This shows that $\lim_{x\to\infty} p(x) = 1$, and, by a similar argument, $\lim_{x\to-\infty} p(x) = 0$. By (4), (5), and (6), $a + b = 1$ and $a - b = 0$, i.e., $a = b = 1/2$. Hence

$$p(x) = \begin{cases} 1 - 2^{-x-1} & \text{for } x \geq 0, \\ 2^{x-1} & \text{for } x \leq 0. \end{cases}$$

In particular, for $X_0 = 1$ we obtain

$$p = \mathbb{P}[\lim X_n = \infty] = p(1) = 3/4.$$

Moreover, by symmetry,

$$\mathbb{P}[\lim X_n = -\infty] = p(-1) = 1/4 = 1 - p.$$

Hence with probability 1, we have either $\lim X_n = +\infty$ or $\lim X_n = -\infty$. In the first case, $X_n - X_{n-1} = Y_n$ for sufficiently large n, and hence

$$\lim(X_n/n) = \lim(S_n/n) = 1/3.$$

Similarly, in the second case,

$$\lim(X_n/n) = -1/3. \qquad \square$$

Also solved by Mihaly Bencze (Romania), Socratis Varelogiannis (France), and Alexander Vauth (Germany).

198. Let $B := (B_t)_{t\geq 0}$ be Brownian motion in the complex plane. Suppose that $B_0 = 1$.

(a) Let T_1 be the first time that B hits the imaginary axis, T_2 be the first time after T_1 that B hits the real axis, T_3 be the first time after T_2 that B hits the imaginary axis, etc. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(b) More generally, let ℓ_n be lines through 0 for $n \geq 1$ such that $1 \notin \ell_1$. Let $T_1 := \inf\{t \geq 0\,;\ B_t \in \ell_1\}$ and recursively define $T_{n+1} := \inf\{t > T_n\,;\ B_t \in \ell_{n+1}\}$ for $n \geq 1$. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(c) In the context of part (b), let α_n be the smaller of the two angles between ℓ_n and ℓ_{n+1}. Show that $\sum_{n=1}^{\infty} \alpha_n = \infty$ iff, for all $\epsilon > 0$, the probability that $\epsilon \leq |B_{T_n}| \leq 1/\epsilon$ tends to 0 as $n \to \infty$.

(d) In the context of part (a), show that

$$\lim_{n\to\infty} \mathbf{P}\Big[\exp(-\delta_n \sqrt{n}) \leq |B_{T_n}| \leq \exp(\delta_n \sqrt{n})\Big] = \int_{-2\delta/\pi}^{2\delta/\pi} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\, du$$

if $\delta_n \geq 0$ tend to $\delta \in [0, \infty]$.

(Russell Lyons, Department of Mathematics, Indiana University, USA [Partially supported by the National Science Foundation under grant DMS-1612363])

Solution by the proposer. We skip (a) and pass directly to (b). Denote inversion in the unit circle by $\phi(z) := 1/\bar{z}$. It is well known that $W := (\phi(B_t))_{t\geq 0}$ is a time-change of Brownian motion. Since ϕ maps each line ℓ_n to itself, $T_1 = \inf\{t \geq 0\,;\ W_t \in \ell_1\}$ and $T_{n+1} = \inf\{t > T_n\,;\ W_t \in \ell_{n+1}\}$ for $n \geq 1$. Thus, B_{T_n} and W_{T_n} have the same distribution. However, $|B_t| \leq 1$ iff $|W_t| \geq 1$. Since the chance that $|B_{T_n}| = 1$ is 0 for each n, we obtain (b).

In light of (b), the conclusion of (c) is equivalent to $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$ for all $\epsilon > 0$.

Let $T := \lim_{n\to\infty} T_n$. If $T < \infty$ a.s., then $\lim_{n\to\infty} B_{T_n} = B_T$, which implies that for some $\epsilon > 0$, $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] \neq 1/2$. Now suppose that $T = \infty$ a.s. Neighbourhood recurrence of B shows that for each $\epsilon > 0$, there is some $t < \infty$ such that $|B_t| \leq \epsilon$. Let S_ϵ be the first such time t. The strong Markov property, scaling, rotational symmetry, and part (b) shows that $\mathbf{P}[|B_{T_n}| < \epsilon \mid T_n > S_\epsilon] = 1/2$. Because $\lim_{n\to\infty} \mathbf{P}[T_n > S_\epsilon] = 1$, this shows that $\lim_{n\to\infty} \mathbf{P}[|B_{T_n}| < \epsilon] = 1/2$.

