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RANDOM WALK IN A RANDOM ENVIRONMENT AND FIRST-PASSAGE PERCOLATION ON TREES[1]

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We show that the transience or recurrence of a random walk in certain random environments on an arbitrary infinite locally finite tree is determined by the branching number of the tree, which is a measure of the average number of branches per vertex. This generalizes and unifies previous work of the authors. It also shows that the point of phase transition for edge-reinforced random walk is likewise determined by the branching number of the tree. Finally, we show that the branching number determines the rate of first-passage percolation on trees, also known as the first-birth problem. Our techniques depend on quasi-Bernoulli percolation and large deviation results.

1. Introduction. A random walk on a tree (by which we always mean an infinite, locally finite tree) is a Markov chain whose state space is the vertex set of the tree and for which the only allowable transitions are between neighboring vertices. We assume throughout that all transition probabilities are nonzero. For a fixed tree, the transition probabilities may be taken as random variables, in which case the resulting mixture of Markov chains is called random walk in a random environment (RWRE). The first theorem proved in this paper is conceptually the "least upper bound" of two previous results obtained by the authors (separately) about RWRE on trees. The notation necessary to describe this is as follows.

Choose an arbitrary vertex as the root and let σ be any other vertex. Let $\overleftarrow{\sigma}$ denote the first vertex on the shortest path from σ to the root. If σ is at distance at least 2 from the root, define A_σ as the transition probability from $\overleftarrow{\sigma}$ to σ divided by the transition probability from $\overleftarrow{\sigma}$ to $\overleftarrow{\overleftarrow{\sigma}}$. We assume the following uniformity in our random environment: All but finitely many of the random variables A_σ are identically distributed. (Since the values of A_σ are determined by the transition probabilities in a way that depends on the choice of root, it may appear that whether this condition is satisfied depends on the choice of root, but actually a different choice of root changes only finitely many of the A_σ's.) Let A denote a random variable with this common distribution.

By the zero–one law, a RWRE is a.s. transient or a.s. recurrent. We shall determine the phase transition boundary; we do not know in general when the cases on the boundary are transient or recurrent, but examples indicate that

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there is no simple rule here. Intuitively, the larger a tree is, the more likely a RWRE is to be transient, not only because the root is harder to find again, but also because there will be more branches along which the values of A_σ are atypically large. This can be quantified by a large deviation calculation which, one way or another, is behind each result in this paper. The *branching number* of a tree Γ, denoted $\mathrm{br}(\Gamma)$, is a real number greater than or equal to 1 that measures the average number of branches per vertex of the tree [4]; the precise definition is given in Section 2. In [4], Theorem 6.6 and Remark 2, it is shown that, when $A \leq 1$, the RWRE is transient or recurrent according to whether $\mathbb{E}[A] \cdot \mathrm{br}(\Gamma)$ is greater or less than 1. In Theorem 2 of [6], it is shown that if Γ is a homogeneous tree or the genealogical tree of a Galton–Watson process on (a subset of full measure of) the event of nonextinction, then RWRE is a.s. transient or a.s. recurrent according to whether $p\,\mathrm{br}(\Gamma)$ is greater or less than 1, where p is a function of the distribution of A (defined in the next section of this paper) and is equal to $\mathbb{E}[A]$ in the case where $A \leq 1$. Theorem 1 of this paper is that, for any tree and any distribution of A, RWRE is transient or recurrent according to whether $p\,\mathrm{br}(\Gamma)$ is greater or less than 1. The solution of the general case combines and simplifies techniques from [4] and [6] and provides as well a simpler expression for p. As shown in [6], edge-reinforced random walk (RRW) on an arbitrary tree is equal in law to a RWRE of the type discussed here. Hence, the present results show too that the phase transition for RRW occurs at a point depending only on the branching number of the tree. Our methods also resolve the boundary case left open in [6].

Our results on RWRE have equivalent formulations for flows in random electrical or capacitated networks. In fact, that will be important for our solution. Such problems are often regarded as part of percolation theory. The second main problem we consider is also part of percolation theory and it illustrates again how the crude behavior of probabilistic processes on trees often involves minimal interaction between the random variables involved and the tree structure as, moreover, the latter enters only as a single number, namely, the branching number of the tree.

Indeed, for our second problem, choose positive i.i.d. random variables for each edge of a tree Γ, regarded as transit times from one end to the other. We shall give the a.s. rate of fastest possible transit from any point to infinity; this is similar to the usual problem of first-passage percolation (or first birth) and is probably the appropriate formulation for this setting. Since this problem explicitly asks for the largest deviation from mean behavior, it is not surprising that the techniques used to find the phase transition for RWRE also solve this problem. Indeed, Theorem 4 gives the a.s. fastest transit rate to infinity as $1/m_1(1/\mathrm{br}(\Gamma))$, where m_1 is an inverse to a rate function m defined in Section 3. A further connection between the two problems is that the proofs of Theorems 1 and 4 both require results on quasi-Bernoulli percolation [3] which show that a configuration of values of A_σ in an appropriate range, once shown to be common enough, must percolate in an appropriate sense. Theorem 4 also gives information concerning the asymptotic profile of the transit times to large distances.

