

RECURRENCE OF RANDOM WALK TRACES¹

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We show that the edges crossed by a random walk in a network form a recurrent graph a.s. In fact, the same is true when those edges are weighted by the number of crossings.

1. Introduction. Let $G = (V, E)$ be a locally finite graph and let $c : E \rightarrow [0, \infty)$ be an assignment of *conductances* to the edges. We call (G, c) a *network*. The associated random walk has transition probabilities $p(x, y) := c(x, y)/\pi(x)$, where $\pi(x) := \sum_y c(x, y)$. Assume that the network random walk is transient when it starts from some fixed vertex o . How big can the trace be, the set of edges traversed by the random walk? We show that they form a.s. a recurrent graph (for a simple random walk).

This fact is already known when G is a Euclidean lattice and $c \equiv 1$ since a.s. the paths there have infinitely many cut-times, a time when the past of the path is disjoint from its future; see [7] and [8]. From this, recurrence follows by the criterion of Nash-Williams [12]. By contrast, Lyons and Peres [9] constructed an example of a transient birth-and-death chain which a.s. has only finitely many cut-times.

A result of similar spirit to ours was proved by Morris [11], who showed that the components of the wired uniform spanning forest are a.s. recurrent. For another a.s. recurrence theorem (for distributional limits of finite planar graphs), see [5].

We expect that a Brownian analogue of the theorem is true, that is, a.s. parabolicity of the Wiener sausage, with reflected boundary conditions. For background on recurrence in the Riemannian context, see, for example, [6]. It would be interesting to prove similar theorems for other processes. For example, consider the trace of a branching random walk on a graph G . Then we conjecture that almost surely the trace is recurrent for a branching random walk with the same branching law. Perhaps a similar result holds for general tree indexed random walks. See Benjamini and Peres [3, 4] for definitions and background.

Perhaps one can strengthen our result as follows. Given a transient network (G, c) , denote by T_n the trace of the first n steps of the network random

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walk. Let $R(n)$ be the maximal effective resistance on T_n between o and another vertex of T_n , where each edge has unit conductance. By our theorem, $R(n) \uparrow \infty$ a.s. [Note, of course, that $R(n) \uparrow \infty$ for growing subgraphs does not imply recurrence of their union, as balls in the binary tree show.] Is there a uniform lower bound over all transient networks for the rate at which $R(n) \uparrow \infty$? That is, does there exist a function f with $\lim_n f(n) = \infty$ such that for any transient network,

$$\limsup_n R(n)/f(n) > 0 \quad \text{a.s.}?$$

In particular, one can speculate that $f(n) = \log^2 n$ might work, which would arise from the graph \mathbb{Z}^2 (although \mathbb{Z}^2 is recurrent, it is on the border of transience). On the other hand, transient wedges in \mathbb{Z}^3 might allow one to prove that there is no such f .

2. Proof. Our proof will demonstrate the following stronger results.

Let $N(x, y)$ denote the number of traversals of the edge (x, y) .

THEOREM 2.1. *The network $(G, \mathbf{E}[N])$ is recurrent. The networks (G, N) and $(G, \mathbf{1}_{\{N>0\}})$ are a.s. recurrent.*

We shall use some facts relating electrical networks to random walks. See [10] for more background.

Let $\mathfrak{g}(x, y)$ be the Green function, that is, the expected number of visits to y for a network random walk started at x .

The *effective resistance* from a vertex o to infinity is defined to be the minimum energy $\frac{1}{2} \sum_{x \neq y} \theta(x, y)^2 / c(x, y)$ of any unit flow θ from o to infinity. This also equals

$$(2.1) \quad \alpha := \mathfrak{g}(o, o) / \pi(o).$$

In particular, the effective resistance is finite iff the network random walk is transient. Its reciprocal, *effective conductance*, is given by Dirichlet’s principle as the infimum of the Dirichlet energy $\frac{1}{2} \sum_{x \neq y} c(x, y) [F(x) - F(y)]^2$ over all functions $F : \mathbf{V} \rightarrow [0, 1]$ that have finite support and satisfy $F(o) = 1$. Since the functional $c \mapsto \sum_{x \neq y} c(x, y) [F(x) - F(y)]^2$ is linear for any given F , we see that effective conductance is concave in c . Thus, if the conductances $\langle \mathbf{E}[N(x, y)]; (x, y) \in \mathbf{E}(G) \rangle$ give a recurrent network, then so a.s. do $\langle N(x, y); (x, y) \in \mathbf{E}(G) \rangle$. Furthermore, Rayleigh’s monotonicity principle implies that if (G, N) is recurrent, then so is $(G, \mathbf{1}_{\{N>0\}})$. (Of course, it follows that any finite union of traces, whether independent or not, is also recurrent a.s.)

