The Annals of Probability
2015, Vol. 43, No. 6, 3337–3358
DOI: 10.1214/14-AOP961

CYCLE DENSITY IN INFINITE RAMANUJAN GRAPHS[1]

BY RUSSELL LYONS AND YUVAL PERES

Indiana University and Microsoft Corporation

We introduce a technique using nonbacktracking random walk for estimating the spectral radius of simple random walk. This technique relates the density of nontrivial cycles in simple random walk to that in nonbacktracking random walk. We apply this to infinite Ramanujan graphs, which are regular graphs whose spectral radius equals that of the tree of the same degree. Kesten showed that the only infinite Ramanujan graphs that are Cayley graphs are trees. This result was extended to unimodular random rooted regular graphs by Abért, Glasner and Virág. We show that an analogous result holds for all regular graphs: the frequency of times spent by simple random walk in a nontrivial cycle is a.s. 0 on every infinite Ramanujan graph. We also give quantitative versions of that result, which we apply to answer another question of Abért, Glasner and Virág, showing that on an infinite Ramanujan graph, the probability that simple random walk encounters a short cycle tends to 0 a.s. as the time tends to infinity.

1. Introduction. A path in a multigraph is called *nonbacktracking* if no edge is immediately followed by its reversal. Note that a loop is its own reversal. Nonbacktracking random walks are almost as natural as ordinary random walks, though more difficult to analyze in most situations. Moreover, they can be more useful than ordinary random walks when random walks are used to search for something, as they explore more quickly, not wasting time immediately backtracking; see Alon et al. (2007). Our aim, however, is to use them to analyze the spectral radius of ordinary random walks on regular graphs.

The *spectral radius* of a (connected, locally finite) multigraph G is defined to be $\rho(G) := \limsup_{n\to\infty} p_n(o,o)^{1/n}$ for a vertex $o \in G$, where $p_n(x,y)$ is the n-step transition probability for simple random walk on G from x to y. It is well known that $\rho(G)$ does not depend on the choice of o.

If $G = \mathbb{T}_d$ is a regular tree of degree d, then $\rho(G) = 2\sqrt{d-1}/d$. Regular trees are Cayley graphs of groups. In general, when G is a Cayley graph of a group, Kesten (1959b) proved that $\rho(G) > \rho(\mathbb{T}_d)$ when G has degree d and $G \neq \mathbb{T}_d$. Kesten (1959a) also proved that for Cayley graphs, $\rho(G) = 1$ iff G is amenable.

If G is a d-regular multigraph, then its universal cover is $\mathbb{T}_d$, whence $\rho(\mathbb{T}_d) \le \rho(G) \le 1$. Using the method of proof due to Cheeger (1970), various researchers

Received October 2013; revised August 2014.

[1]Supported in part by Microsoft Research and NSF Grant DMS-10-07244.

MSC2010 subject classifications. Primary 60G50, 82C41; secondary 05C81.

Key words and phrases. Nonbacktracking random walks, spectral radius, regular graphs.

related $1 - \rho(G)$ to the expansion (or isoperimetric) constant of infinite graphs G, showing that again, G is amenable iff $\rho(G) = 1$; see Dodziuk (1984), Dodziuk and Kendall (1986), Varopoulos (1985), Ancona (1988), Gerl (1988), Biggs, Mohar and Shawe-Taylor (1988) and Kaimanovich (1992).

It appears considerably more difficult to understand the other inequality for $\rho(G)$: when is $\rho(G) = \rho(\mathbb{T}_d)$? This question will be our focus.

For finite graphs, the spectral radius is 1. Of interest instead is the second largest eigenvalue, λ_2, of the transition matrix. An inequality of Alon and Boppana [see Alon (1986) and Nilli (1991)] says that if $\langle G_n; n \geq 1\rangle$ is a family of d-regular graphs whose size tends to infinity, then $\liminf_{n\to\infty} \lambda_2(G_n) \geq \rho(\mathbb{T}_d)$. Regular graphs G such that all eigenvalues have absolute value either 1 or at most $\rho(\mathbb{T}_d)$ were baptized *Ramanujan graphs* by Lubotzky, Phillips and Sarnak (1988), who, with Margulis (1988), were the first to exhibit explicit such families. Moreover, their examples had better expansion properties than the random graphs that had been constructed earlier. See Murty (2003) and Li (2007) for surveys of finite Ramanujan graphs.

Abért, Glasner and Virág (2015) studied the density of short cycles in Ramanujan graphs. One of their tools was graph limits, which led them to define and study *infinite Ramanujan graphs*, which are d-regular infinite graphs whose spectral radius equals $\rho(\mathbb{T}_d)$. Now limits of finite graphs, taken in the appropriate sense, are probability measures on rooted graphs; the probability measures that arise have a property called unimodularity. Theorem 5 of Abért, Glasner and Virág (2015) shows that every unimodular random rooted infinite regular graph that is a.s. Ramanujan is a.s. a tree. Unimodularity is a kind of stochastic homogeneity that, among other things, ensures that simple random walk visits short cycles with positive frequency when they exist.

Abért, Glasner and Virág (2015) asked whether the hypothesis of unimodularity could be weakened to something called stationarity. We answer this affirmatively in a very strong sense, using no extra hypotheses on the graph and including cycles of all lengths at once. To state our result, call a cycle *nontrivial* if it is not purely a backtracking cycle, that is, if when backtracks are erased iteratively from the cycle, some edge remains. For example, a single loop is a nontrivial 1-edge cycle, but a loop followed by the same loop is a trivial 2-edge cycle. Let $X = \langle X_n; n \geq 1\rangle$ be simple random walk on G, where X_n are directed edges and the tail of X_1 is any fixed vertex. Call n a *nontrivial cycle time* of X if there exist $1 \leq s \leq n \leq t$ such that $(X_s, X_{s+1}, \dots, X_t)$ is a nontrivial cycle.

THEOREM 1.1. *If G is an infinite Ramanujan graph of degree at least* 3, *then a.s. the density of nontrivial cycle times of X in* $[1, n]$ *tends to* 0 *as* $n \to \infty$.

Now fix $L \geq 1$. Let q_n be the probability that simple random walk at time n lies on a nontrivial cycle of length at most L. Then the preceding theorem implies that $\liminf_{n\to\infty} q_n = 0$. In their Problem 10, Abért, Glasner and Virág (2015) ask whether $\lim_{n\to\infty} q_n = 0$. We answer it affirmatively.

THEOREM 1.2. *Let G be an infinite Ramanujan graph and $L \geq 1$. Then* $\lim_{n\to\infty} q_n = 0$.

In broad outline, our technique to prove these results is the following: First, we prove that when simple random walk on G has many nontrivial cycle times, then so does nonbacktracking random walk. Second, we deduce that under these circumstances, we may transform nonbacktracking paths to nonbacktracking cycles with controlled length and find that there are many nonbacktracking cycles. The exponential growth rate of the number of nonbacktracking cycles is called the cogrowth of G. Finally, we use the cogrowth formula relating cogrowth to spectral radius to conclude that G is not Ramanujan.

Thus, of central importance to us is the notion of cogrowth. We state the essentials here. Let the number of nonbacktracking cycles of length n starting from some fixed $o \in \mathsf{V}(G)$ be $b_n(o)$. Let

$$\mathtt{cogr}(G) := \limsup_{n\to\infty} b_n(o)^{1/n}$$

be the exponential growth rate of the number of nonbacktracking cycles containing o. This number is called the *cogrowth* of G. The reason for this name is that if we consider a universal covering map $\varphi : T \to G$, then the cogrowth of G equals the exponential growth rate of $\varphi^{-1}(o)$ inside T since φ induces a bijection between simple paths in T and nonbacktracking paths in G. By using this covering map, one can see that $\mathtt{cogr}(G)$ does not depend on o. Note, too, that if $\mathcal{P}$ is a finite path in G that lifts to a path in T from vertex x to vertex y, then erasing backtracks from $\mathcal{P}$ iteratively yields $\varphi[\mathcal{P}']$, where $\mathcal{P}'$ is the shortest path in T from x to y.

Let G be a connected graph. It is not hard to check the following: If G has no simple nonloop cycle and at most one loop, then $\mathtt{cogr}(G) = 0$. If G has one simple cycle and no loop or no simple nonloop cycle and two loops, then $\mathtt{cogr}(G) = 1$. In all other cases, that is, when the fundamental group of G is not virtually abelian, $\mathtt{cogr}(G) > 1$.

The central result about cogrowth is the following formula (1.2), due to Grigorchuk (1980) for Cayley graphs and extended by Northshield (1992) to all regular graphs.

THEOREM 1.3 (Cogrowth formula). *If G is a d-regular connected multigraph, then*

$$\mathtt{cogr}(G) > \sqrt{d-1} \quad \textit{iff} \quad \rho(G) > \frac{2\sqrt{d-1}}{d}, \tag{1.1}$$

in which case

$$d\rho(G) = \frac{d-1}{\mathtt{cogr}(G)} + \mathtt{cogr}(G). \tag{1.2}$$

If (1.1) fails, then $\rho(G) = 2\sqrt{d-1}/d$ and $\mathtt{cogr}(G) \leq \sqrt{d-1}$.

See Lyons and Peres (2015), Section 6.3, for a proof.

Our use of Theorem 1.3 will be mainly via (1.1), rather than (1.2). In order to use (1.1), we shall prove the following result on density of nontrivial cycle times:

THEOREM 1.4. *Suppose that G is a graph all of whose degrees are at least* 3. *If with positive probability the limsup density of nontrivial cycle times of simple random walk in* $[1, n]$ *is positive as* $n \to \infty$, *then the same holds for nonbacktracking random walk.*

Here, nonbacktracking random walk is the random walk that at every time n, chooses uniformly among all possible edges that are not the reversal of the nth edge. In terms of the universal cover $\varphi : T \to G$, if simple random walk $X = \langle X_n; n \geq 1\rangle$ is lifted to a random walk, call it $\widehat{X}$, on T, then $\widehat{X}$ is simple random walk on T. Backtracking on G is the same as backtracking on T. Since all degrees of T are at least 3, $\widehat{X}$ is transient and so there is a unique simple path $\mathcal{P}$ in T with the same starting point as $\widehat{X}$ and having infinite intersection with $\widehat{X}$. The law of $\varphi[\mathcal{P}]$ is that of nonbacktracking random walk on G.

