Determinantal Probability Measures

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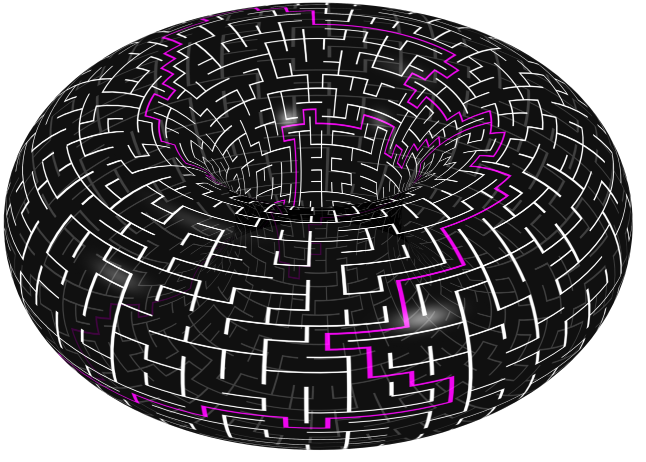


Determinantal Measures

If E is finite and $H \subseteq \ell^2(E)$ is a subspace, it defines the determinantal measure

$$\forall T \subseteq E \text{ with } |T| = \dim H \qquad \mathbf{P}^H(T) := \det[P_H]_{T,T} \,,$$

where the subscript T, T indicates the submatrix whose rows and columns belong to T. This representation has a useful extension, namely,

$$\forall D \subseteq E \qquad \mathbf{P}^H[D \subseteq T] = \det[P_H]_{D,D} \,.$$

In case E is infinite and H is a closed subspace of $\ell^2(E)$, the determinantal probability measure $\mathbf{P}^H$ is defined via the requirement that this equation hold for all finite $D \subset E$.

Matroids

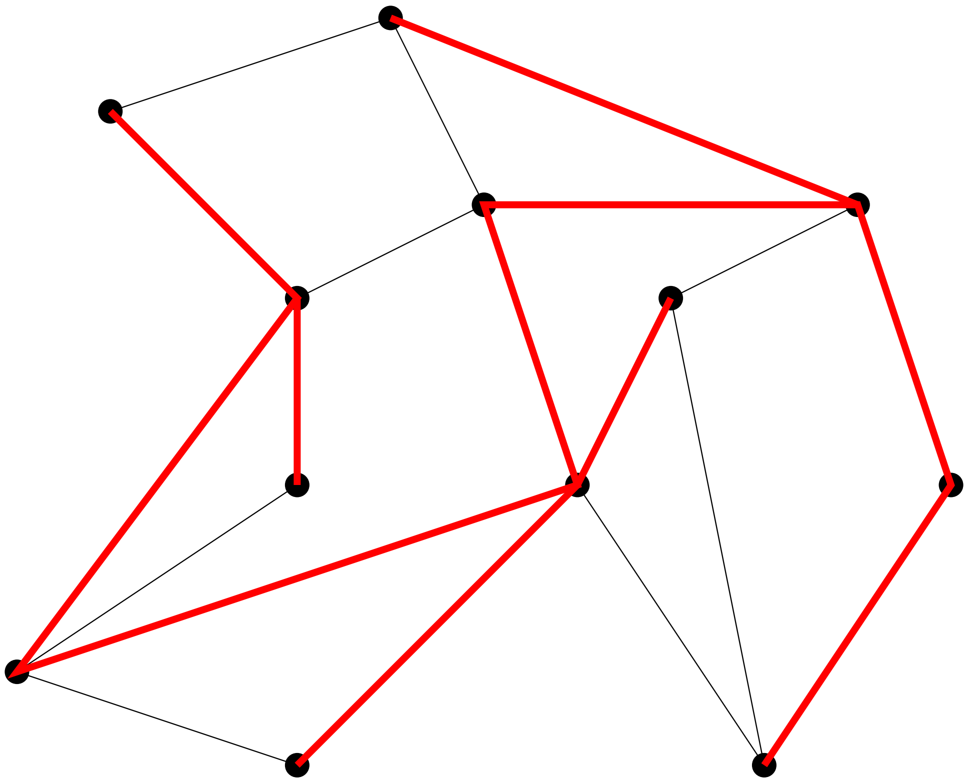
Let E be a finite set, called the **ground set**, and let $\mathscr{B}$ be a nonempty collection of subsets of E. We call the pair $\mathscr{M} := (E, \mathscr{B})$ a **matroid** with **bases** $\mathscr{B}$ if the following exchange property is satisfied:

$$\forall B, B' \in \mathscr{B} \quad \forall e \in B \setminus B' \quad \exists e' \in B' \setminus B$$
$$(B \setminus \{e\}) \cup \{e'\} \in \mathscr{B}\,.$$