2. RWRE. Given a tree Γ, designate one of its vertices as the root, 0. If σ is a vertex, we write $|\sigma|$ for the number of edges on the shortest path from 0 to σ. For vertices σ and τ, we write $\sigma \le \tau$ if σ is on the shortest path from 0 to τ, we write $\sigma < \tau$ if $\sigma \le \tau$ and $\sigma \ne \tau$ and we write $\sigma \to \tau$ if $\sigma \le \tau$ and $|\tau| = |\sigma| + 1$; in this last situation, we call τ a *successor* of σ. If $\sigma \ne 0$, then $\overleftarrow{\sigma}$ denotes, as in Section 1, the vertex such that $\overleftarrow{\sigma} \to \sigma$. The edge *preceding* σ, from $\overleftarrow{\sigma}$ to σ, is denoted $e(\sigma)$. A *cutset* Π is a finite set of vertices not including 0 such that every infinite path from 0 intersects Π and such that there is no pair $\sigma, \tau \in \Pi$ with $\sigma < \tau$. The *branching number* of Γ [4] is defined by

$$\operatorname{br} \Gamma := \inf\left\{\lambda > 0;\ \inf_{\Pi} \sum_{\sigma \in \Pi} \lambda^{-|\sigma|} = 0\right\}.$$

The branching number is a measure of the average number of branches per vertex of Γ. It is less than or equal to $\liminf_{n\to\infty} M_n^{1/n}$, where $M_n := \operatorname{card}\{\sigma \in \Gamma;\ |\sigma| = n\}$, and takes more of the structure of Γ into account than does this growth rate. For sufficiently regular trees, such as homogeneous trees or, more generally, Galton–Watson trees, $\operatorname{br} \Gamma = \lim_{n\to\infty} M_n^{1/n}$ [4].

Given a random environment and the r.v.'s A_σ as described in Section 1, we shall assume without loss of generality that *all* A_σ are identically distributed. Choose further i.i.d. A_σ for $|\sigma| = 1$ and set

$$C_\sigma := \prod_{0<\tau\le\sigma} A_\sigma.$$

Consider an electrical network formed from Γ with conductance C_σ along the edge $e(\sigma)$. The transition probability from σ to τ is recovered by dividing the conductance of the edge joining σ to τ by the sum of the conductances of all edges incident to σ. Actually, this may not be true for $|\sigma| \le 1$, but we may ignore this insofar as our interest lies in the type of the random walk.

We shall use the fact that our random walk is transient if and only if the electrical network has positive conductance from 0 to ∞ (see [2], Proposition 9-131). This, in turn, is closely related to the question of whether the capacitated network (with, say, water flowing instead of electricity) formed by Γ with (channel) capacity C_σ through $e(\sigma)$ admits flow to infinity. In particular, if no water flows, then no current flows. [To see this, note that if the electrical conductance is positive, then a unit potential imposed between the root and infinity induces a current flow that is bounded by C_σ on $e(\sigma)$ for each σ and is hence an admissible water flow.] Moreover, the converse to this is almost true, as made precise in the proof of part (i) of Theorem 1. For more details, see [4].

THEOREM 1. *Consider a random environment on a tree* Γ *as described previously, with* $0 < A < \infty$ *a.s. Let* $p := \min_{0\le x\le 1} \mathbb{E}[A^x]$.

(i) *If* $p \operatorname{br} \Gamma > 1$, *then the RWRE is a.s. transient, the electrical network has positive conductance a.s. and the capacitated network admits flow a.s.*

(ii) *If* $p \operatorname{br} \Gamma < 1$, *then the RWRE is a.s. recurrent, the electrical network has zero conductance a.s. and the capacitated network admits no flow a.s. More generally, it suffices that* $\inf_\Pi \sum_{\sigma\in\Pi} p^{|\sigma|} = 0$.

(iii) *If* $p \limsup_{n\to\infty} M_n^{1/n} < 1$, *then the RWRE is a.s. positive recurrent. More generally, it suffices that* $\sum_{\sigma\in\Gamma} p^{|\sigma|} < \infty$.

REMARK 1. One might expect a part (iv) to this theorem, as there is when A is a.s. constant [4]. However, the following example shows that $p \limsup M_n^{1/n} > 1$ does *not* imply that the RWRE is a.s. null recurrent or transient. Let Γ be a single infinite branch, to which have been added $2^{|\sigma|+1} - 1$ successors of each σ. Each of the added nodes has no successor. Then $M_n^{1/n} = 2$ for all n. It is easily shown that the random walk will be a.s. positive recurrent or not according to whether the geometric mean of A is less or greater than $\frac{1}{2}$. Since we can choose A to have geometric mean less than p, there is no (weak) converse to (iii). Similar examples exist even on trees with every vertex having at least two neighbors.