As it may be of separate interest, we note in passing the following basic elementary bound on the number of nonbacktracking cycles. Write $S(x) := \{n; b_n(x) \neq 0\}$. If a nonbacktracking cycle is a loop or has the property that its last edge is different from the reverse of its first edge, then call the cycle *fully nonbacktracking* (usually called "cyclically reduced" in the case of a Cayley graph). Let the number of fully nonbacktracking cycles of length n starting from x be $b_n^*(x)$.

PROPOSITION 1.5. *Let G be a graph with* $\mathtt{cogr}(G) \geq 1$. *For each* $x \in \mathsf{V}(G)$, *we have that* $\lim_{S(x)\ni n\to\infty} b_n(x)^{1/n}$ *exists and there is a constant c_x such that* $b_n(x) \leq c_x \,\mathtt{cogr}(G)^n$ *for all* $n \geq 1$. *Furthermore, if x belongs to a simple cycle of length L, then* $c_x \leq 2 + 2L\,\mathtt{cogr}(G)^{L-2}$. *If G is d-regular, then G is Ramanujan iff for all vertices x and all* $n \geq 1$, *we have* $b_n^*(x) \leq 2(d-1)^{n/2}$.

We shall illustrate our technique first by giving a short proof of Kesten's theorem (extended to transitive multigraphs). We then prove a version of Theorems 1.1 and 1.4 with a stronger hypothesis on the density of nontrivial cycle times, a hypothesis that holds for stationary random rooted graphs, for example. The proof of the full Theorems 1.1 and 1.4 requires a large number of technical lemmas, which makes the basic idea harder to see. The final section proves Theorem 1.2.

All our graphs are undirected connected infinite multigraphs. However, each edge comes with two orientations, except loops, which come with only one orientation. An edge e is oriented from its tail e^- to its head e^+. These endpoints are the same when e is a loop. A vertex may have many loops and two vertices may be joined by many edges. If e is an oriented edge, then its reversal is the same unoriented edge with the opposite orientation, denoted $-e$. This is equal to e iff e is a loop.

We shall have no need of unimodularity or stationarity, so we do not define those terms.

2. Kesten's theorem. It is easiest to understand the basic ideas behind our proofs in the case of transitive multigraphs. Kesten (1959b) proved the following result and various extensions for Cayley graphs.

THEOREM 2.1. *If $d \geq 3$ and G is a d-regular transitive multigraph that is not a tree, then $\rho(G) > \rho(\mathbb{T}_d)$.*

PROOF. Let L be the length of the shortest cycle in G (which is 1 if there is a loop). Consider a nonbacktracking random walk $\langle Y_n; n \geq 1\rangle$, where each edge Y_{n+1} is chosen uniformly among the edges incident to the head Y_n^+ of Y_n, other than the reversal of Y_n. We are going to handle loops differently than other cycles, so it will be convenient to let

$$L' := \begin{cases} L, & \text{if } L > 1, \\ 3, & \text{if } L = 1. \end{cases}$$

Let A_n be the event that $Y_{n+1}, \dots, Y_{n+L'}$ is a nonbacktracking cycle. Write $b := d-1$. For $n \geq 1$,

$$\mathbf{P}(A_n | Y_1, \dots, Y_n) \geq \frac{1}{db^{L'-1}}$$

since if $L > 1$, then there is a way to traverse a simple cycle starting at Y_n^+ and not using the reversal of Y_n, while if $L = 1$, then the walk can first choose an edge other than the reversal of Y_n, then traverse a loop, and then return by the reversal of Y_{n+1}. Let $Z_k := \mathbf{1}_{A_{kL'}} - \mathbf{P}(A_{kL'} | Y_1, \dots, Y_{kL'})$. Then $\langle Z_k; k \geq 1\rangle$ are uncorrelated, whence by the Strong Law of Large Numbers for uncorrelated random variables, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} Z_k = 0 \qquad \text{a.s.},$$

which implies that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{A_{kL'}} \geq \frac{1}{db^{L'-1}} \qquad \text{a.s.}$$

Therefore, if we choose $\varepsilon < 1/(db^{L'-1})$, then in nL' steps, at least εn events $A_{kL'}$ will occur for $0 \leq k < n$ with probability tending to 1 as $n \to \infty$.

Consider the following transformation of a path $\mathcal{P} = (Y_1, \dots, Y_{nL'})$ to a "reduced" path $\mathcal{P}'$: For each k such that $A_{kL'}$ occurs, remove the edges $Y_{k+1}, \dots, Y_{k+L'}$. Next, combine $\mathcal{P}$ and $\mathcal{P}'$ to form a nonbacktracking cycle $\mathcal{P}''$ by appending to $\mathcal{P}$ a nonbacktracking cycle of length L' that does not begin with the reversal of $Y_{nL'}$, and then by returning to the tail of Y_1 by $\mathcal{P}'$ in reverse order. Note that the map $\mathcal{P} \mapsto \mathcal{P}''$ is 1–1.

When at least εn events $A_{kL'}$ occur, the length of $\mathcal{P}''$ is at most $(2n+1-\varepsilon n)L'$. The number of nonbacktracking paths $Y_1, \ldots, Y_n$ equals db^{n-1}, whence $\sum_{k \le (2n+1-\varepsilon n)L'} b_k(G) \ge db^{nL'-1}/2$ for large n. This gives that $\texttt{cogr}(G) > \sqrt{b}$, which implies the result by Theorem 1.3. □

An alternative way of handling loops in the above proof is to use the following: Consider a random walk on a graph with spectral radius ρ. Suppose that we introduce a delay so that each step goes nowhere with probability p_{delay}, and otherwise chooses a neighbor with the same distribution as before. Then the new spectral radius equals $p_{\text{delay}} + (1 - p_{\text{delay}})\rho$. Hence, if there is a loop at each vertex and G is d-regular, then $\rho(G) \ge 1/(d-1) + (d-2)\rho(\mathbb{T}_d)/(d-1) > \rho(\mathbb{T}_d)$.

For a simple extension of this proof, let G be a d-regular multigraph. Suppose that there are some $L, M < \infty$ such that for every vertex $x \in \mathsf{V}(G)$, there is a simple cycle of length at most L that is at distance at most M from x. Then $\rho(G) > \rho(\mathbb{T}_d)$. Theorem 3 of Abért, Glasner and Virág (2015) gives a quantitative strengthening of this result.

3. Expected frequency. In the case of transitive multigraphs that are not trees, it is clear that simple random walk a.s. has many nontrivial cycle times. The most difficult part of our extension to general regular graphs is to show how this property is inherited by nonbacktracking random walk. This actually does not depend on regularity and is an interesting fact in itself.

Before we prove the general case, which has many complications, it may be helpful to the reader to see how to prove Theorem 1.1 with a stronger assumption on the density of nontrivial cycle times.

Recall that a cycle is *nontrivial* if it is not purely a backtracking cycle, that is, when backtracks are erased iteratively from the cycle, some edge remains. We call such cycles *NT-cycles*.

THEOREM 3.1. *Suppose that G is a graph all of whose degrees lie in some interval* $[3, D]$. *If the limsup expected frequency that simple random walk traverses some nontrivial cycle of length at most L is positive, then the same is true for nonbacktracking random walk. Hence, if G is also d-regular, then $\rho(G) > 2\sqrt{d-1}/d$.*

PROOF. We may assume that simple cycles of length exactly L are traversed with positive expected frequency. Let $X = \langle X_n; n \ge 1 \rangle$ be simple random walk on G and $\widehat{X} = \langle \widehat{X}_n \rangle$ be its lift to the universal cover T of G.

Now consider X. It contains purely backtracking excursions that are erased when we iteratively erase all backtracking. Let the lengths of the successive excursions be $M_1, M_2, \ldots$, where $M_i \ge 0$. Define

$$\Phi(n) := n + \sum_{k=1}^{n} M_k. \tag{3.1}$$

Then the edges that remain after erasing all backtracking are $\langle X_{\Phi(n)}; n \geq 1\rangle$. If we write $Y_n := X_{\Phi(n)}$, then $Y := \langle Y_n \rangle =: \mathtt{NB}(X)$ is the nonbacktracking path created from X. Let $\operatorname{im}\Phi$ be the image of Φ. Thus, $t \in \operatorname{im}\Phi$ iff the edge X_t is not erased from X when erasing all backtracking.

Consider a time t such that X_t completes a traversal of a simple cycle of length L. Because all degrees of T are at least 3, the probability (given the past) that $\widehat{X}$ will never cross the edge $-\widehat{X}_t$ after time t is at least $1/2$. In such a case, the cycle just traversed will not be erased (even in part) by the future. However, erasing backtracks from $X_1, \dots, X_t$ may erase (at least in part) this cycle.

Let $\mathtt{Trav}(Y)$ be the set of times n such that Y_n completes a traversal of a cycle and let $\mathtt{Trav}(X)$ be the set of times t that X_t completes a traversal of a simple cycle of length L.

We divide the rest of the proof into two cases, depending on whether $L > 1$ or not.