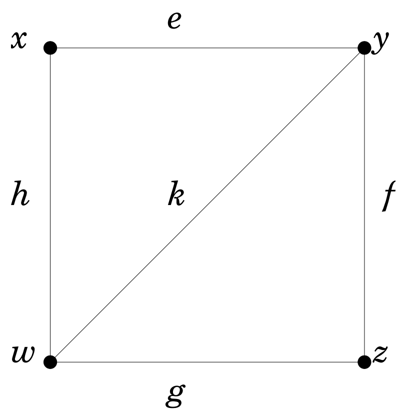
All bases have the same cardinality, called the **rank** of the matroid.

Example: If E is the set of edges of a finite connected graph and $\mathscr{B}$ is the set of spanning trees of the graph, this is called a **graphical matroid**. Proof.

Example: If E is a finite subset of a vector space and $\mathscr{B}$ is the set of maximal linearly independent subsets of E, this is called a **vectorial matroid**. Represent E by columns of a matrix. Example: graphical. Use the incidence matrix.



A spanning tree of a graph with the edges of the tree in red.



$$
\begin{array}{c} \\ x \\ y \\ z \\ w \end{array}
\begin{array}{c}
\begin{array}{ccccc} e & f & g & h & k \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & -1
\end{pmatrix}
\end{array}
$$

A **representable matroid** is one that is isomorphic to a vectorial matroid. A **regular matroid** is one that is representable over every field. For example, graphical matroids are regular.

The **dual** of a matroid $\mathscr{M} = (E, \mathscr{B})$ is the matroid $\mathscr{M}^{\perp} := (E, \mathscr{B}')$, where

$$\mathscr{B}' := \{E \setminus B\,;\; B \in \mathscr{B}\}\,.$$

Determinantal Probability Measures

(Random Matrix Theory 1950s–present, Macchi 1972–75,
Daley, Vere-Jones 1988, many papers since late 1990s; L. 2005)

For representable matroids only. The measure depends on the representation.

The usual way of representing a vectorial matroid $\mathscr{M}$ over $\mathbb{R}$ (or over $\mathbb{C}$) of rank r on a ground set E is by an $(s \times E)$-matrix M whose columns are the vectors in $\mathbb{R}^s$ representing $\mathscr{M}$. The column space of M is r-dimensional, so the rank of M is r, and the row space $H \subseteq \mathbb{R}^E$ of M is r-dimensional. Suppose that the first r rows, say, of M span H. For an r-subset $B \subseteq E$, let M_B denote the $(r \times r)$-matrix determined by the first r rows of M and the columns of M indexed by those e belonging to B. Let $M_{(r)}$ denote the matrix formed by the first r rows of M. Define

$$\mathbf{P}^H[B] := |\det M_B|^2 / \det(M_{(r)} M_{(r)}^T)\,,$$

where the superscript T denotes (conjugate) transpose. This depends only on H. row ops and scale

Simpler formula: Identify $e \in E$ with $\mathbf{1}_{\{e\}} \in \ell^2(E)$. Let P_H be the orthogonal projection onto H. Then

$$\mathbf{P}^H[B] = \det[(P_H e, e')]_{e,e' \in B} = \det[(P_H e, P_H e')]_{e,e' \in B}\,.$$

Thus, for r-element subsets $B \subseteq E$, we have $B \in \mathscr{B}$ iff $P_H B$ is a basis for H.

One also obtains $\forall A \subseteq E$

$$\mathbf{P}^H[A \subseteq \mathfrak{B}] = \det[(P_H e, e')]_{e,e' \in A}\,. \qquad (\dagger)$$

Example: For a graphical matroid, M is the vertex-edge incidence matrix (each edge has a fixed arbitrary orientation). The row space is the space $\bigstar$ spanned by the stars or cuts. The measure $\mathbf{P}^\bigstar$ is uniform measure on spanning trees. Equation ($\dagger$) is called the Transfer Current Theorem of Burton and Pemantle (1993).

Remark. For any given matroid $\mathscr{M}$, there exists some real representation with a row space H such that $\mathbf{P}^H$ is uniform on $\mathscr{B}$ iff $\mathscr{M}$ is regular.

Why is this a probability measure?

Suppose first that H is 1-dimensional ($r = 1$). Choose a unit vector $v \in H$. Then

$$\mathbf{P}^H[\{e\}] = |(v,\, e)|^2 \,.$$

The general case arises from multivectors.

