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ERRATUM TO "LOWER BOUNDS FOR TRACE RECONSTRUCTION"

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We correct the proof of Lemma 3.1 of our paper *Ann. Appl. Probab.* **30**, no. 2 (2020), 503–525.

Lemma 3.1 asserts that $\mathbf{E}_{\mathbf{y}_n}\big[Z(\widetilde{\mathbf{y}}_n)\big] - \mathbf{E}_{\mathbf{x}_n}\big[Z(\widetilde{\mathbf{x}}_n)\big] = \Theta(n^{-1/2})$ and $\mathbf{E}_{\mathbf{y}_n}\big[Z(\widetilde{\mathbf{y}}_n)\big] > \mathbf{E}_{\mathbf{x}_n}\big[Z(\widetilde{\mathbf{x}}_n)\big]$ for all sufficiently large n. Our proof was not correct: As Benjamin Gunby and Xiaoyu He pointed out to us, we missed four terms in the computation of equation (3.3). Those terms contribute a negative amount, so the proof is more delicate. Here is a correct proof.

The intuition behind the result is that a string with a defect of the type we consider, namely, a 10 in a string of 01's, is likely to cause more 11's in the trace than a string without the defect. Since the defect in $\mathbf{y}_n$ is shifted to the right as compared to the defect in $\mathbf{x}_n$, the defect of $\mathbf{y}_n$ is slightly more likely to "fall into" the test window $\{\lceil 2np+1\rceil, \dots, \lfloor 2np+\sqrt{npq}\rfloor\}$ of the trace than is the defect of $\mathbf{x}_n$. More precisely, the difference in probability is of order $n^{-1/2}$. In the proof below, we make this intuition rigorous.

PROOF. We assume throughout the proof that $k \in \{\lceil 2np+1\rceil, \dots, \lfloor 2np+\sqrt{npq}\rfloor\}$. Let $E(m,k)$ denote the event that bit m in the input string is copied to position k in the trace. First observe that

$$\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1\big] = \sum_{m=k}^{4n} \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\,\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \,\big|\, E(m,k)\big],$$

$$\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1\big] = \sum_{m=k}^{4n} \mathbf{P}_{\mathbf{y}_n}\big[E(m,k)\big]\,\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1 \,\big|\, E(m,k)\big],$$

and

$$\mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big] = \mathbf{P}_{\mathbf{y}_n}\big[E(m,k)\big] = (1-q)^k q^{m-k}\binom{m-1}{k-1}, \quad m \in \{k, \dots, 4n\}.$$

Note that the string $\mathbf{x}_n$ centered at m is identical to the string $\mathbf{y}_n$ centered at $m+2$, except for two bits at the ends. Therefore, for every $m \in \{k, \dots, 3n\}$, we have

$$\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \,\big|\, E(m,k)\big] = \mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1 \,\big|\, E(m+2,k)\big] \pm o^{\infty}(n),$$

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where $o^\infty(n)$ denotes something non-negative that decays at least exponentially fast in n. Combining this with $\mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big] = o^\infty(n)$ for $m < k+2$ or $m > 3n$ yields

$$\begin{aligned}
&\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1\big] - \mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1\big] \\
&\quad = \sum_{m=k}^{3n} \Big(\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\Big)\, \mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \bigm| E(m,k)\big] \\
&\qquad \pm o^\infty(n).
\end{aligned}$$

Setting $a_m := qp/(1-q^2) = q/(1+q)$ if m is even and $a_m := 0$ otherwise, we see that

$$\sum_{m=k}^{3n} \Big(\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\Big) a_m = \pm o^\infty(n).$$

Subtracting this from the previous display gives

$$\begin{aligned}
&\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1\big] - \mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1\big] \\
&\quad = \sum_{m=k}^{3n} \Big(\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\Big) \\
&\qquad\qquad \cdot \Big(\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \bigm| E(m,k)\big] - a_m\Big) \pm o^\infty(n).
\end{aligned} \tag{E.1}$$

The second factor in the above summand, modulo an additive error of $o^\infty(n)$, represents the *difference* in probability of the event $\widetilde{x}_k = \widetilde{x}_{k+1} = 1$ given $E(m,k)$ for the string $\mathbf{x}_k$ as compared to a string without any defect. It takes the following explicit form:

$$\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \bigm| E(m,k)\big] - a_m \approx \begin{cases} 0 & \text{if } m \le 2n-3 \text{ is odd,} \\ q^{2n-m-2}(1-q)^2 & \text{if } m \le 2n-2 \text{ is even,} \\ \frac{q^2}{1+q} & \text{if } m = 2n-1, \\ -\frac{q}{1+q} & \text{if } m = 2n, \\ 0 & \text{if } 2n+1 \le m \le 3n, \end{cases} \tag{E.2}$$

where $\approx$ means that we incur an additive error of $\pm o^\infty(n)$.

