.1) holds μ_n-a.s., we have proved that

$$\forall t > 0 \quad \mathbf{E}\Big[p_t\big(o; (G, w_{G,n})\big)\Big] \le \mathbf{E}\Big[p_t\big(o; (H, w_{H,n})\big) \Bigm| o \in \mathsf{V}(H)\Big] .$$

Taking $n \to \infty$ and using the bounded convergence theorem, we get the desired result. ∎

A similar proof shows that (3.6) holds if j is any decreasing convex function.

Proof of Remark 2.2. We may rewrite the right-hand side of (2.2) as $\mathbf{E}\big[\Delta_G(o,o)\big]$ and the left-hand side as $\mathbf{E}\big[\Delta_H(o,o) \bigm| o \in \mathsf{V}(H)\big]$. Now both sides of (2.3) equal 1 for $t = 0$. We claim that the derivative of the left-hand side at $t = 0$ is $-\,\mathbf{E}\big[\Delta_H(o,o) \bigm| o \in \mathsf{V}(H)\big]$ and the derivative of the right-hand side at $t = 0$ is $-\,\mathbf{E}\big[\Delta_G(o,o)\big]$. This clearly implies the remark. To evaluate these derivatives, note that for every fixed G, the spectral representation

$$p_t(o; G) = \int_0^\infty e^{-\lambda t}\, d\big(E_{\Delta_G}(\lambda)\mathbf{1}_{\{o\}},\, \mathbf{1}_{\{o\}}\big)$$

shows that $t \mapsto p_t(o; G)$ is monotone decreasing and convex. By Tonelli's theorem, it follows that for $(G, o) \sim \mathbf{P}$,

$$\mathbf{E}\big[p_t(o; G)\big] - 1 = \mathbf{E}\Big[\int_0^t p'_s(o; G)\, ds\Big] = \int_0^t \mathbf{E}\big[p'_s(o; G)\big]\, ds .$$

The fundamental theorem of calculus and the monotone convergence theorem now yield that

$$\frac{d}{dt}\, \mathbf{E}\big[p_t(o; G)\big]\bigg|_{t=0} = \mathbf{E}\big[p'_0(o; G)\big] = -\,\mathbf{E}\big[\Delta_G(o,o)\big] .$$

A similar calculation applied to the distribution of (H, o) given $o \in \mathsf{V}(H)$ yields the derivative of the left-hand side of (2.3). ∎

REFERENCES

Aldous, D.J. and Lyons, R. (2007). Processes on unimodular random networks. *Electron. J. Probab.*, **12**, paper no. 54, 1454–1508 (electronic).

Antezana, J., Massey, P., and Stojanoff, D. (2007). Jensen's inequality for spectral order and submajorization. *J. Math. Anal. Appl.*, **331**(1), 297–307.

Fack, T. and Kosaki, H. (1986). Generalized s-numbers of τ-measurable operators. *Pacific J. Math.*, **123**(2), 269–300.

Fontes, L.R.G. and Mathieu, P. (2006). On symmetric random walks with random conductances on $\mathbb{Z}^d$. *Probab. Theory Related Fields*, **134**(4), 565–602.

Heicklen, D. and Hoffman, C. (2005). Return probabilities of a simple random walk on percolation clusters. *Electron. J. Probab.*, **10**, paper no. 8, 250–302 (electronic).

Huang, X., Keller, M., Masamune, J., and Wojciechowski, R.a.K. (2013). A note on self-adjoint extensions of the Laplacian on weighted graphs. *J. Funct. Anal.*, **265**(8), 1556–1578.

Lyons, R. (2016). Comparing graphs of different sizes. *Combin. Probab. Comput.* To appear, http://www.arxiv.org/abs/1602.06995.

Department of Mathematics, 831 E. 3rd St., Indiana University, Bloomington, IN 47405-7106
rdlyons@indiana.edu
http://pages.iu.edu/~rdlyons/