First, suppose that $L > 1$. Define a map $\psi : \mathbb{Z}^+ \to \mathtt{Trav}(Y) \cup \{\infty\}$ as follows:

$$\psi(t) := \begin{cases} \Phi^{-1}(t) + L, & \text{if } t \in \operatorname{im}\Phi \cap \mathtt{Trav}(X) \text{ and } \Phi^{-1}(t) + L \in \mathtt{Trav}(Y), \\ \infty, & \text{otherwise.} \end{cases}$$

For $t \in \mathtt{Trav}(X)$, the probability (given the past) that the steps $X_{t+1}, \dots, X_{t+L}$ traverse the same cycle $X_{t-L+1}, X_{t-L+2}, \dots, X_t$ of length L (in only L steps and in the same direction) and then (on the tree) $\widehat{X}_{t+L+1}, \widehat{X}_{t+L+2}, \dots$ never crosses the edge $-\widehat{X}_t$ is at least $1/(2D^L)$; similarly, for traversing the cycle in the opposite direction. In at least one of these two cases, some part of the cycle $X_{t+1}, \dots, X_{t+L}$ will be left after erasing *all* backtracks in X, in which case $\psi(t) \in \mathtt{Trav}(Y)$. Therefore, $\mathbf{P}[\psi(t) \in \mathtt{Trav}(Y) | t \in \mathtt{Trav}(X)] \geq 1/(2D^L)$, that is, $\mathbf{P}[\psi(t) \in \mathtt{Trav}(Y)] \geq \mathbf{P}[t \in \mathtt{Trav}(X)]/(2D^L)$. Hence,

$$\sum_{s \leq t} \mathbf{P}\bigl[\psi(s) \in \mathtt{Trav}(Y)\bigr] \geq \sum_{s \leq t} \mathbf{P}\bigl[s \in \mathtt{Trav}(X)\bigr] / \bigl(2D^L\bigr).$$

Note that $\psi(t_1) = \psi(t_2) \in \mathtt{Trav}(Y)$ implies that $t_1 = t_2$. Since $\psi(s) \leq s + L$, it follows that

$$\sum_{k \leq t+L} \mathbf{1}_{[k \in \mathtt{Trav}(Y)]} \geq \sum_{s \leq t} \mathbf{1}_{[\psi(s) \in \mathtt{Trav}(Y)]},$$

whence

$$\begin{aligned} &\limsup_{n \to \infty} n^{-1} \sum_{k \leq n} \mathbf{P}\bigl[k \in \mathtt{Trav}(Y)\bigr] \\ &\quad \geq \limsup_{t \to \infty} t^{-1} \sum_{s \leq t} \mathbf{P}\bigl[s \in \mathtt{Trav}(X)\bigr] / \bigl(2D^L\bigr) \\ &\quad = \limsup_{t \to \infty} t^{-1} \mathbf{E}\Bigl[\sum_{s \leq t} \mathbf{1}_{[s \in \mathtt{Trav}(X)]}\Bigr] \Big/ \bigl(2D^L\bigr) > 0 \end{aligned}$$

by assumption. Now the method of proof of Theorem 2.1 applies when G is regular.

Finally, suppose that $L = 1$. This means that $t \in \texttt{Trav}(X)$ iff X_t is a loop, and similarly for $\texttt{Trav}(Y)$. Define a map $\psi : \mathbb{Z}^+ \to \mathbb{Z}^+ \cup \{\infty\}$ as follows:

$$\psi(t) := \begin{cases} \Phi^{-1}(t), & \text{if } t \in \operatorname{im}\Phi \cap \texttt{Trav}(X), \\ \Phi^{-1}(t+1), & \text{if } t \in \texttt{Trav}(X) \setminus \operatorname{im}\Phi \text{ and } t+1 \in \operatorname{im}\Phi \cap \texttt{Trav}(X), \\ \infty, & \text{otherwise.} \end{cases}$$

Consider $t \in \texttt{Trav}(X)$. If erasing backtracks from $X_1, \ldots, X_t$ does not erase X_t, then the probability (given the past) that (on the tree) $\widehat{X}_{t+1}, \widehat{X}_{t+2}, \ldots$ never crosses the edge $-\widehat{X}_t$ is at least $1/2$, in which case $t \in \operatorname{im}\Phi$. On the other hand, if erasing backtracks from $X_1, \ldots, X_t$ *does* erase X_t, then the probability (given the past) that X_{t+1} is a loop and (on the tree) $\widehat{X}_{t+2}, \widehat{X}_{t+3}, \ldots$ never crosses the edge $-\widehat{X}_{t+1}$ is at least $1/(2D)$, in which case $t \notin \operatorname{im}\Phi$ and $t+1 \in \operatorname{im}\Phi \cap \texttt{Trav}(X)$. In each of these two cases, $\psi(t) \in \texttt{Trav}(Y)$. Therefore, $\mathbf{P}[\psi(t) \in \texttt{Trav}(Y) | t \in \texttt{Trav}(X)] \geq 1/(2D)$, that is, $\mathbf{P}[\psi(t) \in \texttt{Trav}(Y)] \geq \mathbf{P}[t \in \texttt{Trav}(X)]/(2D)$. Now the rest of the proof goes through as when $L > 1$, with the small change that instead of injectivity, we have that $|\psi^{-1}(n)| \leq 2$ for $n \in \texttt{Trav}(Y)$. $\square$

4. Proofs of Theorems 1.1 and 1.4. Here, we remove from Theorem 3.1 the upper bound on the degrees in G that was assumed and we weaken the assumption on the nature of nontrivial cycle frequency.

Consider a finite path $\mathcal{P} = \langle e_t; 1 \leq t \leq n \rangle$. Say that a time t is a *cycle time of* $\mathcal{P}$ if there exist $1 \leq s \leq t \leq u \leq n$ such that $(e_s, e_{s+1}, \ldots, e_u)$ is a cycle. If the cycle is required to be an NT-cycle, then we will call t an NT-cycle time, and likewise for other types of cycles. Call a cycle *fully nontrivial* if it is a loop or is nontrivial and its first edge is not the reverse of its last edge. Such cycles will be called FNT-cycles. For a finite or infinite path $\mathcal{P}$, we denote by $\mathcal{P} \restriction n$ its initial segment of n edges.

We state a slightly different version of Theorems 1.1 and 1.4 here. At the end of the section, we shall deduce the theorems as originally stated in Section 1.

THEOREM 4.1. *Suppose that G is a graph all of whose degrees are at least* 3. *Let $X = \langle X_t; t \geq 1 \rangle$ be simple random walk on G. If with positive probability the limsup frequency of NT-cycle times of $X \restriction n$ is positive as $n \to \infty$* (*i.e.*, *the expected limsup frequency is positive*), *then the same is true for nonbacktracking random walk. If G is also d-regular, then $\rho(G) > 2\sqrt{d-1}/d$.*

For $n \in \mathbb{Z}^+$, $\alpha > 0$, and a path $\mathcal{P}$ of length $> n$, let $C_n(\alpha, \mathcal{P})$ be the indicator that the number of NT-cycle times of $\mathcal{P} \restriction n$ is $>\alpha n$. We shall prove the following finitistic version of Theorem 4.1, which will be useful to us later.

THEOREM 4.2. *There exist $\zeta, \gamma > 0$ with the following property. Suppose that G is a graph all of whose degrees are at least 3. Then for all n and α,*

$$\mathbf{E}\big[C_n(\alpha, X)\big(1 - C_n\big(\hat{\alpha}, \mathtt{NB}(X)\big)\big)\big] < 3e^{-\zeta n},$$

where

$$\hat{\alpha} := \gamma\alpha / \log^2(10{,}368/\alpha).$$

There exists $\zeta' > 0$ such that if G is also d-regular, $\mathtt{cogr}(G) > 1$, and

$$\mathbf{E}\big[C_n(\alpha, X)\big] > \frac{c_o n e^{-\zeta' n}}{\mathtt{cogr}(G) - 1},$$

where c_o is as in Proposition 1.5, then $\rho(G) > 2\sqrt{d-1}/d$. If G is d-regular and

$$\limsup_{n\to\infty}\big[\mathbf{E}C_n(\alpha, X)\big]^{1/n} = 1,$$

then

$$\rho(G) > \frac{\sqrt{d-1}}{d}\big((d-1)^{\hat{\alpha}/24} + (d-1)^{-\hat{\alpha}/24}\big).$$

We shall use the following obvious fact.

LEMMA 4.3. *If $(e_1, \dots, e_k)$ and $(f_1, \dots, f_m)$ are paths without backtracking, the head of e_k equals the tail of f_1, and e_k is not the reverse of f_1, then $(e_1, \dots, e_k, f_1, \dots, f_m)$ is a path without backtracking.*

We shall apply the following well-known lemma to intervals with integer endpoints.

LEMMA 4.4 (Vitali covering). *Let I be a finite collection of subintervals of $\mathbb{R}$. Write $\|I\|$ for the sum of the lengths of the intervals in I. Then there exists a subcollection J of I consisting of pairwise disjoint intervals such that $\|J\| \geq \|I\|/3$.*

This lemma is immediate from choosing iteratively the largest interval disjoint from previously chosen intervals.

The following is a simple modification of a standard bound on large deviations.

LEMMA 4.5. *Suppose that $c > 0$. There exist $\varepsilon \in (0, 1)$ and $\beta > 0$ such that whenever $Z_1, \dots, Z_n$ are random variables satisfying the inequalities $P[Z_k > z | Z_1, \dots, Z_{k-1}] \leq e^{-cz}$ for all $z > 0$, we have*

$$\mathbf{P}\left[\sum_{k=1}^{\varepsilon n} Z_k \geq n\right] \leq e^{-\beta n}.$$

PROOF. Write $S_t := \sum_{j=1}^{\lfloor t \rfloor} Z_j$. Given $\lambda := c/2$ and $k \in [1, n]$, we have

$$\begin{aligned}\mathbf{E}\big[e^{\lambda Z_k} | Z_1, \dots, Z_{k-1}\big] &= \int_0^\infty \mathbf{P}\big[e^{\lambda Z_k} > z | Z_1, \dots, Z_{k-1}\big]\, dz \\ &\le 1 + \int_1^\infty z^{-c/\lambda}\, dz = 2,\end{aligned}$$

whence

$$\mathbf{E}\big[e^{\lambda S_k} | Z_1, \dots, Z_{k-1}\big] \le 2e^{\lambda S_{k-1}}.$$

By induction, therefore, we have that $\mathbf{E}[e^{\lambda S_k}] \le 2^k$. It follows by Markov's inequality that

$$\mathbf{P}\left[\sum_{k=1}^{\varepsilon n} Z_k \ge n\right] = \mathbf{P}\big[e^{\lambda S_{\varepsilon n}} \ge e^{\lambda n}\big] \le 2^{\varepsilon n} e^{-\lambda n}.$$

Thus, if we choose $\varepsilon := \min\{1/4, c/(4\log 2)\}$ and $\beta := c/4$, the desired bound holds. $\square$

Several lemmas now follow that will be used to handle various possible behaviors of simple random walk on G.