REMARK 2. The behavior on the phase transition boundary itself will be discussed after the proof of the theorem.

REMARK 3. Our assumptions of independence concerning the random environment are stricter than necessary. For example, the same proof is valid in the situation where, for some root 0, the environments at σ and τ are independent whenever $|\sigma| \neq |\tau|$.

REMARK 4. In case we wish to allow the transition probabilities to be zero with probability in $]0,1[$, then we must make the following changes to the statement of the theorem. First, in calculating p, we use the conventions that $0^0 = 0$ and $\infty^0 = 1$. Second, we change "a.s." in (i) to "with positive probability"; alternatively, instead of starting the RWRE at the root, we can say in case (i) that there is a.s. *some* vertex in Γ at which the RWRE is transient and from which conductance to infinity is positive. To see this, suppose first that the transition probabilities $p_{\sigma,\tau}$ satisfy $p_{\overleftarrow{\sigma},\sigma} > 0$ a.s. and $\mathbb{P}[p_{\sigma,\overleftarrow{\sigma}} = 0] > 0$ for $\sigma \neq 0$. In this case, $\mathbb{P}[A = \infty] > 0$, so that $p = 1$ and we must show that the RWRE is a.s. transient. Indeed, it is immediately apparent that this is the case for any subtree consisting of a single infinite branch, hence that it is true for the whole tree.

More generally, now, without changing the law of the random environment, we may assume that the transition probabilities $p_{\sigma,\tau}$ have the form $p_{\sigma,\tau} = b_{\sigma,\tau}\bar{p}_{\sigma,\tau}$, where $b_{\sigma,\tau}$ takes only the values 0 and 1 and, in fact, $b_{\sigma,\overleftarrow{\sigma}} \equiv 1$; $\bar{p}_{\overleftarrow{\sigma},\sigma}$ is never 0; $\{b_{\sigma,\tau}\}$ are jointly independent of $\{\bar{p}_{\sigma,\tau}\}$; $b_{\sigma,\tau}[\bar{p}_{\sigma,\tau}]$ is independent of $b_{\rho,\psi}[\bar{p}_{\rho,\psi}]$ for $\sigma \neq \rho$. Thus, the random environment $\{p_{\sigma,\tau}\}$ can be considered as the compound process of percolation via $\{b_{\overleftarrow{\sigma},\sigma}\}$ followed by use of the transition probabilities $\{\bar{p}_{\sigma,\tau}\}$. If $\bar{A}$ corresponds to $\{\bar{p}_{\sigma,\tau}\}$ and $q := \mathbb{E}[b_{\overleftarrow{\sigma},\sigma}]$, then $\mathbb{E}[A^x] = q\mathbb{E}[\bar{A}^x]$, whence $p \operatorname{br}\Gamma = \min_{0\le x\le 1} \mathbb{E}[\bar{A}^x](q \operatorname{br}\Gamma)$. We now combine Theorem 1 with the fact that percolation via $\{b_{\overleftarrow{\sigma},\sigma}\}$ leaves subtrees, the supremum of whose branching number is a.s. $q \operatorname{br}\Gamma$ ([4], Corollary 6.3); here, we interpret a branching number less than 1 to mean that the tree is finite.

The proof of our theorem depends on the Chernoff–Cramér theorem. We have not found this theorem stated in the literature in the form and generality which we require, so we state it here. The reader may check that it follows from the material in Chapter 1, Section 9 of [1] by a truncation argument. Note that we make no assumptions on the existence even of $\mathbb{E}[X]$.

THE CHERNOFF–CRAMÉR THEOREM. *Let X be a real-valued random variable and define*

$$\phi(\theta) := \mathbb{E}[e^{\theta X}] \quad \text{and} \quad \gamma(a) := \inf_{\theta \geq 0}(-a\theta + \log \phi(\theta)).$$

If S_n denotes the sum of n independent copies of X, then, for all $a \in \mathbb{R}$, the quantity

$$\frac{1}{n} \log \mathbb{P}[S_n \geq na]$$

approaches $\gamma(a)$ from below (though not necessarily monotonically) as $n \to \infty$.

Note that in the following lemma, as well as in the proof of the theorem, no exceptions need be made when $\mathbb{E}[A^x] = \infty$ for some x.

LEMMA. $\min_{0 \leq x \leq 1} \mathbb{E}[A^x] = \max_{0 < y \leq 1} \inf_{x \geq 0} y^{1-x} \mathbb{E}[A^x]$.