Recall that

$$(u_1 \wedge \cdots \wedge u_k,\, v_1 \wedge \cdots \wedge v_k) = \det \big[(u_i,\, v_j)\big]_{i,j \in [1,k]} \,.$$

Also, vectors $u_1, \ldots, u_k \in \ell^2(E)$ are linearly independent iff $u_1 \wedge \cdots \wedge u_k \neq 0$.

If $\dim H = r$, then $\bigwedge^r H$ is a 1-dimensional subspace of $\mathrm{Ext}\big(\ell^2(E)\big)$; denote by ξ_H a unit multivector in this subspace.

Review of Exterior Algebra

$E_k :=$ choice of ordered k-subsets of E

$\bigwedge^k \ell^2(E) := \ell^2\Big(\big\{e_1 \wedge \cdots \wedge e_k\,;\ \langle e_1, \ldots, e_k\rangle \in E_k\big\}\Big) =:$ **multivectors** of **rank** k.

$$\bigwedge_{i=1}^{k} e_{\sigma(i)} = \operatorname{sgn}(\sigma) \bigwedge_{i=1}^{k} e_i \qquad \text{for any permutation } \sigma \text{ of } \{1, 2, \ldots, k\}$$

$$\bigwedge_{i=1}^{k} \sum_{e \in E} a_i(e) e = \sum_{e_1, \ldots, e_k \in E} \prod_{j=1}^{k} a_j(e_j) \bigwedge_{i=1}^{k} e_i \quad \text{for any scalars } a_i(e)\ (i \in [1, k],\ e \in E).$$

$\operatorname{Ext}\big(\ell^2(E)\big) := \bigoplus_{k=1}^{|E|} \bigwedge^k \ell^2(E)$, orthogonal summands

For $H \subseteq \ell^2(E)$, we identify $\operatorname{Ext}(H)$ with its inclusion in $\operatorname{Ext}\big(\ell^2(E)\big)$, that is, $\bigwedge^k H$ is the linear span of

$$\big\{v_1 \wedge \cdots \wedge v_k\,;\ v_1, \ldots, v_k \in H\big\}.$$

Why $\mathbf{P}^H$ is a probability measure:

$$\mathbf{P}^H[\{e_1, \ldots, e_r\}] = \left|\left(\xi_H, \bigwedge_{i=1}^{r} e_i\right)\right|^2 .$$

To prove this, we use:

LEMMA. *For any subspace $H \subseteq \ell^2(E)$, any $k \geq 1$, and any $u_1, \ldots, u_k \in \ell^2(E)$,*

$$P_{\wedge^k H}(u_1 \wedge \cdots \wedge u_k) = (P_H u_1) \wedge \cdots \wedge (P_H u_k) .$$

Proof. Write

$$u_1 \wedge \cdots \wedge u_k = (P_H u_1 + P_H^{\perp} u_1) \wedge \cdots \wedge (P_H u_k + P_H^{\perp} u_k)$$

and expand the product. All terms but

$$P_H u_1 \wedge \cdots \wedge P_H u_k$$

have a factor of $P_H^{\perp} u_i$ in them, making them orthogonal to $\bigwedge^k H$. ∎

Proof that

$$\mathbf{P}^{H}[\{e_1, \ldots, e_r\}] = \left|\left(\xi_H, \bigwedge_{i=1}^{r} e_i\right)\right|^2 :$$

We have

$$\begin{aligned}
\left|\left(\xi_H, \bigwedge_{i=1}^{r} e_i\right)\right|^2 &= \|P_{\wedge^r H}(\bigwedge_i e_i)\|^2 \\
&= \left(P_{\wedge^r H}(\bigwedge_i e_i),\ P_{\wedge^r H}(\bigwedge_i e_i)\right) \\
&= \left(P_{\wedge^r H}(\bigwedge_i e_i),\ \bigwedge_i e_i\right) \\
&= \left(\bigwedge_i P_H e_i,\ \bigwedge_i e_i\right) \\
&= \det[(P_H e_i,\ e_j)]\,.
\end{aligned}$$

Let the ith row of M be m_i. For some constant c, we thus have

$$\xi_H = c \bigwedge_{i=1}^{r} m_i \,,$$

whence

$$\begin{aligned}\mathbf{P}^H[B] &= \left|\left(\xi_H, \bigwedge_{e \in B} e\right)\right|^2 \\ &= |c|^2 \left|\det \left[(m_i, e)\right]_{i \le r,\ e \in B}\right|^2 \\ &= |c|^2 \,|\det M_B|^2 \,.\end{aligned}$$