LEMMA 4.6. *Suppose that G is a graph all of whose degrees are at least* 3. *Let X record the oriented edges taken by simple random walk on G. Let $\Phi(n)$ index the nth edge of X that remains in* $\mathrm{NB}(X)$, *so that* $\mathrm{NB}(X) = \langle X_{\Phi(n)} \rangle$: *see* (3.1). *Write* $\Phi(0) := 0$. *Then there exists $t_0 < \infty$ such that for all n and all $t > t_0$, we have*

$$\mathbf{P}\big[\Phi(n+1) - \Phi(n) > t\big] < (8/9)^{t/2}.$$

In addition, there exists r such that for every n and λ,

$$\mathbf{P}\big[\Phi(n) > (r + \lambda)n\big] < (8/9)^{\lambda n/4}.$$

More generally, for all $L > 0$, there exists $r_L \le 36^2(8/9)^{L/4}$ such that

$$\mathbf{P}\left[\sum_{k<n} \big(\Phi(k+1) - \Phi(k)\big)\mathbf{1}_{[\Phi(k+1)-\Phi(k)>L]} > (r_L + \lambda)n\right] < (8/9)^{\lambda n/4}.$$

PROOF. Let $\widehat{X}$ be the lift of X to T. Then Φ also indexes the edges that remain in $\mathrm{NB}(\widehat{X})$. Since the distance of $\mathrm{NB}(\widehat{X})$ from $\widehat{X}_1^-$ increases by 1 at each step, the times $\Phi(n+1) - \Phi(n)$ are dominated by the times between escapes for random walk on $\mathbb{N}$ that has probability $2/3$ to move right and $1/3$ to move left, reflecting at 0. These in turn are dominated by the time to the first escape for random walk S

on $\mathbb{Z}$ with the same bias. Such an escape can happen only at an odd time, t. The chance of an escape at time exactly t is

$$\begin{aligned}\mathbf{P}\big[S_{t-1}=0, S_t=1, \forall t'>t\ S_{t'}>0\big]\\ &= \binom{t-1}{(t-1)/2}\left(\frac{2}{3}\right)^{(t-1)/2}\left(\frac{1}{3}\right)^{(t-1)/2}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\\ &\sim c\frac{(\sqrt{8/9})^t}{\sqrt{t}}\end{aligned}$$

for some constant c. This proves the first inequality.

Now $\Phi(n)=\sum_{k<n}(\Phi(k+1)-\Phi(k))$ and these summands are dominated by the corresponding inter-escape times for the biased random walk on $\mathbb{Z}$. The latter are i.i.d. with some distribution ν (which we bounded in the last paragraph), whence if we choose $a := (8/9)^{1/4} \in (\sqrt{8/9}, 1)$ and put $b := \sum_{j\geq 1} a^{-j}\nu(j) \in (1,\infty)$, we obtain that for all n, we have $\mathbf{E}[a^{-\Phi(n)}]\leq b^n$. By Markov's inequality, this implies that

$$\mathbf{P}\big[\Phi(n) > (c+\lambda)n\big] \leq a^{(c+\lambda)n}b^n,$$

so if we choose $c=r$ with $a^r b=1$, then we obtain the second inequality.

The third inequality follows similarly: let $B_k := (\Phi(k+1)-\Phi(k)) \times \mathbf{1}_{[\Phi(k+1)-\Phi(k)>L]}$. Put $b_L := \nu[1,L]+\sum_{j>L}a^{-j}\nu(j) \in (1, 1+36a^L)$. Then $\mathbf{E}[a^{-\sum_{k=0}^{n-1}B_k}]\leq b_L^n$ for all n. By Markov's inequality, this implies that

$$\mathbf{P}\Bigg[\sum_{k=0}^{n-1} B_k > (c_L+\lambda)n\Bigg] \leq a^{(c_L+\lambda)n}b_L^n,$$

so if we choose $c_L = r_L$ with $a^{r_L}b_L=1$, then we obtain the third inequality. We have the estimate $r_L \leq 36^2 a^L$. $\square$

It follows that

$$\mathbf{P}\big[\Phi\big(n/(r+\lambda)\big) > n\big] < a^{\lambda n}.$$

That is, except for exponentially small probability, there are at least $n/(r+\lambda)$ nonbacktracking edges by time n. Similarly, except for exponentially small probability, there are at most $(r_L+\lambda)n$ edges by time n that are in intervals of length $> L$ that have no escapes.

The following is clear.

LEMMA 4.7. *With notation as in Lemma 4.6, if $s \leq \Phi(n) \leq t$ satisfy $X_s^- = X_t^+$, then n is a cycle time of* $\mathtt{NB}(X)\restriction t$.

We call a time t an *escape time* for $\widehat{X}$ if $-\widehat{X}_{t+1}\notin \mathtt{NB}(\widehat{X}_1,\dots,\widehat{X}_t)$ and $-\widehat{X}_{t+1}\notin \{\widehat{X}_s; s>t+1\}$. We let $\mathtt{Esc}(\widehat{X})$ be the set of escape times for $\widehat{X}$. Then $\mathtt{Esc}(\widehat{X}) = \mathrm{im}(\Phi-1)$.

LEMMA 4.8. *Suppose that $\langle \tau_k; 1 \le k < K\rangle$ is a strictly increasing sequence of stopping times for X, where K is random, possibly ∞. Then there exist $\eta, \delta > 0$ such that for all $n > 0$,*

$$\mathbf{P}\big[K > n, \big|\{k \le n; \tau_k \in \mathtt{Esc}(\widehat{X})\}\big| < \eta n\big] < e^{-\delta n}.$$

PROOF. Define random variables σ_j, λ_j recursively. First, we describe in words what they are. Start by setting $\lambda_1 := 1$ and by examining what happens after time τ_1. If $\widehat{X}$ escapes, define $\sigma_1 := 1$, $\lambda_2 := 2$, and look at time τ_2. If not, then look at the first time τ_j that occurs after the first time $\ge t+1$ we know that $\widehat{X}$ has not escaped, that is, $t+1$ if $-\widehat{X}_{t+1} \in \mathtt{NB}(\widehat{X}_1, \ldots, \widehat{X}_t)$ or else $\min\{s > t+1; -\widehat{X}_{t+1} = \widehat{X}_s\}$, and define $\sigma_1 := \tau_j - \tau_1$, $\lambda_2 := j$. Now repeat from time τ_{λ_1} to define σ_2 and λ_3, etc.

The precise definitions are as follows. Suppose that $K > n$ (otherwise we do not define these random variables). Define A_j to be the event that one of the following holds:

$$-\widehat{X}_{\tau_j+1} \in \mathtt{NB}(\widehat{X}_1, \ldots, \widehat{X}_{\tau_j}) \quad \text{or} \quad \tau_j \in \mathtt{Esc}(\widehat{X}).$$

Write $\lambda_1 := 1$. To recurse, suppose that λ_k has been defined. Let

$$\lambda_{k+1} := \begin{cases} \lambda_k + 1, & \text{if } A_{\lambda_k}, \\ \min\{j; -\widehat{X}_{\tau_{\lambda_k}+1} \in \{\widehat{X}_s; \tau_{\lambda_k} + 1 < s < \tau_j\}\}, & \text{otherwise} \end{cases}$$

and

$$\sigma_k := \begin{cases} 1, & \text{if } A_{\lambda_k}, \\ \tau_{\lambda_{k+1}} - \tau_{\lambda_k}, & \text{otherwise.} \end{cases}$$

Let $J := \max\{j; \lambda_j \le n\}$. This is the number of times we have looked for escapes up to the nth stopping time. Each stopping time until the nth is covered by one of the intervals $[\tau_1, \tau_{\lambda_2}), \ldots, [\tau_{\lambda_J}, \tau_{\lambda_{J+1}})$, which have lengths $\sigma_1, \ldots, \sigma_J$. Therefore, we have that

$$\sum_{j=1}^{J} \sigma_j \ge n.$$

We claim that this forces J to be large with high probability:

$$\mathbf{P}[J \le \varepsilon n] \le e^{-\gamma n} \tag{4.1}$$

for some $\varepsilon, \gamma > 0$. Indeed, we claim that for each $k \le \varepsilon n$,

$$\mathbf{P}\left[\sum_{j=1}^{k} \sigma_j \ge n\right] \le e^{-\beta n},$$

where ε and β are given by Lemma 4.5 with c (in that lemma) to be determined. This would imply that

$$\mathbf{P}[J \le \varepsilon n] \le \varepsilon n e^{-\beta n}.$$

Now $\tau_{\lambda_{k+1}} - \tau_{\lambda_k} \geq \lambda_{k+1} - \lambda_k$. Thus, it suffices to show that there is some $c > 0$ for which

$$\mathbf{P}[\lambda_{k+1} - \lambda_k \geq z | \sigma_1, \ldots, \sigma_{k-1}] \leq e^{-cz}$$

for all $z > 1$. Now the event $\lambda_{k+1} - \lambda_k \geq z > 1$ implies the event B that $-\widehat{X}_t \notin \mathtt{NB}(\widehat{X}_1, \ldots, \widehat{X}_{\tau_{\lambda_k}})$ for all $t \in (\tau_{\lambda_k}, \tau_{\lambda_k} + z)$ and that $-\widehat{X}_t = \widehat{X}_{\tau_{\lambda_k}}$ for some $t \geq \tau_{\lambda_k} + z$. Because the distance from $\widehat{X}_t$ to $\widehat{X}_{\tau_{\lambda_k}}$ has a probability at least $2/3$ to get larger at all times, this is exponentially unlikely in z. What we need, however, is that this is exponentially unlikely even under the given conditioning. For every event A in the σ-field on which we are conditioning, we always have that $A \supseteq [\tau_{\lambda_k} \in \mathtt{Esc}(\widehat{X})]$. Furthermore, $\mathbf{P}[\tau_{\lambda_k} \in \mathtt{Esc}(\widehat{X})] \geq 1/2$. Hence, $\mathbf{P}(B|A) \leq 2\mathbf{P}(B)$, so that the bound on the unconditional probability of B also gives an exponential bound on the conditional probability of B. Thus, we have proved (4.1).