PROOF. This can be verified directly by a case analysis of the point x_0 where $\mathbb{E}[A^x]$ is minimum, but it also follows immediately from Fenchel's duality theorem ([7], Theorem 31.1): Let $f(x) := \log \mathbb{E}[A^x]$ for $x \geq 0$ and $f(x) := +\infty$ for $x < 0$. Then $f(x)$ is convex by Hölder's inequality and lower semicontinuous by Fatou's lemma. Let $f^*(r) := \sup_x(rx - f(x))$ be the convex conjugate of f. Similarly, let $g(x)$ be the concave function that is 0 for $x \leq 1$ and $-\infty$ elsewhere, and let

$$g^*(r) := \inf_x (rx - g(x)) = \begin{cases} r, & \text{if } r \leq 0, \\ -\infty, & \text{if } r > 0, \end{cases}$$

be its concave conjugate. Then Fenchel's theorem asserts that $\inf_x(f(x) - g(x)) = \max_r(g^*(r) - f^*(r))$, which is the same as the statement of the lemma. Indeed,

$$\inf_x (f(x) - g(x)) = \log \min_{0 \leq x \leq 1} \mathbb{E}[A^x]$$

and

$$\max_r (g^*(r) - f^*(r)) = \max_{r \leq 0}(r - f^*(r)) = \log \max_{0 < y \leq 1} \inf_{x \geq 0} y^{1-x} \mathbb{E}[A^x]. \quad \square$$

PROOF OF THEOREM 1. The assertions will be demonstrated in reverse order.

(iii) We shall use the fact that the random walk is positive recurrent if and only if the electrical conductances have finite sum ([2], Proposition 9-131).

Suppose that $\sum_{\sigma \in \Gamma} p^{|\sigma|} < \infty$ and that $p = \mathbb{E}[A^x]$, $0 < x \leq 1$. Then

$$\mathbb{E}\left[\sum_{0 \neq \sigma \in \Gamma} C_\sigma^x\right] = \sum_{\sigma \neq 0} \mathbb{E}\left[\prod_{0 < \tau \leq \sigma} A_\tau^x\right] = \sum_{\sigma \neq 0} \prod_{0 < \tau \leq \sigma} \mathbb{E}[A_\tau^x] = \sum_{\sigma \neq 0} p^{|\sigma|} < \infty,$$

whence $\sum_{\sigma \neq 0} C_\sigma^x < \infty$ a.s. In particular, $C_\sigma < 1$ for all but finitely many σ a.s. Since $C_\sigma^x \geq C_\sigma$ for $C_\sigma < 1$, it follows that $\sum_{\sigma \neq 0} C_\sigma < \infty$ a.s.

(ii) Here we use the fact that the random walk is recurrent if and only if no electrical current flows. Suppose that $\inf_\Pi \sum_{\sigma \in \Pi} p^{|\sigma|} = 0$ and that $p = \mathbb{E}[A^x]$, $0 < x \leq 1$. Then, as before,

$$\mathbb{E}\left[\sum_{\sigma \in \Pi} C_\sigma^x\right] = \sum_{\sigma \in \Pi} p^{|\sigma|},$$

whence if $\sum_{\sigma \in \Pi_n} p^{|\sigma|} \to 0$, we have, as before,

$$\liminf_{n \to \infty} \sum_{\sigma \in \Pi_n} C_\sigma = 0 \quad \text{a.s.},$$

by virtue of Fatou's lemma. By (the trivial half of) the max-flow min-cut theorem, the capacitated network admits no flow a.s. Hence, no electrical current flows, and the random walk is a.s. recurrent.

(i) This part uses the fact that if water flows even when C_σ is reduced exponentially in $|\sigma|$, then electrical current flows and the random walk is transient ([4], Corollary 4.2). If $p \operatorname{br} \Gamma > 1$, let $y \in]0, 1]$ be such that $p = \inf_{x \in \mathbb{R}} y^{1-x} \mathbb{E}[A^x]$. By the Chernoff–Cramér theorem, there exists $k \geq 1$ such that, for $|\sigma| = k$,

$$\mathbb{P}\left[C_\sigma \geq y^k\right] > (y \operatorname{br} \Gamma)^{-k}.$$

Let $\varepsilon > 0$ be sufficiently small that, for $|\sigma| = k$,

$$q := \mathbb{P}\left[C_\sigma \geq y^k \text{ and } \forall\, 0 < \tau \leq \sigma,\ A_\tau \geq \varepsilon\right] > (y \operatorname{br} \Gamma)^{-k}.$$

Let Γ^k be the tree whose vertices are $\{\sigma \in \Gamma;\ k \| \sigma|\}$ such that $\sigma \to \tau$ in Γ^k if and only if $\sigma \leq \tau$ and $|\sigma| + k = |\tau|$ in Γ. It is easily verified that $\operatorname{br} \Gamma^k = (\operatorname{br} \Gamma)^k$.