Now let j_0 be a sufficiently large positive integer that

$$1 - q - q^{2j_0} > 0. \tag{E.3}$$

Note that j_0 depends on q but can be chosen so that it does not depend on n. We suppose in the rest of the proof that $n > j_0$. By (E.1) and (E.2),

$$\begin{aligned}
&\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1\big] - \mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1\big] \\
&\quad \ge \sum_{m=2n-2j_0}^{2n} \Big(\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\Big) \\
&\qquad\qquad \cdot \Big(\mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1 \bigm| E(m,k)\big] - a_m\Big) - o^\infty(n).
\end{aligned} \tag{E.4}$$

For $m \in \{2n-2j_0, \ldots, 2n+2\}$ and with $\xi := k - 2np$, we have

$$\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big] = \mathbf{P}_{\mathbf{x}_n}\big[E(2n,k)\big]\Big(\frac{\xi}{nq} \pm O\Big(\frac{1}{n}\Big)\Big), \tag{E.5}$$

because for $m \in \{2n-2j_0, \dots, 2n\}$,

$$\begin{aligned}\frac{\mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]}{\mathbf{P}_{\mathbf{x}_n}\big[E(2n,k)\big]} &= \frac{(m-k+1)(m-k+2)\cdots(2n-k)}{m(m+1)\cdots(2n-1)\cdot q^{2n-m}} \\ &= \frac{(\frac{m-2np+1}{2nq}-\frac{\xi}{2nq})(\frac{m-2np+2}{2nq}-\frac{\xi}{2nq})\cdots(1-\frac{\xi}{2nq})}{\frac{m}{2n}\cdot\frac{m+1}{2n}\cdots(1-\frac{1}{2n})} \\ &= 1-\xi(2n-m)/(2nq) \pm O(1/n) \pm O(\xi^2/n^2) \\ &= 1-\xi(2n-m)/(2nq) \pm O(1/n);\end{aligned}$$

the same result holds for $m \in \{2n+1, 2n+2\}$ by a similar estimate.

Combining (E.2) and (E.5), we get that the right-hand side of (E.4) is equal to

$$\mathbf{P}_{\mathbf{x}_n}\big[E(2n,k)\big]\Big(\frac{\xi}{nq} \pm O\Big(\frac{1}{n}\Big)\Big)\cdot\frac{(1-q)(1-q-q^{2j_0})}{1+q}. \tag{E.6}$$

Summing the left-hand side of (E.4) over $k \in \{\lceil 2np+1\rceil, \dots, \lfloor 2np+\sqrt{npq}\rfloor\}$ and using the last display along with $\mathbf{P}_{\mathbf{x}_n}\big[E(2n,k)\big] = \Theta(n^{-1/2})$ and (E.3), we get the lower bounds in the lemma, namely, $\mathbf{E}_{\mathbf{y}_n}\big[Z(\widetilde{\mathbf{y}}_n)\big] - \mathbf{E}_{\mathbf{x}_n}\big[Z(\widetilde{\mathbf{x}}_n)\big] = \Omega(n^{-1/2})$ and $\mathbf{E}_{\mathbf{y}_n}\big[Z(\widetilde{\mathbf{y}}_n)\big] > \mathbf{E}_{\mathbf{x}_n}\big[Z(\widetilde{\mathbf{x}}_n)\big]$.

It remains to prove the upper bound, namely, $\mathbf{E}_{\mathbf{y}_n}\big[Z(\widetilde{\mathbf{y}}_n)\big] - \mathbf{E}_{\mathbf{x}_n}\big[Z(\widetilde{\mathbf{x}}_n)\big] = O(n^{-1/2})$. Let $b_{m,n}$ denote the absolute value of the right-hand side of (E.2). By (E.1) and (E.2), we have

$$\begin{aligned}&\big|\mathbf{P}_{\mathbf{y}_n}\big[\widetilde{y}_k = \widetilde{y}_{k+1} = 1\big] - \mathbf{P}_{\mathbf{x}_n}\big[\widetilde{x}_k = \widetilde{x}_{k+1} = 1\big]\big| \\ &\quad\le \sum_{m=\lceil 2np-1\rceil}^{3n}\Big|\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\Big|\cdot b_{m,n} + o^{\infty}(n).\end{aligned}$$

Now sum over k; (2.7) of Lemma 2.2 yields $\sum_k\big|\mathbf{P}_{\mathbf{x}_n}\big[E(m+2,k)\big] - \mathbf{P}_{\mathbf{x}_n}\big[E(m,k)\big]\big| = O(m^{-1/2}) = O(n^{-1/2})$. In addition, $\sum_m b_{m,n} = O(1)$. Combining these bounds, we arrive at the upper bound of the lemma. □

We remark that one can get a more precise bound in (E.6) that gives something positive for all $q \in (0,1)$ simultaneously by not truncating the sum on the right-hand side of (E.1) and by using a more precise version of (E.5). The result, in fact, gives lower *and* upper bounds for the left-hand side of (E.4) of the form

$$\mathbf{P}_{\mathbf{x}_n}\big[E(2n,k)\big]\Big(\frac{\xi}{nq} \pm O\Big(\frac{1}{n}\Big)\Big)\cdot\frac{(1-q)^2}{1+q}.$$

Finally, we note that in the proof of Proposition 1.4 on p. 519, the definitions of X and Y should be slightly modified: c should be $\sqrt{c}$ both times.

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