Define $E_k := [\tau_{\lambda_k} \in \mathtt{Esc}(\widehat{X})]$. We claim that

$$\mathbf{P}\big(E_k | \sigma(E_1, \ldots, E_{k-1})\big) \geq 1/2. \tag{4.2}$$

Indeed, let Z_t be the distance of $\widehat{X}_t^+$ to $\widehat{X}_1^-$. Note that $t \in \mathtt{Esc}(\widehat{X})$ iff $Z_s > Z_t$ for all $s > t$. Write $F_t(j)$ for the event that $Z_s > j$ for all $s > t$. We claim that

$$\mathbf{P}\big(E_k | \sigma(E_1, \ldots, E_{k-1}, \widehat{X}_1, \ldots, \widehat{X}_{\lambda_k}, \lambda_1, \ldots, \lambda_k)\big) \geq 1/2,$$

which is stronger than (4.2). By choice of $\lambda_1, \ldots, \lambda_k$, we have that for every event $E \in \sigma(E_1, \ldots, E_{k-1}, \widehat{X}_1, \ldots, \widehat{X}_{\lambda_k}, \lambda_1, \ldots, \lambda_k)$,

$$\mathbf{P}(E_k|E) = \mathbf{P}\big(F_t(j_m) | F_t(j_1), \ldots, F_t(j_{m-1})\big)$$

for some $j_1, \ldots, j_{m-1} < j_m$ and some t, where $m \geq 1$. Since $F_t(j_i) \supseteq F_t(j_m)$ and $\mathbf{P}(F_t(j_m)) \geq 1/2$, the claim follows.

Therefore, by (4.2), we may couple the events E_k to Bernoulli trials with probability $1/2$ each so that the kth successful trial implies E_k. This shows that there exists $\delta > 0$ such that

$$\mathbf{P}\big[J > \varepsilon n, \big|\{k \leq J; E_k\}\big| < \varepsilon n/3\big] < e^{-\delta n}.$$

Hence,

$$\mathbf{P}\big[K = \infty, \big|\{j \leq n; \tau_j \in \mathtt{Esc}(\widehat{X})\}\big| < \varepsilon n/3\big] < e^{-\delta n}.$$

Thus, we may choose $\eta := \varepsilon/3$. $\square$

Call a cycle of $> L$ edges an *L^+-cycle*. Define $I(n, L)$ to be the set of times $t \in [1, n]$ for which there exist $1 \leq s \leq t \leq u \leq n$ such that $(X_s, X_{s+1}, \ldots, X_u)$ is a nontrivial L^+-cycle.

LEMMA 4.9. *If $n, L \geq 1$ and $\beta \in (0, 1)$, then*

$$\mathbf{E}\big[\mathbf{1}_{[|I(n,L)| \geq \beta n]}\big(1 - C_n\big(\beta/(2L), \mathtt{NB}(X)\big)\big)\big] < (8/9)^{\beta n/16}.$$

PROOF. Let $\texttt{Long}$ be the event $[|I(n,L)| \geq \beta n]$. Let J be the union of intervals in $[1,n]$ that have length $> L$ and are disjoint from $\texttt{Esc}(X)$. Let $\texttt{Bad}$ be the event that $|J| > \beta n/2$. By Lemma 4.6, we have $\mathbf{P}(\texttt{Bad}) < (8/9)^{\beta n/16}$ (use $\lambda := \beta/4$ there). On the event $\texttt{Long} \setminus \texttt{Bad}$, the set $I(n,L)$ contains at least $\beta n/2$ times that are within distance $L/2$ of an escape. Therefore, on the event $\texttt{Long} \setminus \texttt{Bad}$, there are at least $\beta n/(2L)$ escapes in nontrivial cycles, whence $\texttt{NB}(X)\restriction n$ has $\geq \beta n/(2L)$ NT-cycle times. $\square$

Let $I_\circ(n,L)$ be the (random) set of times $t \in [1,n] \setminus I(n,L)$ such that X_t is a loop.

LEMMA 4.10. *There exist $\eta, \delta > 0$ such that if $n, L \geq 1$ and $\beta \in (0,1)$, then*

$$\mathbf{E}\big[\mathbf{1}_{[|I_\circ(n,L)|\geq \beta n]}\big(1 - C_n\big(\eta\beta/(L+1), \texttt{NB}(X)\big)\big)\big] < e^{-\delta n}.$$

PROOF. Let $\texttt{Loop} := [|I_\circ(n,L)| \geq \beta n]$. Note that if there are 3 times at which a given loop in G is traversed, then necessarily the first of those times belongs to a nontrivial cycle with at least one of the other times. In particular, if a given loop is traversed at least $L+2$ times, then it belongs to a nontrivial long cycle. Therefore, on $\texttt{Loop}$, there are $\geq \beta n/(L+1)$ times spent in distinct loops. If we take the first traversal of a loop as a stopping time, then Lemma 4.8 supplies us with η, δ such that on the event $\texttt{Loop}$, except for probability $< e^{-\delta n}$, the number of new loops at escape times is at least $\eta\beta n/(L+1)$. Necessarily, all such loops remain in $\texttt{NB}(X)$. Therefore, $\texttt{NB}(X)\restriction n$ also has at least $\eta\beta n/(L+1)$ loops on the event $\texttt{Loop}$ except for probability $< e^{-\delta n}$. $\square$

Let $D(n)$ denote the maximal number of disjoint FNT-cycles in $X\restriction n$, other than loops.

LEMMA 4.11. *There exist $\eta, \delta > 0$ such that if $n \geq 1$ and $\beta \in (0,1)$, then*

$$\mathbf{E}\big[\mathbf{1}_{[|D(n)|\geq \beta n]}\big(1 - C_n\big(\eta\beta, \texttt{NB}(X)\big)\big)\big] < e^{-\delta n}.$$

PROOF. Fix $n \geq 1$. Let $\texttt{Cycs}$ be a (measurable) set of pairs of times $1 \leq s < t \leq n$ such that $(X_s, X_{s+1}, \ldots, X_t)$ is an FNT-cycle other than a loop, chosen so that $|\texttt{Cycs}| = D(n)$. Let $\texttt{Cycs\&Esc} := \{(s,t) \in \texttt{Cycs}; t \in \texttt{Esc}(\widehat{X})\}$. By Lemma 4.8, on the event $D(n) \geq \beta n$, we have $|\texttt{Cycs\&Esc}| > \eta'\beta n$ for some $\eta' > 0$ except for exponentially small probability.

Let $\texttt{Sofar}$ be the set of $(s,t) \in \texttt{Cycs}$ such that $X_{s-1} \neq -X_t$ or $s = 1$.

Note that for $(s,t) \in \texttt{Cycs}$, the cycle from X_s to X_t can be traversed in either order, both being equally likely given $X_1, \ldots, X_{s-1}$, and at least one of them has the property that $(s,t) \in \texttt{Sofar}$ [see Lemma 4.3, where we concatenate $\texttt{NB}(X_1, \ldots, X_{s-1})$ with either $\texttt{NB}(X_s, \ldots, X_t)$ or $\texttt{NB}(X_t, X_{t-1}, \ldots, X_s)$, as appropriate]. In fact, the same holds even conditioned on $\texttt{Cycs\&Esc}$. Therefore, we may couple to Bernoulli trials and conclude that on the event $D(n) \geq \beta n$, we have $|\texttt{Cycs\&Esc} \cap \texttt{Sofar}| > \eta'\beta n/3$ except for exponentially small probability. Note that for $t \in \texttt{Cycs\&Esc} \cap \texttt{Sofar}$, some edge in the cycle $(X_s, \dots, X_t)$ belongs to $\texttt{NB}(X)$ (see Lemma 4.3 again)—more precisely, $u \in \operatorname{im}\Phi$ for some $u \in [s, t]$—, whence on the event $D(n) \geq \beta n$, we have $\texttt{NB}(X)\restriction n$ has $> \eta\beta n$ cycle times except for exponentially small probability, where $\eta := \eta'/3$. $\square$

LEMMA 4.12. *Suppose that* $\rho := \rho(G) < 1$, $n \geq 2$, $\varepsilon > 0$ *and* $L > 2e^2$. *Then*

$$\mathbf{P}\big[\big|\{L^+\text{-cycle times of } X\restriction n\}\big| \geq \varepsilon n\big] < e^{(6n/L)\log L}\rho^{\varepsilon n/3}/(1-\rho).$$

PROOF. For every n and k, the chance that X_n begins a cycle of length k is at most ρ^k. Suppose that the number of L^+-cycle times of $X\restriction n$ is at least εn. Then there are disjoint L^+-cycles in $X_1, \dots, X_n$ the sum of whose lengths is at least $\varepsilon n/3$ by Lemma 4.4. There are fewer than n/L starting points and fewer than n/L ending points for those cycles since each has length $> L$ and they are disjoint. The number of collections of subsets of $[0, n]$ of size at most $2n/L$ is $< e^{(n+1)h(2/L)} < e^{(6n/L)\log L}$, where $h(\alpha) := -\alpha\log\alpha - (1-\alpha)\log(1-\alpha)$. This is because $h(\alpha) < -2\alpha\log\alpha$ for $\alpha < e^{-2}$. For each such collection of starting and ending points giving total length k, the chance that they do start L^+-cycles is at most ρ^k, whence summing over collections and total lengths that are $\geq \varepsilon n/3$, we get the result. $\square$

Call a nonbacktracking cycle an *NB-cycle*. If an NB-cycle is a loop or has the property that its last edge is different from the reverse of its first edge, then call the cycle *fully nonbacktracking*, abbreviated *FNB-cycle*. Recall that the number of NB-cycles of length n starting from $x \in \mathtt{V}(G)$ is $b_n(x)$. We also say that a cycle starting from x is "at x". Let the number of FNB-cycles of length n at x be $b_n^*(x)$. Recall that $S(x) := \{n; b_n(x) \neq 0\}$. We shall need the following bounds on $b_n(x)$.