Form a random subgraph $\Gamma^k(\omega)$ of Γ^k by deleting those edges $\sigma \to \tau$ where

$$\prod_{\substack{\sigma < \rho \leq \tau \\ \rho \in \Gamma}} A_\rho < y^k \quad \text{or} \quad \exists\, \rho \in \Gamma, \qquad \sigma < \rho \leq \tau \text{ and } A_\rho < \varepsilon.$$

This is an edge percolation on Γ^k for which each edge is present with probability q and the presence of edges with distinct "preceding" vertices are mutually independent. In particular, it is a quasi-Bernoulli percolation process on Γ^k (see [4] for the general definition of quasi-Bernoulli percolation). Choose $w \in](yq^{1/k} \operatorname{br} \Gamma)^{-1}, 1[$. Since $q \operatorname{br} \Gamma^k > (wy)^{-k} > 1$, there is almost surely a subtree Γ^* of $\Gamma^k(\omega)$, not necessarily beginning at the root, that has branching number larger than $(wy)^{-k}$ (combine the method of proof of Corollary 6.3 or Proposition 6.4 of [4] with Theorem 3.1 of [3]). Any subtree Γ^* of Γ^k induces a subtree Γ' of Γ whose vertices are those $\rho \in \Gamma$ for which $\exists\, \sigma, \tau \in \Gamma^*$ such that $\sigma \leq \rho \leq \tau$. Thus, there is almost surely a subtree Γ' of Γ induced by $\Gamma^k(\omega)$ with the following three properties: $\operatorname{br} \Gamma' > (wy)^{-1}$; $A_\sigma \geq \varepsilon$ along each edge;

for every $\sigma, \tau \in \Gamma'$ such that $k \| |\sigma| = |\tau| - k$, we have $\prod_{\sigma < \rho \leq \tau} A_\rho \geq y^k$. It follows that

$$\inf_{\Pi'} \sum_{\sigma \in \Pi'} w^{|\sigma|} C_\sigma \geq \inf_{\Pi'} \varepsilon^{k-1} \sum_{\sigma \in \Pi'} (wy)^{|\sigma|} > 0,$$

where Π' is any cutset of Γ'. By [4], Corollary 4.2, Γ' has positive conductance, hence so does Γ (a.s.). We may now deduce that the RWRE is transient a.s. and that the capacitated network admits flow a.s. (e.g., the current flow from an imposed unit potential from 0 to ∞). □

REMARK 5. The proofs of (i) and (ii) use the different expressions of p given by the lemma. Since the equality of the two expressions is not intuitive, the fact that (i) and (ii) meet (cover all possibilities except $p \operatorname{br} \Gamma = 1$) may seem miraculous. The following discussion is intended to explain "why" (i) and (ii) meet. Part (i) is true because there are enough branches of the tree along which the geometric mean of the A_σ's exceeds a certain value y in order to force transience. This is shown directly from the second expression for p, which is just a large deviation rate. For each y individually, there is a converse to this, which is that the conductances to those parts of a cutset where the geometric mean of the A_σ's is close to y approach zero as the cutset gets far from the root. This uses the other half (i.e., the upper bound) of the Chernoff–Cramér theorem not used in (i). It is possible to show that the total conductance to a cutset goes to zero by "integrating" this fact over y. Such an approach can be used to derive (ii) from the second expression for p, but it involves several calculations (see [6], where this is done for positive recurrence). These are avoided by using the simpler expression for p provided by the lemma.

We now turn our attention to the behavior of the random walk on the phase transition boundary, $p \operatorname{br} \Gamma = 1$. Here, the type of the walk depends on further structure of the tree and of A. First, we remark that, even when A is constant, the walk may be either transient or recurrent [4]. Furthermore, there are trees Γ for which $p = \operatorname{br}(\Gamma) = 1$ and RWRE is a.s. transient, yet simple random walk on Γ (i.e., $A = p = 1$ almost surely) is recurrent. On the other hand, there are trees Γ for which $\operatorname{br}(\Gamma) > 1$ and for which there are a.s. recurrent RWRE's with $p = \operatorname{br}(\Gamma)^{-1}$, even though the RWRE with the deterministic environment $A = p$ is transient. We do not know whether the difference in behavior of the walk for random as compared to deterministic environments always depends, as before, on $\operatorname{br} \Gamma$. We hope to clarify this behavior at a later date, but for now, suffice it to record the following extension to part (ii) of Theorem 1, which covers many of the boundary cases, $p \operatorname{br}(\Gamma) = 1$.