Now we calculate $|c|^2$:

$$\begin{aligned}1 = \|\xi_H\|^2 = |c|^2 \,\|\bigwedge_{i=1}^{r} m_i\|^2 &= |c|^2 \det\left[(m_i, m_j)\right]_{i,j \le r} \\ &= |c|^2 \det(M_{(r)} M_{(r)}^T) \,.\end{aligned}$$

THE MATRIX-TREE THEOREM. *Let G be a finite connected graph and $o \in \mathsf{V}$. Then the number of spanning trees of G equals*

$$\det\left[(\star_x, \star_y)\right]_{x \neq o, y \neq o}.$$

Proof. In other words, we want to show that if $\mathbf{u}$ is the wedge product (in some order) of the stars at all the vertices other than o, then $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$ is the number of spanning trees. Any set of all the stars but one is a basis for $\bigstar$. Thus, $\mathbf{u}$ is a multiple of $\xi_\bigstar$. Since $\bigstar$ represents the graphic matroid, the only non-zero coefficients of $\mathbf{u}$ are those in which choosing one edge in each $\star_x$ for $x \neq o$ yields a spanning tree; moreover, each spanning tree occurs exactly once since there is exactly one way to choose an edge incident to each $x \neq o$ to get a given spanning tree. This means that its coefficient is ± 1. ∎

This proof also shows that $\mathbf{P}^\bigstar$ is uniform.

Additional Probabilities: Recall that

$$\mathbf{P}^{H}[A \subseteq \mathfrak{B}] = \det[(P_{H}e,\, e')]_{e,e' \in A} = \left(P_{\mathrm{Ext}(H)}\theta_{A},\, \theta_{A}\right),$$

where, for a finite subset $A = \{e_1, \ldots e_k\} \subseteq E$, we write

$$\theta_{A} := \bigwedge_{i=1}^{k} e_i \,.$$

This is proved by proving an extension:

For any $A_1, A_2 \subseteq E$,

$$\mathbf{P}^{H}[A_1 \subseteq \mathfrak{B}, A_2 \cap \mathfrak{B} = \varnothing] = \left(P_{\mathrm{Ext}(H)}\theta_{A_1} \wedge P_{\mathrm{Ext}(H^{\perp})}\theta_{A_2},\, \theta_{A_1} \wedge \theta_{A_2}\right).$$

(First prove when $A_1 \cup A_2 = E$, then sum over partitions.) Therefore

$$\mathbf{P}^{H^{\perp}}[B] = \mathbf{P}^{H}[E \setminus B]\,.$$

Orthogonal subspaces thus correspond to dual matroids.

Additional Property: Extend $\mathbf{P}^H$ from $\mathscr{B}$ to the collection 2^E of all subsets of E.

An event $\mathcal{A}$ is called **increasing** if whenever $A \in \mathcal{A}$ and $e \in E$, we have also $A \cup \{e\} \in \mathcal{A}$.

Given two probability measures $\mathbf{P}^1$, $\mathbf{P}^2$ on 2^E, we say that **$\mathbf{P}^1$ is stochastically dominated by $\mathbf{P}^2$** and write $\mathbf{P}^1 \preccurlyeq \mathbf{P}^2$ if

$$\mathbf{P}^1[\mathcal{A}] \le \mathbf{P}^2[\mathcal{A}] \qquad \text{for all increasing } \mathcal{A}\,.$$

THEOREM (L.). *If $H' \subseteq H \subseteq \ell^2(E)$, then $\mathbf{P}^{H'} \preccurlyeq \mathbf{P}^H$.*

A **monotone coupling** of two probability measures $\mathbf{P}^1$, $\mathbf{P}^2$ on 2^E is a probability measure μ on $2^E \times 2^E$ whose coordinate projections give $\mathbf{P}^1$, $\mathbf{P}^2$ and which is concentrated on the set $\{(A_1, A_2)\,;\; A_1 \subseteq A_2\}$. That is,

$$\forall A_1 \subseteq E \qquad \sum_{A_2 \subseteq E} \mu(A_1, A_2) = \mathbf{P}^1[A_1]\,,$$

$$\forall A_2 \subseteq E \qquad \sum_{A_1 \subseteq E} \mu(A_1, A_2) = \mathbf{P}^2[A_2]\,,$$

$$\forall A_1, A_2 \subseteq E \qquad \mu(A_1, A_2) \neq 0 \quad \Longrightarrow \quad A_1 \subseteq A_2\,.$$

Strassen's theorem (proved, say, by Max Flow-Min Cut Theorem) says that stochastic domination is equivalent to existence of a monotone coupling.

Open Question: Find an explicit monotone coupling of $\mathbf{P}^{H'}$ and $\mathbf{P}^{H}$ when $H' \subseteq H$.