PROPOSITION 1.5. *Let G be a graph with* $\texttt{cogr}(G) \geq 1$. *For each* $x \in \mathtt{V}(G)$, *we have that* $\lim_{S(x)\ni n\to\infty} b_n(x)^{1/n}$ *exists and there is a constant* c_x *such that* $b_n(x) \leq c_x \,\texttt{cogr}(G)^n$ *for all* $n \geq 1$. *Furthermore, if x belongs to a simple cycle of length L, then* $c_x \leq 2 + 2L\,\texttt{cogr}(G)^{L-2}$. *If G is d-regular, then G is Ramanujan iff for all vertices x and all* $n \geq 1$, *we have* $b_n^*(x) \leq 2(d-1)^{n/2}$.

PROOF. Write $S^*(x) := \{n; b_n^*(x) \neq 0\}$. Given two FNB-cycles starting at x, we may concatenate the first with either the second or the reversal of the second to obtain an FNB-cycle at x, unless both FNB-cycles are the same loop. Therefore, if $b_n^*(x)$ is the number of FNB-cycles at x, we have $b_m^*(x)b_n^*(x)/2 \leq b_{m+n}^*(x)$ for $m + n \geq 3$, whence $\langle b_n^*(x)/2; n \geq 2\rangle$ is supermultiplicative and Fekete's lemma implies that $\lim_{S^*(x)\ni n\to\infty} b_n^*(x)^{1/n}$ exists and $b_n^*(x) \leq 2\,\texttt{cogr}(G)^n$ for $n \geq 2$. It is easy to check that the same inequality holds for $n = 1$. Together with Theorem 1.3, this also implies that if G is d-regular and Ramanujan, then for all vertices x and all $n \geq 1$, we have $b_n^*(x) \leq 2(d-1)^{n/2}$.

Let $\widehat{b}_n(x) := b_n(x) - b_n^*(x)$ be the number of nonloop NB-cycles at x whose last edge equals the reverse of its first edge, that is, NB-cycles that are not FNB-cycles. We shall bound $\widehat{b}_n(x)$ when x belongs to a simple cycle, say, $\mathcal{P}_0 = (e_1, \ldots, e_L)$ with L edges. Let $\mathcal{P}$ be a nonloop NB-cycle at x whose last edge is e' and whose first edge is $-e'$. If e' is a loop, then removing e' at the end of $\mathcal{P}$ gives an FNB-cycle $\mathcal{P}'$ at x. Otherwise, decompose $\mathcal{P}$ as $\mathcal{P}_1.\mathcal{P}_2$, where . indicates concatenation, and $\mathcal{P}_2$ is maximal containing only edges e such that $e \in \mathcal{P}_0$ or $-e \in \mathcal{P}_0$. By reversing $\mathcal{P}_0$ if necessary, we may assume the former: all edges of $\mathcal{P}_2$ lie in $\mathcal{P}_0$. Suppose the first edge of $\mathcal{P}_2$ is e_k. Then $\mathcal{P}_2$ traverses the remainder of $\mathcal{P}_0$ and possibly the whole of $\mathcal{P}_0$ several times. Thus, write $\mathcal{P}_2 = \mathcal{P}_3.\mathcal{P}_4$, where $\mathcal{P}_3 = (e_k, \ldots, e_L)$ has length $\le L$. Finally, the NB-cycle $\mathcal{P}' := \mathcal{P}_1.(-e_{k-1}, \ldots, -e_1).\overline{\mathcal{P}}_4$ is FNB, where the bar indicates path reversal. In addition, the length of $\mathcal{P}'$ differs from the length of $\mathcal{P}$ by at most $L - 2$. Since the map $\mathcal{P} \mapsto \mathcal{P}'$ is injective, $\widehat{b}_n(x) \le \sum_{i=-1}^{L-2} b_{n+i}^*(x) \le 2L\,\mathtt{cogr}(G)^{n+L-2}$.

Combining the results of the previous two paragraphs, we obtain that if x belongs to a simple cycle, then there is a constant c_x such that for all $n \ge 1$, we have $b_n(x) \le c_x\,\mathtt{cogr}(G)^n$. We also get the bound claimed for c_x.

We now prove the same for x that do not belong to a simple cycle. We claim that if y is a neighbor of x, then $b_n(x) \le b_{n-2}(y) + b_n(y) + b_{n+2}(y)$. Indeed, let $\mathcal{P}$ be an NB-cycle at x. Suppose the first edge of $\mathcal{P}$ goes to y. If $\mathcal{P}$ is not FNB, then removing the first and last edges of $\mathcal{P}$ yields an NB-cycle at y of length $n-2$. If $\mathcal{P}$ is FNB, then shifting the starting point from x to y yields an FNB-cycle at y of length n. Lastly, if the first edge of $\mathcal{P}$ does not go to y, then we may prepend to $\mathcal{P}$ an edge from y to x and either append an edge from x to y if the last edge of $\mathcal{P}$ was not from y, or else delete the last edge of $\mathcal{P}$, yielding an NB-cycle at y of length $n + 2$ or n. This map of NB-cycles at x to NB-cycles at y is injective, which gives the claimed inequality. It follows that $b_n(x) \le c_x b_n(z)$, where z is the nearest point to x that belongs to a simple cycle and c_x does not depend on n.

Finally, if $\lim_{S(x)\ni n\to\infty} b_n(x)^{1/n}$ exists for one x, then it exists for all x by the covering-tree argument we used earlier in Section 1. Suppose that for all x belonging to a simple cycle, $\lim_{S^*(x)\ni n\to\infty} b_n^*(x)^{1/n} < \mathtt{cogr}(G)$. Then the bounds in the preceding paragraphs show that $\lim_{S^*(x)\ni n\to\infty} b_n(x)^{1/n} < \mathtt{cogr}(G)$. It is not hard to see that therefore $\limsup_{S(x)\ni n\to\infty} b_n(x)^{1/n} < \mathtt{cogr}(G)$ as well, which is a contradiction to the definition of $\mathtt{cogr}(G)$. Hence for some x, we have $\lim_{S^*(x)\ni n\to\infty} b_n^*(x)^{1/n} = \mathtt{cogr}(G)$ and, therefore, $\lim_{S(x)\ni n\to\infty} b_n(x)^{1/n} = \mathtt{cogr}(G)$ as well. Together with Theorem 1.3, this also implies that if G is d-regular and for all vertices x and all $n \ge 1$, we have $b_n^*(x) \le 2(d-1)^{n/2}$, then G is Ramanujan, which completes the proof of the last sentence of the proposition. □

Let $Y := \mathtt{NB}(X)$. Let $A_n^L(\beta)$ be the event that there are $\ge \beta n$ times $t \in [1, n]$ for which there exist $1 \le s \le t \le u \le n$ such that $(Y_s, Y_{s+1}, \ldots, Y_u)$ is a cycle with $u - s < L$.

LEMMA 4.13. *Let G be d-regular with* $\mathtt{cogr}(G) > 1$ *and* $\beta \in (0, 1)$. *For every* $L < \infty$, *if*

$$\mathbf{P}\big[A_n^L(\beta)\big] > \frac{c_o n (d-1)^{-\beta^2 n/6 + L/2}}{\mathtt{cogr}(G) - 1},$$

where c_o is as in Proposition 1.5, then $\rho(G) > 2\sqrt{d-1}/d$. *If*

$$\limsup_{n\to\infty} \mathbf{P}\big[A_n^L(\beta)\big]^{1/n} = 1, \tag{4.3}$$

then

$$\rho(G) > \frac{\sqrt{d-1}}{d}\big((d-1)^{\beta/24} + (d-1)^{-\beta/24}\big).$$

PROOF. We may suppose that $\rho(G) < 1$, as there is nothing to prove otherwise.

Let $A_n(\beta, L)$ be the event that $Y \restriction n$ has at least βn cycle times and that Y_n completes a cycle of length $\le L$. Note that $\mathbf{P}[A_k(\beta, L)] \ge \mathbf{P}[A_n^L(\beta)]/n$ for some $k \in [\beta n, n]$ by considering the last cycle completed.

Consider the following transformation $\mathcal{P} \mapsto \mathcal{P}'$ of finite nonbacktracking paths $\mathcal{P}$: let I be the collection of cycles in $\mathcal{P}$. Choose (measurably) a maximal subcollection J as in Lemma 4.4. Excise the edges in J from $\mathcal{P}$, concatenate the remainder, and remove backtracks to arrive at $\mathcal{P}'$. Then $\mathcal{P}'$ is a nonbacktracking path without cycles and $|\mathcal{P}| - |\mathcal{P}'|$ is at least $1/3$ the number of cycle times of $\mathcal{P}$.

