Weak and Weak* Topologies

by Russell Lyons

For us, the key duality relationship is the Riesz representation theorem, identifying $M(\mathbb{R})$ (i.e., the Banach space of finite, signed measures on $\mathbb{R}$ with norm being total variation) as the dual of $C_0(\mathbb{R})$ (i.e., the space of continuous functions on $\mathbb{R}$ that vanish—tend to 0—at $\pm\infty$, with the sup norm), with $\mu \in M(\mathbb{R})$ defining the linear functional $f \mapsto \int f \, \mathrm{d}\mu$. This theorem was extended by Kakutani from $\mathbb{R}$ to every locally compact Hausdorff space, such as $\mathbb{R}^d$. Thus, we recall some basic facts from functional analysis.

If X is a Banach space, then its dual, X^*, denotes the linear space of bounded (= continuous) linear functionals on X. The norm on X^* is

$$\|x^*\| := \sup\bigl\{|x^*(x)| \, ; \, \|x\| = 1\bigr\} = \sup\bigl\{x^*(x) \, ; \, \|x\| = 1\bigr\} = \sup\bigl\{x^*(x)/\|x\| \, ; \, x \neq 0\bigr\},$$

which makes X^* into a Banach space. Note that when $X = C_0(\mathbb{R})$, this says that for $\mu \in M(\mathbb{R})$,

$$\|\mu\| = \sup\biggl\{\int f \, \mathrm{d}\mu \, ; \, f \in C_0(\mathbb{R}), \; \|f\| = 1\biggr\}.$$

The weak* topology on X^* is the smallest topology with respect to which $x^* \mapsto x^*(x)$ is continuous for each $x \in X$. If X is separable (i.e., X has a countable, dense subset), then the weak* topology of X^* is metrizable, so $x_n^* \overset{w^*}{\to} x^*$ iff $\forall x \in X \;\; x_n^*(x) \to x^*(x)$. For example, $\mu_n \overset{w^*}{\to} \mu$ iff $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ for all $f \in C_0(\mathbb{R})$. (If we required this for all $f \in C_{\mathrm{b}}(\mathbb{R})$, then this would be an extension of weak convergence for probability measures, by the portmanteau theorem. If we required this for all $f \in C_{\mathrm{c}}(\mathbb{R})$—where "c" means with compact support—, then this would be what is called ***vague convergence***, which is important for infinite measures. For finite measures with bounded norm, vague convergence is the same as weak* convergence.)

The Banach–Alaoglu theorem says that the unit ball of X^* is weak* compact (i.e., compact in the weak* topology). Recall that a metric space is compact iff it is sequentially compact. Thus, when X is separable, the unit ball of X^* is weak* sequentially compact.

We call a (positive) measure a ***subprobability measure*** if its norm is at most 1. Here are three homework problems about $M(\mathbb{R})$:

Homework 1. *Let μ_n and μ be subprobability measures. Show that $\mu_n \overset{w^*}{\to} \mu$ as $n \to \infty$ iff $\mu_n(a, b] \to \mu(a, b]$ for all $a, b \in \mathbb{R}$ with $\mu\bigl(\{a, b\}\bigr) = 0$.*

Homework 2. *Let μ_n be probability measures with $\mu_n \overset{w^*}{\to} \mu$ as $n \to \infty$. Show that $\{\mu_n \, ; \, n \geq 1\}$ is tight iff μ is a probability measure.*

Homework 3. *Let μ_n and μ be probability measures. Show that $\mu_n \Rightarrow \mu$ iff $\mu_n \overset{w^*}{\to} \mu$.*

Corollary. *Let μ_n and μ be probability measures. If $\langle \mu_n \, ; \, n \geq 1 \rangle$ is a tight sequence with only one weak sequential limit point, μ, then $\mu_n \Rightarrow \mu$ as $n \to \infty$.*

Proof. Assume $\langle \mu_n \, ; \, n \geq 1 \rangle$ is a tight sequence with only one weak sequential limit point. By the Banach–Alaoglu theorem, $\langle \mu_n \, ; \, n \geq 1 \rangle$ has weak* sequential limit points, each of which, by Homework 2, is a probability measure. By Homework 3, μ is the only weak* sequential limit point of $\langle \mu_n \, ; \, n \geq 1 \rangle$. Thus, $\mu_n \overset{w^*}{\to} \mu$ and by Homework 3 again, $\mu_n \Rightarrow \mu$. ∎