PROPOSITION 2. *Under the same hypotheses as in Theorem* 1, *if* $\liminf_{\Pi \to \infty} \sum_{\sigma \in \Pi} p^{|\sigma|} < \infty$ (*in other words, if there are cutsets* Π_n *such that*

$\inf\{|\sigma|;\ \sigma \in \Pi_n\} \to \infty$ and $\sup_n \sum_{\sigma \in \Pi_n} p^{|\sigma|} < \infty$), then the RWRE is a.s. recurrent and the electrical network has zero conductance a.s.

PROOF. Let Π_n be as indicated. We have

$$\mathbb{E}\left[\sup_n \sum_{\sigma\in\Pi_n} C_\sigma^x\right] \le \sup \mathbb{E}\left[\sum_{\sigma\in\Pi_n} C_\sigma^x\right] = \sup \sum_{\sigma\in\Pi_n} p^{|\sigma|} < \infty,$$

whence

$$\sup \sum_{\sigma\in\Pi_n} C_\sigma \le \sup\left(\sum_{\sigma\in\Pi_n} C_\sigma^x\right)^{1/x} < \infty \quad \text{a.s.}$$

This implies a.s. recurrence by virtue of [4], Corollary 4.2. □

The capacitated network may admit flow in these circumstances, as the case where Γ is a binary tree and $A \equiv \frac{1}{2}$ shows. A complete answer for both electrical and capacitated networks may be given when Γ is homogeneous or produced by a Galton–Watson process. The following theorem simplifies, refines and extends Theorem 2 of [6]. (The assumptions there that the progeny distribution be bounded and that $\mathbb{E}[\log A]$ exists are now seen to be unnecessary.)

THEOREM 3. *Let Γ be the genealogical tree of a Galton–Watson branching process with mean $m > 1$. Consider a random environment as in Theorem 1.*

(i) *If $pm > 1$, then, given nonextinction, the RWRE is a.s. transient, the electrical network has positive conductance a.s. and the capacitated network admits flow a.s.*

(ii) *If $pm \le 1$, then, given nonextinction, the RWRE is a.s. recurrent, the electrical network has zero conductance a.s. and, unless both A and Γ are constant, the capacitated network admits no flow a.s.*

(iii) *If $pm < 1$, then the RWRE is a.s. positive recurrent.*

REMARK 6. When $pm = 1$, the RWRE may be either null or positive recurrent ([5], Theorem 3.2).

PROOF OF THEOREM 3. Part (i) follows immediately from Theorem 1, since, given nonextinction, the genealogical tree has branching number m a.s. ([4], Proposition 6.4). The first two parts of (ii) follow from the same argument used in the proof of Proposition 2:

$$\mathbb{E}\left[\sum_{|\sigma|=n} C_\sigma^x\right] = m^n p^n \le 1,$$

whence

$$\sup_n \sum_{|\sigma|=n} C_\sigma < \infty \quad \text{a.s.}$$

Likewise, the proof of (iii) is similar to the proof of part (iii) of Theorem 1. It remains to establish the assertion in (ii) on the capacitated network when $pm = 1$.

For $0 < t \leq 1$, let $F^{(t)}$ be the maximum flow to infinity in the network on Γ with capacities C_σ^t along $e(\sigma)$. Thus, we are interested in showing that $F^{(1)} = 0$ almost surely. Now, as in Proposition 2,

$$(F^{(1)})^x = \inf_\Pi \Big(\sum_{\sigma \in \Pi} C_\sigma \Big)^x \leq \inf_\Pi \Big(\sum_{\sigma \in \Pi} C_\sigma^x \Big) = F^{(x)},$$

so that it suffices to show that $F^{(x)} = 0$ almost surely. For $|\sigma| = 1$, let $F_\sigma^{(x)}$ be the maximum flow in the subtree $\{\tau \in \Gamma; \sigma \leq \tau\}$ with capacities C_τ^x / A_σ^x. Thus, $F_\sigma^{(x)}$ has the same law as $F^{(x)}$ does. It is easily seen that

$$F^{(x)} = \sum_{|\sigma|=1} A_\sigma^x \big(1 \wedge F_\sigma^{(x)}\big).$$

Taking expectations yields

$$\mathbb{E}[F^{(x)}] = mp\mathbb{E}[1 \wedge F^{(x)}] = \mathbb{E}[1 \wedge F^{(x)}].$$

Therefore $F^{(x)} \leq 1$ almost surely. In addition, we have, by independence,

$$\|F^{(x)}\|_\infty = \Big\| \sum_{|\sigma|=1} A_\sigma^x \Big\|_\infty \|F^{(x)}\|_\infty.$$

Since $\|\sum_{|\sigma|=1} A_\sigma^x\|_\infty > 1$ unless both A and Γ are constant, this shows that $F^{(x)} = 0$ almost surely unless both A and Γ are constant. □