Fix n. Let $p_n(\beta, L) := \mathbf{P}(A_n(\beta, L))$. Let $q_n(\beta, L)$ be the probability that the length of $\mathcal{P}'$ is at most $n - \beta n/3$. By the last paragraph, we have $q_n(\beta, L) \ge p_n(\beta, L)$.

We define another transformation $\mathcal{P} \mapsto \mathcal{P}''$ as follows, where $\mathcal{P}''$ will be a nonbacktracking cycle when Y_n completes a cycle: Let $m := \min\{i; Y_i^+ = Y_n^+\}$. Let s be minimal with $\mathcal{P}'$ ending in $(Y_s, Y_{s+1}, \dots, Y_n)$ and define $\widehat{\mathcal{P}}$ by $\mathcal{P}' = \widehat{\mathcal{P}}.(Y_s, Y_{s+1}, \dots, Y_n)$, where . indicates concatenation. Since $\mathcal{P}'$ has no cycles, if $m < n$ (which it is if Y_n completes a cycle), then $m < s$. Now define $\mathcal{P}'' := \mathcal{P}.\overline{\mathcal{P}'}$ if $s = n$, where the bar indicates path reversal, or else $\mathcal{P}'' := \mathcal{P}.(Y_{m+1}, Y_{m+2}, \dots, Y_s).\overline{\widehat{\mathcal{P}}}$.

Write $b := d - 1$. On the event $A_n(\beta, L)$, we have that $\mathcal{P}''$ is a nonbacktracking cycle with length at most $2n - \beta n/3 + L$. Furthermore, the map $\mathcal{P} \mapsto \mathcal{P}''$ is injective because the first part of $\mathcal{P}''$ is simply $\mathcal{P}$. Therefore, Proposition 1.5 provides a constant c_o such that

$$d b^{n-1} q_n(\beta, L) \le \sum_{k \le 2n - \beta n/3 + L} c_o b_k(o),$$

whence

$$q_n(\beta, L) \le \frac{c_o\, \mathtt{cogr}(G)^{2n - \beta n/3 + L}}{b^n(\mathtt{cogr}(G) - 1)}.$$

For some $k \geq \beta n$, we have

$$\frac{\mathbf{P}[A_n^L(\beta)]}{n} \leq p_k(\beta, L) \leq q_k(\beta, L) \leq \frac{c_o\,\mathtt{cogr}(G)^{2k-\beta k/3+L}}{b^k(\mathtt{cogr}(G)-1)}.$$

It follows that if $\mathtt{cogr}(G) \leq \sqrt{b}$, then the last quantity above is

$$\leq \frac{c_o b^{-\beta k/6+L/2}}{\mathtt{cogr}(G)-1} \leq \frac{c_o b^{-\beta^2 n/6+L/2}}{\mathtt{cogr}(G)-1},$$

which proves the first part of the lemma. Similarly, if (4.3) holds, then

$$\mathtt{cogr}(G) \geq b^{1/(2-\beta/3)} > b^{1/2+\beta/12},$$

whence by Theorem 1.3,

$$\rho(G) > \frac{b^{1/2+\beta/12} + b^{1/2-\beta/12}}{d}. \qquad \square$$

We remark that with more work, we may let $L := \infty$ in (4.3).

PROOF OF THEOREM 4.2. Let $X = \langle X_n \rangle$ be simple random walk on G and $\widehat{X} = \langle \widehat{X}_n \rangle$ be its lift to the universal cover T of G.

Fix n. Let $\mathtt{Good}$ be the event that $C_n(\alpha, X) = 1$. We may choose $L \leq 34\log(10{,}368/\alpha)$ so that the number r_L of Lemma 4.6 satisfies $r_L < \alpha/8$. Fix such an L.

Let $\mathtt{Long} := [|I(n,L)| \geq \alpha n/2]$. By Lemma 4.9 (using $\beta := \alpha/2$), we have that

$$\mathbf{E}\big[\mathbf{1}_{\mathtt{Long}}\big(1 - C_n(\alpha/(4L), \mathrm{NB}(X))\big)\big] < (8/9)^{\alpha n/32}.$$

Let $\mathtt{Loop} := [|I_\circ(n,L)| \geq \alpha n/(8L)]$. Then by Lemma 4.10 (using $\beta := \alpha/4$),

$$\mathbf{E}\big[\mathbf{1}_{\mathtt{Loop}}\big(1 - C_n\big(\eta_1\alpha/\big(8L^2+8L\big), \mathrm{NB}(X)\big)\big)\big] < e^{-\delta_1 n}$$

for some $\eta_1, \delta_1 > 0$.

On the event $(\mathtt{Good} \setminus \mathtt{Long} \setminus \mathtt{Loop})$, there are $\geq \alpha n/4$ times $t \in [1,n]$ for which there exist $1 \leq s \leq t \leq u \leq n$ such that $(X_s, X_{s+1}, \ldots, X_u)$ is an NT-cycle with $1 \leq u - s < L$ and that does not contain any loops; this is because every loop can be contained in at most $2L$ NT-cycles of length at most L in $X\!\restriction\! n$. By Lemma 4.4, on the event $(\mathtt{Good} \setminus \mathtt{Long} \setminus \mathtt{Loop})$, there are $\geq \alpha n/(12L)$ disjoint nonloop NT-cycles in $X\!\restriction\! n$. Within every NT-cycle, there is an FNT-cycle. Thus, on the event $(\mathtt{Good} \setminus \mathtt{Long} \setminus \mathtt{Loop})$, there are $\geq \alpha n/(12L)$ disjoint nonloop FNT-cycles in $X\!\restriction\! n$, that is, $D(n) > \alpha n/(12L)$ in the notation of Lemma 4.11. Applying that lemma with $\beta := \alpha/(12L)$ yields

$$\mathbf{E}\big[\mathbf{1}_{\mathtt{Good}\setminus\mathtt{Long}\setminus\mathtt{Loop}}\big(1 - C_n\big(\eta_2\alpha/(12L), \mathrm{NB}(X)\big)\big)\big] < e^{-\delta_2 n}$$

for some $\eta_2, \delta_2 > 0$. Thus, the statement of the theorem holds with $\zeta := \min\{(\alpha/32)\log(9/8), \delta_1, \delta_2\}$ and $\gamma := \min\{\eta_1/12, \eta_2\}$.

Now we prove the second part of the theorem.

Suppose that G is d-regular. We may also suppose that $\rho(G) < (8/9)^{1/4}$, as there is nothing to prove otherwise. Choose L so that $L/\log L \geq 2853/\alpha$, which is $>84/(\alpha\log(1/\rho))$. Lemma 4.12 then ensures that the above event $\texttt{Long}$ has exponentially small probability:

$$\mathbf{P}(\texttt{Long}) < \frac{\rho^{\alpha n/84}}{1-\rho}.$$

Let $Y := \texttt{NB}(X)$. Although we did not state it, our proofs of Lemmas 4.10 and 4.11 provide many cycle times of Y that occur in cycles of length $\leq L$, that is, they show that the event $A_n^L(\beta)$ occurs with high probability for certain β. Thus,

$$\mathbf{P}\big[\texttt{Good} \setminus \texttt{Long} \setminus A_n^L(\hat{\alpha})\big] < \frac{\rho^{\alpha n/84}}{1-\rho} + e^{-\zeta n}.$$

It follows by Lemma 4.13 that if

$$\mathbf{P}(\texttt{Good}) \geq \frac{c_o n(d-1)^{-\alpha^2 n/24+L/2}}{\texttt{cogr}(G)-1} + \frac{\rho^{\alpha n/84}}{1-\rho} + e^{-\zeta n},$$

then $\rho(G) > 2\sqrt{d-1}/d$.

Finally, if $\limsup_{n\to\infty}[\mathbf{E}C_n(\alpha, X)]^{1/n} = 1$, then $\limsup_{n\to\infty}\mathbf{P}[A_n^L(\hat{\alpha}, Y)]^{1/n} = 1$, so Lemma 4.13 completes the proof. $\square$

REMARK 4.14. Instead of requiring all degrees in G to be at least 3, one could require that $\rho(G) < 1$. A similar result holds.

PROOF OF THEOREM 1.4. Let $\mathcal{P}$ be an infinite path. Write α_n for the number of NT-cycle times $\leq n$ in $\mathcal{P}$, divided by n. Since we count here cycles that may end after time n, this may be larger than the density β_n of NT-cycle times in $\mathcal{P}\restriction n$. However, we claim that $\limsup_{n\to\infty}\beta_n \geq \limsup_{n\to\infty}\alpha_n$, whence the limsups are equal.

Suppose that $\alpha_n > \beta_n$. Then there is some NT-cycle time $t \leq n$ that belongs to an NT-cycle that ends at some time $s > n$. Every time in $[t, s]$ then is an NT-cycle time for $\mathcal{P}$. It follows that $\beta_s \geq \alpha_n$, and this proves the claim.

It is now clear that Theorem 1.4 follows from Theorem 4.1. $\square$

PROOF OF THEOREM 1.1. The proof follows just as for Theorem 1.4. $\square$

5. Cycle encounters. Here, we prove Theorem 1.2. We first sketch the proof that $q_n \to 0$. Assume the random walk has a good chance of encountering a short cycle at a large time n. Because of the inherent fluctuations of random walk, the time it reaches such a short cycle cannot be precise; there must be many times around n with approximately the same chance. This means that there are actually many short cycles and if we look at how many are encountered at times around n, we will have a good chance of seeing many. This means the cycles are relatively dense (for random walk) in that part of the graph, which boosts the cogrowth and hence the spectral radius.

We begin by proving the following nonconcentration property of simple random walk on regular graphs.