Thus, it remains to prove that $(G, \mathbf{E}[N])$ is recurrent. We shall, however, also show how the proof that (G, N) is a.s. recurrent follows from a simpler argument. Another mostly probabilistic proof of this is due to Benjamini and Gurel-Gurevich [2].

The *effective resistance* from a finite set of vertices A to infinity is defined to be the effective resistance from a to infinity when A is identified to a single vertex, a . The *effective resistance* from an infinite set of vertices A to infinity is defined to be the infimum of the effective resistance from B to infinity among all finite subsets $B \subset A$. Its reciprocal, the *effective conductance* from A to infinity in the network (G, c) , will be denoted by $\mathcal{C}(A, G, c)$. From the above, we have

$$(2.2) \quad \mathcal{C}(A, G, c) = \sup_B \inf \left\{ \frac{1}{2} \sum_{x \neq y} c(x, y) [F(x) - F(y)]^2; \right. \\ \left. F \upharpoonright B \equiv 1, F \text{ has finite support} \right\},$$

where the supremum is over finite subsets B of A .

Let the original voltage function be $v(\bullet)$ throughout this article, where $v(o) = 1$ and $v(\bullet)$ is 0 “at infinity.” Then $v(x)$ is the probability of ever visiting o for a random walk starting at x .

Note that

$$\mathbf{E}[N(x, y)] = \mathcal{G}(o, x)p(x, y) + \mathcal{G}(o, y)p(y, x) \\ = (\mathcal{G}(o, x)/\pi(x) + \mathcal{G}(o, y)/\pi(y))c(x, y)$$

and

$$\pi(o)\mathcal{G}(o, x) = \pi(x)\mathcal{G}(x, o) = \pi(x)v(x)\mathcal{G}(o, o).$$

Thus, we have [from the definition (2.1)]

$$(2.3) \quad \mathbf{E}[N(x, y)] = \alpha c(x, y)[v(x) + v(y)] \\ \leq 2\alpha \max\{v(x), v(y)\}c(x, y) \leq 2\alpha c(x, y).$$

In a finite network (H, c) , we write $\mathcal{C}(A, z; H, c)$ for the effective conductance between a subset A of its vertices and a vertex z . This is given by Dirichlet’s principle as the infimum of the Dirichlet energy of F over all functions $F : \mathbf{V}(H) \rightarrow [0, 1]$ that satisfy $F \upharpoonright A \equiv 1$ and $F(z) = 0$. Clearly, $A \subset B \subset \mathbf{V}$ implies that $\mathcal{C}(A, z; H, c) \leq \mathcal{C}(B, z; H, c)$. The function that minimizes the Dirichlet energy is the voltage function, v . The amount of current that flows from A to z in this case is defined as $\sum_{x \in A, y \notin A} [v(x) - v(y)]c(x, y)$; it equals $\mathcal{C}(A, z; H, c)$. The voltage function that is t on A instead of 1 has Dirichlet energy equal to $t^2\mathcal{C}(A, z; H, c)$ and gives a current that is t times as large as the unit-voltage current, which shows that $\mathcal{C}(A, z; H, c)$ is the amount of current that flows from A to z divided by the voltage on A .

LEMMA 2.1. *Let (H, c) be a finite network and $a, z \in \mathbf{V}(H)$. Let v be the voltage function that is 1 at a and 0 at z . For $0 < t < 1$, let A_t be the set of*

vertices x with $v(x) \geq t$. Then $\mathcal{C}(A_t, z; H, c) \leq \mathcal{C}(a, z; H, c)/t$. Thus, for every $A \subset V(H) \setminus \{z\}$, we have

$$\mathcal{C}(A, z; H, c) \leq \frac{\mathcal{C}(a, z; H, c)}{\min v \upharpoonright A}.$$

PROOF. We subdivide edges as follows. If any edge (x, y) is such that $v(x) > t$ and $v(y) < t$, then subdividing the edge (x, y) with a vertex z by giving resistances

$$r(x, z) := \frac{v(x) - t}{v(x) - v(y)}r(x, y)$$

and

$$r(z, y) := \frac{t - v(y)}{v(x) - v(y)}r(x, y)$$

will result in a network such that $v(z) = t$ while no other voltages change. Doing this for all such edges gives a possibly new graph H' and a new set A'_t whose internal vertex boundary is a set W'_t on which the voltage is identically equal to t . We have $\mathcal{C}(A_t, z; H, c) = \mathcal{C}(A_t, z; H', c) \leq \mathcal{C}(A'_t, z; H', c)$. Now $\mathcal{C}(A'_t, z; H', c) = \mathcal{C}(a, z; H, c)/t$ since the amount of current that flows is $\mathcal{C}(a, z; H, c)$ and the voltage difference is t . Therefore, $\mathcal{C}(A_t, z; H, c) \leq \mathcal{C}(a, z; H, c)/t$, as desired.