3. First-passage percolation. Given a tree Γ rooted at 0, the *boundary* $\partial\Gamma$ of Γ is the space of infinite paths beginning at 0 which go through no vertex more than once. This is a compact space with metric $d(s,t) = e^{-n}$, where n is the number of edges common to s and t. Changing the root gives essentially the same boundary with an equivalent metric. Suppose we are given real-valued i.i.d. r.v.'s X_σ for each edge $e(\sigma)$. Let $S_\sigma = \sum_{0 < \tau \leq \sigma} X_\tau$. As explained in Section 1, the random variable

$$\inf_{s \in \partial\Gamma} \limsup_{\sigma \in s} \frac{S_\sigma}{|\sigma|}$$

may be thought of as the reciprocal of the fastest sustainable transit rate to infinity if $X_\sigma > 0$. The calculation of this rate depends on the nondecreasing function

$$m(y) = \inf_{x \leq 0} \mathbb{E}[e^{x(X-y)}],$$

where X has the same law as every X_σ. As the infimum of linear functions, $\log m$ is concave. Write

$$m_1(z) := \sup\{y; m(y) < z\},$$

so that m_1 is a sort of inverse function to m. Note that m cannot be constant unless $m \equiv 1$ nor have range $\{0, 1\}$. Hence m is strictly increasing where

]0, 1[-valued by log-concavity and

$$m_1(z) = \inf\{y;\, m(y) > z\},$$

for $0 \le z \le 1$, except if $m \equiv 1$ and $z = 1$.

THEOREM 4. *Unless* $\operatorname{br}\Gamma = 1$ *and* $m \equiv 1$, *we have*

$$\inf_{s\in\partial\Gamma} \limsup_{\sigma\in s} \frac{S_\sigma}{|\sigma|} = \inf\left\{ \lim_{\sigma\in s} \frac{S_\sigma}{|\sigma|};\ s \in \partial\Gamma \text{ and } \lim_{\sigma\in s}\frac{S_\sigma}{|\sigma|} \text{ exists}\right\}$$
$$= m_1\big((\operatorname{br}\Gamma)^{-1}\big) \quad a.s.$$

and

$$\dim\left\{ s \in \partial\Gamma;\ \limsup_{\sigma\in s} \frac{S_\sigma}{|\sigma|} \le y\right\} = \dim\left\{ s\in\partial\Gamma;\ \lim_{\sigma\in s}\frac{S_\sigma}{|\sigma|} = y\right\}$$
$$= \log(m(y)\operatorname{br}\Gamma) \quad a.s.$$

for $m_1((\operatorname{br}\Gamma)^{-1}) \le y < \sup_n \mathbb{E}[X \wedge n]$.

REMARK 7. The statement concerning Hausdorff dimension (cf. [4], Section 7) may be interpreted as giving information on the asymptotic profile of transit times. [We are grateful to Yuval Peres for the statements and proofs concerning lim (rather than lim sup). Peres also has examples showing that when $\operatorname{br}\Gamma = 1$ and $m \equiv 1$, the rate depends on more information.]

PROOF OF THEOREM 4. Suppose first that $m(y)\operatorname{br}\Gamma < 1$. Then by the Chernoff–Cramér theorem,

$$\inf_\Pi \mathbb{E}\left[\sum_{\sigma\in\Pi} \mathbf{1}_{S_\sigma \le y|\sigma|}\right] \le \inf_\Pi \sum_{\sigma\in\Pi} m(y)^{|\sigma|} = 0,$$

whence there are cutsets $\Pi_n \to \infty$ (i.e., $\min\{|\sigma|;\, \sigma\in\Pi_n\}\to\infty$) such that

$$\liminf_{n\to\infty} \operatorname{card}\{\sigma\in\Pi_n;\, S_\sigma \le y|\sigma|\} = 0 \quad \text{a.s.}$$

In other words, a.s. for infinitely many n,

$$\forall\, \sigma\in\Pi_n, \qquad S_\sigma > y|\sigma|.$$

Therefore, a.s.,

$$\forall\, s\in\partial\Gamma, \qquad \limsup_{\sigma\in s}\frac{S_\sigma}{|\sigma|} \ge y.$$

On the other hand, if $m(y)\operatorname{br}\Gamma > 1$, then, by the Chernoff–Cramér theorem, there is a $k \ge 1$ such that, for $|\sigma| = k$,

$$\mathbb{P}[S_\sigma \le ky] > (\operatorname{br}\Gamma)^{-k}.$$

Let M be sufficiently large that, for $|\sigma| = k$,

$$q := \mathbb{P}[S_\sigma \le ky \text{ and } \forall\, 0 < \tau \le \sigma,\, X_\tau \le M] > (\operatorname{br}\Gamma)^{-k}. \tag{3.1}$$

Let Γ^k be as in the proof of Theorem 1. Form a random subgraph $\Gamma^k(\omega)$ of Γ^k by deleting those edges $\sigma \to \tau$ where

$$\sum_{\substack{\sigma<\rho\le\tau \\ \rho\in\Gamma}} X_\rho > ky \quad \text{or} \quad \exists\, \rho \in \Gamma, \qquad \sigma < \rho \le \tau \text{ and } X_\rho > M.$$