LEMMA 5.1. *Write $p_n(\cdot,\cdot)$ for the n-step transition probability of simple random walk on a given graph. Let $d<\infty$ and $\varepsilon>0$. There exists $c>0$ such that for every d-regular graph G, every $o\in \mathbb{V}(G)$, and every $n\ge 1$, there exists $A\subseteq \mathbb{V}(G)$ that has the property that*

$$p_n(o,A)>1-\varepsilon \tag{5.1}$$

and

$$\forall x\in A, \forall k\in[0,\sqrt{n}] \qquad p_{n+2k}(o,x)\ge cp_n(o,x). \tag{5.2}$$

PROOF. Write $Q_n(j)$ for the probability that a binomial random variable with parameters $\lfloor n/2\rfloor$ and $1/d$ takes the value j. Given ε, define c' so that

$$\sum_{|j-n/(2d)|\le c'\sqrt{n}} Q_n(j)>1-\varepsilon^2.$$

It has been known since the time of de Moivre that

$$Q_{n+2k}(j+k)\ge cQ_n(j)$$

whenever $n\ge 0$, $k\in[0,\sqrt{n}]$, and $|j-n/(2d)|\le c'\sqrt{n}$.

Given G, $o\in\mathbb{V}(G)$, and $n\ge 1$, let $X_1,\dots,X_n$ be n steps of simple random walk on G starting with $X_1^-=o$. Define

$$Z:=\bigl|\{i\in[1,n/2];\,X_{2i-1}=-X_{2i}\}\bigr|.$$

The events $[X_{2i-1}=-X_{2i}]$ are Bernoulli trials with probability $1/d$ each, whence Z has a binomial distribution with parameters $\lfloor n/2\rfloor$ and $1/d$. Thus,

$$\mathbf{P}\bigl[\bigl|Z-n/(2d)\bigr|\le c'\sqrt{n}\bigr]>1-\varepsilon^2$$

by choice of c'. Define

$$A:=\bigl\{x\in\mathbb{V}(G);\,\mathbf{P}\bigl[\bigl|Z-n/(2d)\bigr|\le c'\sqrt{n}|X_n^+=x\bigr]>1-\varepsilon\bigr\}.$$

Since

$$\begin{aligned}1-\varepsilon^2&<\mathbf{P}\bigl[\bigl|Z-n/(2d)\bigr|\le c'\sqrt{n}\bigr]\\&=\sum_{x\in\mathbb{V}(G)}p_n(o,x)\mathbf{P}\bigl[\bigl|Z-n/(2d)\bigr|\le c'\sqrt{n}|X_n^+=x\bigr]\\&\le p_n(o,A)+\bigl(1-p_n(o,A)\bigr)(1-\varepsilon),\end{aligned}$$

we obtain (5.1).

If we excise all even backtracking pairs (X_{2i-1}, X_{2i}) $(1 \le i \le n/2)$ from the path $(X_1, \ldots, X_n)$, then we obtain simple random walk for $n - 2Z$ steps conditioned not to have any even-time step be a backtrack.

Given $k \in [0, \sqrt{n}]$, let $X'_1, \ldots, X'_{n+2k}$ be simple random walk from o coupled with X as follows: Define

$$Z' := \bigl|\bigl\{i \in [1, (n+2k)/2]; X'_{2i-1} = -X'_{2i}\bigr\}\bigr|.$$

By choice of c, we have $\mathbf{P}[Z' = j + k] \ge c\mathbf{P}[Z = j]$ whenever $|j - n/(2d)| \le c'\sqrt{n}$. Thus, we may couple X' and X so that $Z' = Z + k$ with probability at least c whenever $X_n^+ \in A$. Furthermore, we may assume that the coupling is such that when $Z' = Z + k$ and we excise from each path the even backtracking pairs, then what remains in X' is the same as in X. This implies that with probability at least c, we have ${X'}^+_{n+2k} = X_n^+$ whenever $X_n^+ \in A$. This gives (5.2). □

THEOREM 1.2. *Let G be an infinite Ramanujan graph and $L \ge 1$. Let q_n be the probability that simple random walk at time n lies on a nontrivial cycle of length at most L. Then* $\lim_{n\to\infty} q_n = 0$.

PROOF. Let S be the set of vertices that lie on a *simple* cycle of length at most L, so that $q'_n := \mathbf{P}[X_n^- \in S] = \Omega(q_n)$ for $n \ge L$. Suppose that $q'_n > 2\varepsilon$. Choose A and c as in the lemma. Then $\mathbf{P}[X_n^- \in A \cap S] \ge \varepsilon$, whence $\mathbf{P}[X_{n+2k}^- \in A \cap S] \ge c\varepsilon$ for $k \in [0, \sqrt{n}]$.

Let $I^L(n_1, n_2)$ be the number of times $t \in [n_1, n_2]$ for which there exist $n_1 \le s \le t \le u \le n_2$ such that $(X_s, X_{s+1}, \ldots, X_u)$ is a cycle with $u - s \le L$. Then $\mathbf{E}_o[I^L(n, n + \sqrt{n} - 1), X_n^- \in S] \ge c'\sqrt{n}$ for some constant $c' > 0$ (depending only on $c\varepsilon$). Thus, there is some vertex $x \in S$ (a value of X_n^-) for which $\mathbf{E}_x[I^L(1, \sqrt{n})] \ge c'\sqrt{n}$. This also means

$$\mathbf{P}_x\bigl[I^L(1, \sqrt{n}) \ge c'\sqrt{n}/2\bigr] \ge c'/2.$$

Then Theorem 4.2 completes the argument when n is sufficiently large since $c_x \le 2 + 2L\,\mathtt{cogr}(G)^{L-2}$. [The case $\mathtt{cogr}(G) = 1$ is immediate.] Alternatively, one can appeal to Lemmas 4.11 and 4.13 instead of Theorem 4.2. □

Acknowledgement. We are grateful to the referee for various suggestions.

REFERENCES

ABÉRT, M., GLASNER, Y. and VIRÁG, B. (2015). The measurable Kesten theorem. *Ann. Probab.* To appear.

ALON, N. (1986). Eigenvalues and expanders. *Combinatorica* **6** 83–96. Theory of computing (Singer Island, Fla., 1984). MR0875835

ALON, N., BENJAMINI, I., LUBETZKY, E. and SODIN, S. (2007). Non-backtracking random walks mix faster. *Commun. Contemp. Math.* **9** 585–603. MR2348845

ANCONA, A. (1988). Positive harmonic functions and hyperbolicity. In *Potential Theory—Surveys and Problems* (*Prague*, 1987) (J. Král, J. Lukeš, I. Netuka and J. Veselý, eds.). *Lecture Notes in Math*. **1344** 1–23. Springer, Berlin. MR0973878

BIGGS, N. L., MOHAR, B. and SHAWE-TAYLOR, J. (1988). The spectral radius of infinite graphs. *Bull*. *Lond*. *Math*. *Soc*. **20** 116–120. MR0924236

CHEEGER, J. (1970). A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis* (*Papers Dedicated to Salomon Bochner*, 1969) (R. C. Gunning, ed.) 195–199. Princeton Univ. Press, Princeton, NJ. MR0402831

DODZIUK, J. (1984). Difference equations, isoperimetric inequality and transience of certain random walks. *Trans*. *Amer. Math*. *Soc*. **284** 787–794. MR0743744

DODZIUK, J. and KENDALL, W. S. (1986). Combinatorial Laplacians and isoperimetric inequality. In *From Local Times to Global Geometry*, *Control and Physics* (*Coventry*, 1984/85) (K. D. Elworthy, ed.). *Pitman Res*. *Notes Math*. *Ser.* **150** 68–74. Longman, Harlow. MR0894523

GERL, P. (1988). Random walks on graphs with a strong isoperimetric property. *J*. *Theoret*. *Probab*. **1** 171–187. MR0938257

GRIGORCHUK, R. I. (1980). Symmetrical random walks on discrete groups. In *Multicomponent Random Systems* (R. L. Dobrushin, Ya. G. Sinaĭ and D. Griffeath, eds.). *Adv. Probab. Related Topics* **6** 285–325. Dekker, New York. MR0599539

KAIMANOVICH, V. A. (1992). Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators. *Potential Anal*. **1** 61–82. MR1245225

KESTEN, H. (1959a). Full Banach mean values on countable groups. *Math*. *Scand*. **7** 146–156. MR0112053

KESTEN, H. (1959b). Symmetric random walks on groups. *Trans*. *Amer. Math*. *Soc*. **92** 336–354. MR0109367

LI, W.-C. W. (2007). Ramanujan graphs and Ramanujan hypergraphs. In *Automorphic Forms and Applications* (P. Sarnak and F. Shahidi, eds.). *IAS/Park City Math*. *Ser.* **12** 401–427. Amer. Math. Soc., Providence, RI. MR2331350

LUBOTZKY, A., PHILLIPS, R. and SARNAK, P. (1988). Ramanujan graphs. *Combinatorica* **8** 261–277. MR0963118

LYONS, R. and PERES, Y. (2015). *Probability on Trees and Networks*. Preprint, Cambridge Univ. Press. Available at http://pages.iu.edu/~rdlyons/.

MARGULIS, G. A. (1988). Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii* **24** 51–60. MR0939574

MURTY, M. R. (2003). Ramanujan graphs. *J*. *Ramanujan Math*. *Soc*. **18** 33–52. MR1966527

NILLI, A. (1991). On the second eigenvalue of a graph. *Discrete Math*. **91** 207–210. MR1124768

NORTHSHIELD, S. (1992). Cogrowth of regular graphs. *Proc*. *Amer. Math*. *Soc*. **116** 203–205. MR1120509

VAROPOULOS, N. TH. (1985). Isoperimetric inequalities and Markov chains. *J*. *Funct*. *Anal*. **63** 215–239. MR0803093

DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
831 E 3RD ST
BLOOMINGTON, INDIANA 47405-7106
USA
E-MAIL: rdlyons@indiana.edu
URL: http://pages.iu.edu/~rdlyons/

MICROSOFT CORPORATION
ONE MICROSOFT WAY
REDMOND, WASHINGTON 98052-6399
USA
E-MAIL: peres@microsoft.com
URL: http://research.microsoft.com/en-us/um/people/peres/