Solved and Unsolved Problems

Michael Th. Rassias (University of Zürich, Switzerland)

Probability theory is nothing but common sense reduced to calculation.

Pierre-Simon Laplace (1749–1827)

The column this month is devoted to probability theory. The proposed problems range from basic to fairly demanding so a wide range of our readers should be able to tackle them. As always, there is also a proposed open research problem. The open problem, along with the relevant discussion, is provided by Martin Hairer.

Probability theory traces back to the 16th century, when the Italian polymath Gerolamo Cardano attempted to mathematically analyse games of chance. More specifically, his book about games of chance, published in 1663 (written ca. 1564), contains the first systematic treatqment of probability. Probability theory also traces back to 17th century France, when Blaise Pascal and Pierre de Fermat corresponded about problems of games of chance. In modern mathematics, probability theory is an extremely applicable and versatile field, which is used in a surprisingly broad spectrum of areas, such as weather prediction, medicine/biology, equity trading, machine perception, music, etc.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

197. In a game, a player moves a counter on the integers according to the following rules. During each round, a fair die is thrown. If the die shows "5" or "6", the counter is moved up one position and if it shows "1" or "2", it is moved down one position. If the die shows "3" or "4", the counter is moved up one position if the current position is positive, down one position if the current position is negative and stays at the same position if the current position is 0. Let X_n denote the position of the player after n rounds when starting at $X_0 = 1$. Find the probability p that $\lim X_n = +\infty$ and show that $X_n/n \to 1/3$ with probability p and $X_n/n \to -1/3$ with probability $1-p$.

(Andreas Eberle, Institute for Applied Mathematics, Probability Theory, Bonn, Germany)

198. Let $B := (B_t)_{t\geq 0}$ be Brownian motion in the complex plane. Suppose that $B_0 = 1$.

(a) Let T_1 be the first time that B hits the imaginary axis, T_2 be the first time after T_1 that B hits the real axis, T_3 be the first time after T_2 that B hits the imaginary axis, etc. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(b) More generally, let ℓ_n be lines through 0 for $n \geq 1$ such that $1 \notin \ell_1$. Let $T_1 := \inf\{t \geq 0\,;\ B_t \in \ell_1\}$ and recursively define $T_{n+1} := \inf\{t > T_n\,;\ B_t \in \ell_{n+1}\}$ for $n \geq 1$. Prove that, for each $n \geq 1$, the probability that $|B_{T_n}| \leq 1$ is 1/2.

(c) In the context of part (b), let α_n be the smaller of the two angles between ℓ_n and ℓ_{n+1}. Show that $\sum_{n=1}^{\infty} \alpha_n = \infty$ iff, for all $\epsilon > 0$, the probability that $\epsilon \leq |B_{T_n}| \leq 1/\epsilon$ tends to 0 as $n \to \infty$.

(d) In the context of part (a), show that

$$\lim_{n\to\infty} \mathbf{P}\Big[\exp(-\delta_n\sqrt{n}) \leq |B_{T_n}| \leq \exp(\delta_n\sqrt{n})\Big] = \int_{-2\delta/\pi}^{2\delta/\pi} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

if $\delta_n \geq 0$ tend to $\delta \in [0,\infty]$.

(Russell Lyons, Department of Mathematics, Indiana University, USA. [Partially supported by the National Science Foundation under grant DMS-1612363])

199. Suppose that each carioca (native of Rio de Janeiro) likes at least half of the other 2^{23} cariocas. Prove that there exists a set A of 1000 cariocas with the following property: for each pair of cariocas in A, there exists a *distinct* carioca who likes both of them.

(Rob Morris, IMPA, Rio de Janeiro, Brazil)

200. Let X, Y, Z be independent and uniformly distributed in $[0,1]$. What is the probability that three sticks of length X, Y and Z can be assembled together to form a triangle?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

201. Suppose that each hour, one of the following four events may happen to a certain type of cell: it may die, it may split into two cells, it may split into three cells or it may remain a single cell. Suppose these four events are equally likely. Start with a population consisting of a single cell. What is the probability that the population eventually goes extinct?

(Sebastien Vasey, Department of Mathematics, Harvard University, Cambridge, Massachusetts, USA)

202. We flip a fair coin repeatedly and record the outcomes.

(1) How many coin flips do we need on average to see three tails in a row?

(2) Suppose that we stop when we first see heads, heads, tails (H, H, T) or tails, heads, tails (T, H, T) come up in this order on three consecutive flips. What is the probability that we stop at H, H, T?

(Benedek Valkó, Department of Mathematics, University of Wisconsin Madison, Madison, Wisconsin, USA)

II An Open Problem, by Martin Hairer (Mathematics Institute, Imperial College London, UK)

Before trying to formulate this open problem, I would like to start by introducing one of the most important objects in probability theory, namely Brownian motion. One way of viewing Brownian motion is as a random variable B taking values in the space $\mathbb{C}$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and satisfying the following two properties.

Claim 1 (i) One has $B(0) = 0$ almost surely.

(ii) For any finite sequence of times $(t_1,\ldots,t_n)$, the $\mathbb{R}^n$-valued random variable $(B(t_1),\ldots,B(t_n))$ is a centred Gaussian random variable such that $\mathbf{E}(B(t_i) - B(t_j))^2 = |t_i - t_j|$ for any $i, j \in \{1,\ldots,n\}$.