Extension to Infinite E

Let $E = \{e_1, e_2, \ldots\}$. If $H \subset \ell^2(E)$ is finite-dimensional, then write H_k for the image of the orthogonal projection of H onto the span of $\{e_1, e_2, \ldots, e_k\}$. Then the matrix entries of P_{H_k} converge to those of P_H, whence we may define $\mathbf{P}^H$ to be the weak* limit of $\mathbf{P}^{H_k}$.

If $H \subseteq \ell^2(E)$ is closed and infinite-dimensional, then let H_k be finite-dimensional subspaces of H that are increasing with union dense in H. Again, the matrix entries of P_{H_k} converge to those of P_H, whence we may define $\mathbf{P}^H$ to be the weak* limit of $\mathbf{P}^{H_k}$.

Theorem (L.). *Let E be finite or infinite and let $H \subseteq H'$ be closed subspaces of $\ell^2(E)$. Then $\mathbf{P}^H \preccurlyeq \mathbf{P}^{H'}$, with equality iff $H = H'$.*

This means that there is a probability measure on the set $\big\{(B, B')\,;\ B \subseteq B'\big\}$ that projects in the first coordinate to $\mathbf{P}^H$ and in the second to $\mathbf{P}^{H'}$.

Let $G = (\mathsf{V}, \mathsf{E})$ be a finite graph. Choose one orientation for each edge $e \in \mathsf{E}$. Let $\bigstar = B^1(G)$ denote the subspace in $\ell^2(\mathsf{E})$ spanned by the stars (coboundaries) and let $\diamondsuit = Z_1(G)$ denote the subspace spanned by the cycles. Then $\ell^2(\mathsf{E}) = \bigstar \oplus \diamondsuit$.

For an infinite graph, let $\bigstar := \bar{B}^1_c(G)$ be the closure in $\ell^2(\mathsf{E})$ of the span of the stars.

For an infinite graph, Benjamini, Lyons, Peres, and Schramm (2001) showed that WUSF is the determinantal measure corresponding to orthogonal projection on $\bigstar$, while FUSF is the determinantal measure corresponding to $\diamondsuit^\perp$.

Thus, $\mathsf{WUSF} \preccurlyeq \mathsf{FUSF}$, with equality iff $\bigstar = \diamondsuit^\perp$.

Open Questions: Orthogonal Decomposition

Suppose that $H = H_1 \oplus H_2$. Is there a disjoint coupling of $\mathbf{P}^{H_1}$ with $\mathbf{P}^{H_2}$ whose union marginal is $\mathbf{P}^H$? I.e., is there a probability measure μ on $2^E \times 2^E$ such that

$$\forall A_1 \subseteq E \qquad \sum_{A_2 \subseteq E} \mu(A_1, A_2) = \mathbf{P}^{H_1}[A_1]\,,$$

$$\forall A_2 \subseteq E \qquad \sum_{A_1 \subseteq E} \mu(A_1, A_2) = \mathbf{P}^{H_2}[A_2]\,,$$

$$\forall A_1, A_2 \subseteq E \qquad \mu(A_1, A_2) \neq 0 \quad \Longrightarrow \quad A_1 \cap A_2 = \varnothing\,,$$

$$\forall A \subseteq E \qquad \sum_{A_1 \cup A_2 = A} \mu\{(A_1, A_2)\} = \mathbf{P}^H[A]\,?$$

E.g., if $H = \ell^2(E)$, then "yes" since then $\mathbf{P}^{H_1}$ and $\mathbf{P}^{H_2}$ correspond to dual matroids and complementary subsets. In general, there is some computer evidence.

Open Questions: Group Representations

We can ask for even more. Suppose that E is a group. Then $\ell^2(E)$ is the group algebra. Invariant subspaces H give subrepresentations of the regular representation and give invariant probability measures $\mathbf{P}^H$. There is a canonical decomposition

$$\ell^2(E) = \bigoplus_{i=1}^{s} H_i \,,$$

where each H_i is an invariant subspace containing all isomorphic copies of a given irreducible subrepresentation. Can we disjointly couple all measures $\mathbf{P}^{H_i}$ so that every partial union has marginal equal to $\mathbf{P}^H$ for H the corresponding partial sum? P_{H_i} given by characters

Consider the case $E = \mathbb{Z}_n$. All irreducible representations are 1-dimensional and there are n of them: for each $k \in \mathbb{Z}_n$, we have the representation

$$j \mapsto e^{2\pi i k j / n} \,.$$

Thus, a coupling as above would be a random permutation of $\mathbb{Z}_n$ with special properties.

comp evid to $n = 7$.

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