For a general A , let $t := \min v \upharpoonright A$. Since $A \subset A_t$, we have $\mathcal{C}(A, z; H, c) \leq \mathcal{C}(A_t, z; H, c)$. Combined with the inequality just reached, this yields the final conclusion. \square

For $t \in (0, 1)$, let $V_t := \{x \in V; v(x) < t\}$. Let W_t be the external vertex boundary of V_t , that is, the set of vertices outside V_t that have a neighbor in V_t . Write G_t for the subgraph of G induced by $V_t \cup W_t$.

We will refer to the conductances c as the *original* ones and the conductances $\mathbf{E}[N]$ as the *new* ones for convenience.

LEMMA 2.2. *The effective conductance from W_t to ∞ in the network $(G_t, \mathbf{E}[N])$ is at most 2.*

PROOF. If any edge (x, y) is such that $v(x) > t$ and $v(y) < t$, then subdividing the edge (x, y) with a vertex z as in the proof of Lemma 2.1 and consequently adding z to W_t has the effect of raising the conductance of the edge (x, y) to $c(z, y) = c(x, y)[v(x) - v(y)]/[t - v(y)]$ and also, by (2.3), of raising its conductance in the new network from $\mathbf{E}[N(x, y)]$ to

$$\begin{aligned} \alpha c(z, y)[t + v(y)] &= \alpha c(z, y)[t - v(y) + 2v(y)] \\ &= \alpha c(x, y)[v(x) - v(y)] + 2\alpha c(z, y)v(y) \\ &> \alpha c(x, y)[v(x) - v(y)] + 2\alpha c(x, y)v(y) = \mathbf{E}[N(x, y)]. \end{aligned}$$

Since raising edge conductances clearly raises effective conductance, it suffices to prove the lemma in the case that $v(x) = t$ for all $x \in W_t$. Thus, we assume this case for the remainder of the proof.

Suppose that $\langle (H_n, c); n \geq 1 \rangle$ is an increasing exhaustion of (G, c) by finite networks that include o . Identify the boundary (in G) of H_n to a single vertex, z_n . Let v_n be the corresponding voltage functions with $v_n(o) = 1$ and $v_n(z_n) = 0$. Then $\mathcal{C}(o, z_n; H_n, c) \downarrow 1/\alpha$ and $v_n(x) \uparrow v(x)$ as $n \rightarrow \infty$ for all $x \in V(G)$. Let A be a finite subset of W_t . By Lemma 2.1, as soon as $A \subset V(H_n)$, we have that the effective conductance from A to z_n of H_n is at most $\mathcal{C}(o, z_n; H_n, c) / \min\{v_n(x); x \in A\}$. Therefore by Rayleigh’s monotonicity principle, $\mathcal{C}(A, G_t, c) \leq \mathcal{C}(A, G, c) = \lim_{n \rightarrow \infty} \mathcal{C}(A, z_n; H_n, c) \leq 1/(\alpha t)$. Since this holds for all such A , we have

$$(2.4) \quad \mathcal{C}(W_t, G_t, c) \leq 1/(\alpha t).$$

By (2.3), the new conductances on G_t are obtained by multiplying the original conductances by factors that are at most $2\alpha t$. Combining this with (2.4), we obtain that the new effective conductance from W_t to infinity in G_t is at most 2. \square

When the complement of V_t is finite for all t , which is the case for “most” networks, this completes the proof by the following lemma (and by the fact that $\bigcap_{t>0} V_t = \emptyset$):

LEMMA 2.3. *If H is a transient network, then for all $m > 0$, there exists a finite subset $K \subset V(H)$ such that for all finite $K' \supseteq K$, the effective conductance from K' to infinity is more than m .*

PROOF. Let θ be a unit flow of finite energy from a vertex o to ∞ . Since θ has finite energy, there is some $K \subset V(G)$ such that the energy of θ on the edges with some endpoint not in K is less than $1/m$. That is, the effective resistance from K to infinity is less than $1/m$. \square