It remains to show that if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $T < \infty$ a.s., whereas if $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $T = \infty$ a.s. If $\limsup_{n\to\infty} \alpha_n > 0$, as in (a), then this is clear from the fact that then a.s. there is no limiting argument of B_{T_n}. In general, we use the skew-product representation of B as $B_t = \exp(X_{H_t} + iY_{H_t})$, where X and Y are independent real Brownian motions started at 0 and $H_t := \int_0^t ds/|B_s|^2$ (the form of H will not matter to us). Note that H_{T_n} are functions of Y and thus independent of X. Also, $H_\infty = \infty$ a.s. It is well known that the expected time for Y to visit either $\alpha > 0$ or $-\beta < 0$ is $\alpha\beta$, whence the expectation of $H_{T_{n+1}} - H_{T_n}$ equals $(\pi - \alpha_n)\alpha_n$. Note that $H_{T_{n+1}} - H_{T_n}$ are independent (and nonnegative). Therefore, if $\sum_{n=1}^{\infty} \alpha_n < \infty$, then $\sum_n (H_{T_{n+1}} - H_{T_n}) < \infty$ a.s., and otherwise (by Kolmogorov's three-series theorem) $\sum_n (H_{T_{n+1}} - H_{T_n}) = \infty$ a.s. This is the same as $H_T < \infty$ a.s. or $H_T = \infty$ a.s. respectively, which in turn is equivalent to $T < \infty$ a.s. or $T = \infty$ a.s.

For part (d), note that $H_{T_n}/n \to \pi^2/4$ in probability by the weak large of large numbers. With $\epsilon_n := \exp(-\delta_n \sqrt{n})$, we have

$$\begin{aligned}\mathbf{P}[\epsilon_n \leq |B_{T_n}| \leq 1/\epsilon_n] &= \mathbf{P}[-\delta_n \sqrt{n} \leq X_{H_{T_n}} \leq \delta_n \sqrt{n}] \\ &= \mathbf{P}\Big[-\frac{2\delta_n}{\pi} \leq X_{\frac{4H_{T_n}}{\pi^2 n}} \leq \frac{2\delta_n}{\pi}\Big]\end{aligned}$$

by Brownian scaling. Now let $n \to \infty$. $\square$

Remark. The proof of (c) could be shortened by using the skew-product representation throughout, but the proof given is more elementary in the context of (a). Part (b) could also be proved with the skew-product representation.

Also solved by Mihaly Bencze (Romania), Sotirios E. Louridas (Greece), and Socratis Varelogiannis (France).

199. Suppose that each carioca (native of Rio de Janeiro) likes at least half of the other 2^{23} cariocas. Prove that there exists a set A of 1000 cariocas with the following property: for each pair of cariocas in A, there exists a *distinct* carioca who likes both of them.

(Rob Morris, IMPA, Rio de Janeiro, Brazil)

Solution by the proposer. Choose 10 random cariocas (possibly with repetition), and consider the set X of cariocas that they all like. Observe that, writing V for the set of all n cariocas, we have

$$\mathbb{E}[|X|] \geq \sum_{v\in V} \left(\frac{d(v)}{n}\right)^{10} \geq 2^{-11} n \geq 2000,$$

where $d(v)$ is the number of cariocas that like carioca v. Indeed, the first inequality follows by linearity of expectation, together with the fact that $v \in X$ if and only if the 10 random cariocas all like v, the second inequality uses the convexity of the function $f(x) = x^{10}$ and the fact that $\sum_{v\in V} d(v) \geq \binom{n}{2}$, and the third holds since $n \geq 2^{23}$.

Now, let Y be the set of pairs of cariocas in X such that fewer than $\binom{1000}{2}$ cariocas like both of them. Observe that

$$\mathbb{E}[|Y|] \leq \binom{n}{2} \left(\frac{\binom{1000}{2}}{n}\right)^{10} \leq \frac{2^{189}}{n^8} \leq 1000,$$

again using the fact that $n \geq 2^{23}$. It follows that $\mathbb{E}[|X| - |Y|] \geq 1000$, and hence there exists a choice of 10 cariocas such that $|X| - |Y| \geq$