This is a quasi-Bernoulli percolation process on Γ^k [3]. Since $q \operatorname{br} \Gamma^k > 1$, percolation occurs a.s. That is, there is a.s. an $s \in \partial\Gamma$ whose image in $\partial\Gamma^k$ is, except for a finite set, contained in $\Gamma^k(\omega)$; for such s, we have $\limsup_{\sigma\in s} S_\sigma/|\sigma| \le y$. This establishes that $\inf_{s\in\partial\Gamma} \limsup_{\sigma\in s} S_\sigma/|\sigma| = m_1((\operatorname{br}\Gamma)^{-1})$ a.s.

Additional information can be extracted from the argument of the last paragraph. Embed Γ in the upper half-plane with its root at the origin and order $\partial\Gamma$ clockwise. Given y such that $m(y)\operatorname{br}\Gamma > 1$, let Γ_y^k be the tree denoted Γ^k previously and let $s(y)$ be the minimal element of $\partial\Gamma$ whose tail lies in $\Gamma_y^k(\omega)$. Thus, $s(y)$ is defined a.s. For any $|\sigma| = k$, set

$$\psi(y) := \mathbb{E}\left[\frac{S_\sigma}{k} \,\middle|\, S_\sigma \le ky \text{ and } \forall\, 0 < \tau \le \sigma,\ X_\tau \le M \right].$$

Recall that k and M depend on y. By the strong law of large numbers, we have

$$\lim_{\sigma\in s(y)} \frac{S_\sigma}{|\sigma|} = \psi(y) \quad \text{a.s.}$$

Since $\psi(y) \le y$ and y is arbitrary, subject only to $m(y)\operatorname{br}\Gamma > 1$, this establishes the remainder of the first assertion of the theorem. We claim, moreover, that, for any $y < \sup_n \mathbb{E}[X \wedge n]$ such that $m(y)\operatorname{br}\Gamma > 1$,

$$\text{(3.2)} \qquad \text{a.s. } \exists\, s \in \partial\Gamma, \qquad \lim_{\sigma\in s} \frac{S_\sigma}{|\sigma|} = y.$$

Indeed, given such y, find k and M such that (3.1) holds and $y < \mathbb{E}[X \wedge M]$. We may write

$$y = \alpha\psi(y) + (1-\alpha)\mathbb{E}[X \mid X \le M]$$

for some $\alpha \in [0,1]$. Choose a sequence $\mathscr{N}$ of density α in $\mathbb{N}$ and percolate as before on Γ^k, *except* that the edges preceding vertices $\sigma \in \Gamma^k$ for which $|\sigma| \notin \mathscr{N}$ survive a.s., rather than with probability q. We now find by similar reasoning to the above that

$$\lim_{\sigma\in s(y)} \frac{S_\sigma}{|\sigma|} = y \quad \text{a.s.,}$$

thereby validating (3.2).

The remainder of the theorem follows from general considerations. Namely, denote the sets in question by

$$E(y) := \left\{ s \in \Gamma;\ \limsup_{\sigma \in s} \frac{S_\sigma}{|\sigma|} \le y \right\},$$

$$F(y) := \left\{ s \in \Gamma;\ \lim_{\sigma \in s} \frac{S_\sigma}{|\sigma|} = y \right\}.$$

By the 0–1 law, dim $E(y)$ and dim $F(y)$ are constant a.s. As $E(y)$ and $F(y)$ are clearly Borel and $F(y) \subseteq E(y)$, [4], Section 7, implies that it suffices to show that if independent Bernoulli percolation with survival parameter p is performed on Γ, then, for $pm(y)\mathrm{br}\,\Gamma < 1$, a.s. no point of $E(y)$ survives, while, for $pm(y)\mathrm{br}\,\Gamma > 1$, with positive probability some point of $F(y)$ does survive. These conditions in fact follow from [4], Corollary 6.3, from what was shown previously and from Fubini's theorem. □

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REFERENCES

[1] Durrett, R. (1991). *Probability: Theory and Examples.* Wadsworth, Belmont, Calif.

[2] Kemeny, J. G., Snell, J. L. and Knapp, A. W. (1976). *Denumerable Markov Chains*, 2nd ed. Springer, New York.

[3] Lyons, R. (1989). The Ising model and percolation on trees and tree-like graphs. *Comm. Math. Phys.* **125** 337–353.

[4] Lyons, R. (1990). Random walks and percolation on trees. *Ann. Probab.* **18** 931–958.

[5] Lyons, R. (1990). Random walks, capacity, and percolation on trees. Unpublished manuscript.

[6] Pemantle, R. (1988). Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* **16** 1229–1241.

[7] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

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