Even when the complement of V_t is not finite for all t , this is enough to show that the network (G, N) is a.s. recurrent: If X_n denotes the position of the random walk on (G, c) at time n , then $v(X_n) \rightarrow 0$ a.s. by Lévy’s 0–1 law. Thus, the path is a.s. contained in V_t after some time, no matter the value of $t > 0$. By Lemma 2.3, if (G, N) is transient with probability $p > 0$, then $\mathcal{C}(B_n, G, N)$ tends in probability, as $n \rightarrow \infty$, to a random variable that is infinite with probability p , where B_n is the ball of radius n about o . In particular, this effective conductance is at least $6/p$ with probability at least $p/2$ for all large n . Fix n with this property. Let $t > 0$ be such that $V_t \cap B_n = \emptyset$. Write D for the (finite) set of vertices in G incident to an edge $e \notin G_t$ with $N(e) > 0$. Then $\mathcal{C}(W_t, G_t, N) = \mathcal{C}(W_t \cup D, G, N) \geq \mathcal{C}(B_n, G, N)$. However, this implies that $\mathcal{C}(W_t, G_t, \mathbf{E}[N]) \geq \mathbf{E}[\mathcal{C}(W_t, G_t, N)] \geq 3$, which contradicts Lemma 2.2.

To complete the proof that $(G, \mathbf{E}[N])$ is recurrent in general, we show that although V_t may not separate the source o from infinity, its complement in the network is recurrent:

LEMMA 2.4. *The vertices $V \setminus V_t$ induce a recurrent network for the original and for the new conductances.*

PROOF. Condition that the original random walk on G returns to its starting point, o . Of course, the corresponding Doob-transformed Markov chain is recurrent. This corresponds to transformed transition probabilities $p(x, y)v(y)/v(x)$ for $x \neq o$, whence to transformed conductances $c'(x, y) := c(x, y)v(x)v(y)$. Rayleigh’s monotonicity principle gives that when we delete V_t , we still have a recurrent network. But off of V_t , the conductances c' differ by a bounded factor from the original conductances and also from the new conductances. This means that the part remaining after we delete V_t is recurrent for both the original and new conductances. \square

PROOF OF THEOREM 2.1. The function $x \mapsto v(x)$ has finite Dirichlet energy for the original network, hence for the new (since conductances are multiplied by a bounded factor). Assume (for a contradiction) that the new random walk is transient. Then by Ancona, Lyons and Peres [1], $\langle v(X_n) \rangle$ converges a.s. for the new random walk. By Lemma 2.4, it a.s. cannot have a limit $> t$ for any $t > 0$, so it converges to 0 a.s.

This means that the unit current flow i for the new network (which is the expected number of signed crossings of edges) has total flow 1 through W_t into G_t for all $t > 0$. Thus, we may choose a finite subset A_t of W_t through which at least $1/2$ of the new current enters. With the notation $(d_t^* i)(x) := \sum_{y \in V(G_t)} i(x, y)$, this means that $\sum_{x \in A_t} d_t^* i(x) \geq 1/2$. By Lemma 2.2, there is a function $F_t : V_t \cup W_t \rightarrow [0, 1]$ with finite support and with $F_t \equiv 1$ on A_t whose Dirichlet energy on the network $(G_t, \mathbf{E}[N])$ is at most 3. Write $(dF_t)(x, y) := F_t(x) - F_t(y)$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left[\sum_{x \neq y \in V(G_t)} i(x, y) dF_t(x, y) \right]^2 &\leq \sum_{x \neq y \in V(G_t)} i(x, y)^2 / c(x, y) \\ &\quad \times \sum_{x \neq y \in V(G_t)} c(x, y) dF_t(x, y)^2 \\ &\leq 3 \sum_{x \neq y \in V(G_t)} i(x, y)^2 / c(x, y). \end{aligned}$$

On the other hand, summation by parts yields that

$$\sum_{x \neq y \in V(G_t)} i(x, y) dF_t(x, y) = \sum_{x \in V(G_t)} d_t^* i(x) F_t(x) \geq \sum_{x \in A_t} d_t^* i(x) \geq 1/2.$$

Therefore, $\sum_{x \neq y \in V(G_t)} i(x, y)^2 / c(x, y) \geq 1/12$, which contradicts $\bigcap_t V(G_t) = \emptyset$ and the fact that i has finite energy. \square

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